SPECTRUM OF PRIME FUZZY HYPERIDEALS

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Abstract. Let R be a commutative hyperring with identity. We introduce and study prime fuzzy hyperideals of R. We investigate the Zariski topology on FHspec(R), the spectrum of prime fuzzy hyperideals of R.

1. Introduction

In the last few years a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals in particular, and some interesting topological properties of the spectrum of fuzzy prime ideals of a ring are obtained. As it is well known that Marty in 1934 ([16]) introduced the notion of hypergroup. Zadeh in 1965 ([24]) introduced the notion of a fuzzy subset of a non-empty set X as a function from X to the unit real interval I = [0, 1]. The study of fuzzy hyperstructure is an interesting research topic of fuzzy sets. In this paper we introduce the notion of prime fuzzy hyperideals of a commutative hyperring with identity and will investigate some basic properties of prime fuzzy hyperideals and characterize the prime fuzzy hyperideals. Finally we investigate the topology on FHSpec(R), the set of all prime fuzzy hyperideals of R.

2. Preliminaries

Recall that a fuzzy subset μ of a non-empty set X is a function μ from X to $[0,1]$. F^X denotes the set of all fuzzy subsets of X. Let A be a subset of X and $y \in [0,1]$. Define $y_A \in F^X$ as follows:

$$y_A(x) = \begin{cases} y & x \in A; \\ 0 & otherwise \end{cases}$$

In particular, if $A = \{a\}$ we denote $y_{\{a\}}$ by y_a, and it is called a fuzzy point of X.

For $\mu \in F^X$ and $a \in [0,1]$, define μ_a as follows:

$$\mu_a = \{x \in X | \mu(x) \geq a\},$$

μ_a is called the *a-cut* or *a-level subset* of μ

For $\mu, \nu \in F^X$ we say that μ is contained in ν and we write $\mu \subseteq \nu$ if for all $x \in X$, $\mu(x) \leq \nu(x)$.

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For $\mu, \nu \in F^X$, union and intersection, $\mu \cup \nu$, $\mu \cap \nu \in F^X$, are defined respectively by

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \quad and \quad (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x),$$

for all $x \in X$.

A map $\circ : H \times H \longrightarrow \mathcal{P}_*(H)$ is called *hyperoperation* or *join operation.* A *hypergroupoid* is a set H with together a (binary) hyperoperation $\circ$. A hypergroupoid $(H, \circ)$, which is associative, that is, $x \circ (y \circ z) = (x \circ y) \circ z, \forall x, y, z \in H$ is called a *semihypergroup.* A *hypergroup* is a semihypergroup such that $\forall x \in H$, we have $x \circ H = H = H \circ x$, which is called *reproduction axiom* (for more details see [5]).

Let H be a hypergroup and K a nonempty subset of H. Then K is said to be a *subhypergroup* of H if it is a hypergroup under hyperoperation "$\circ$". Hence it is clear that a subset K of H is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on H. A subhypergroup K of H is called invertible if for all $x, y \in H$ we have

$$x \in yK \Leftrightarrow y \in xK \quad and \quad x \in Ky \Leftrightarrow y \in Kx.$$

Definition 2.1. [17] A semihypergroup $(R, +)$ is called a *canonical hypergroup* if the following conditions are satisfied:

(i) $x + y = y + x \quad \forall x, y \in R$;
(ii) $\exists 0 \in R$ (unique) such that for every $x \in R, x \in 0 + x = x$;
(iii) for every $x \in R$, there exists a unique element, say x' such that $0 \in x+x'$.(we denote x' by $-x$);
(iv) for every $x, y, z \in R, z \in x + y \Longleftrightarrow x \in z - y \Longleftrightarrow y \in z - x.$

Definition 2.2. A triple $(R, +, .)$ is called a *hyperring* if the following conditions are satisfied:

(i) $(R, +)$ is a canonical hypergroup;
(ii) $(R, .)$ is a semi-hypergroup;
(iii) $x.(y + z) = x.y + x.z, (y + z).x = y.x + z.x, \forall x, y, z \in R.$

Definition 2.3. A hyperring $(R, +, .)$ is said to be:

(i) *commutative* if $x + y = y + x, \forall x, y \in R$;
(ii) with *identity* if there exists a unique element, say $1_R \in R$, such that $x \in 1_R.x \cap x.1_R, \forall x \in R.$

Definition 2.4. A canonical subhypergroup I of R is a *right(resp. left) hyperideal* of R provided that $r.x \subseteq I$ (resp. $x.r \subseteq I), \forall r \in R, \forall x \in I$. Also I is called a *hyperideal* if I is both left and right hyperideal.

In the sequel by R we mean a commutative hyperring with an identity.

Definition 2.5. [3] Let R be a hyperring. A hyperideal I of R is called *prime* if $a.b \subseteq I$ implies that either $a \in I$ or $b \in I$, for $a, b \in R$.

Definition 2.6. [3] Let R be a hyperring. A hyperideal I of R is called *maximal* if I is a maximal element in the set of all hyperideals of R with respect to the inclusion relation.

3. Some Results on Hyperideals

In the sequel by R we mean a commutative hyperring with an identity, say 1.

Lemma 3.1. *Every maximal hyperideal is prime.*

Proof. Straight forward $\square$

Lemma 3.2. *Every hyperideal is contained in a maximal hyperideal.*

Proof. Straight forward $\square$

Definition 3.3. Let R be a hyperideal and A be a subset of R. The hyperideal *generated* by A denoted by $< A >$, is the smallest hyperideal of R containing A, that is

$$< A >= \bigcap\{I \ \mid \ A \subseteq I \text{ and } I \text{ is hyperideal of } R\}.$$

It is easy to see that

$$< A >= \bigcup\{\sum\nolimits_{i=1}^{n} r_i a_i \ \mid \ r_i \in R, a_i \in A, n \in \mathbb{N}\}.$$

Corollary 3.4. *Let R be a hyperideal and $x \in R$, then*

$$< x >= \bigcup\{rx | r \in R\}.$$

For hyperideals I and J of a hyperring R we define multiplication of I and J as follows:

$$IJ = \bigcup\{\sum\nolimits_{i=1}^{n} r_i s_i | r_i \in I, s_i \in J, n \in \mathbb{N}\}.$$

It is easy to see that IJ is a hyperideal of R contained in $I \cap J$.

Lemma 3.5. *Let $x, y \in R$, then $< x > . < y > \subseteq < x.y >$.*

Proof. Let $z \in < x > . < y >$, then $z \in \displaystyle\sum_{i=1}^{n} x_i y_i$ for $x_i \in < x >$ and $y_i \in < y >$. But for each $1 \leq i \leq n$ there exist $r_i, r_i' \in R$, such that $x_i \in r_i x$ and $y_i \in r_i' y$. So

$$z \in \sum_{i=1}^{n} (r_i x)(r_i' y) = \sum_{i=1}^{n} (r_i r_i')(xy).$$

But

$$\sum_{i=1}^{n} (r_i r_i')(xy) = \bigcup\{c.b | c \in \bigcup_{a_i \in r_i r_i'} a_i, b \in x.y\}.$$

Now there exist $c \in \bigcup_{a_i \in r_i r'_i} a_i$ and $b \in x.y$, such that $z \in c.b$. Since $c \in \bigcup_{a_i \in r_i r'_i} a_i$, then for $i = 1, ..., n$, there exist $a_i \in r_i.r'_i$, such that $c \in \sum_{i=1}^{n} a_i$. Therefore

$$z \in (\sum_{i=1}^{n} a_i).b = \sum_{i=1}^{n} a_i.b,$$

this means that $z \in < x.y >$. Thus $< x >< y > \subseteq < x.y >$. □

4. **Fuzzy Hyperideals**

Definition 4.1. [3] A fuzzy subset μ of R is called a *fuzzy hyperideal* of R if the following conditions hold:

(*i*) $\bigwedge_{z \in x+y} \mu(z) \geq \mu(x) \wedge \mu(y), \forall x, y \in R$;

(*ii*) $\mu(-x) \geq \mu(x), \forall x \in R$;

(*iii*) $\bigwedge_{z \in x.y} \mu(z) \geq \mu(x) \vee \mu(y), \forall x, y \in R$.

In the sequel by $FHI(R)$ we mean the set of all fuzzy hyperideals of R.

Theorem 4.2. [3] *Let μ be a fuzzy subset of R. Then μ is fuzzy hyperideal of R if and only if every non-empty level subset of R is a hyperideal of R.*

Corollary 4.3. [3] *Let μ be a fuzzy hyperideal of R. Then for all $x \in R, \mu(x) = \mu(-x)$.*

Proposition 4.4. [3] *Let μ be a fuzzy hyperideal of R. Then $\mu(x) \leq \mu(0), \forall x \in R$, and if R has an identity say 1_R, then $\mu(1_R) \leq \mu(x), \forall x \in R$.*

Definition 4.5. [3] Let $\mu, \nu \in FHI(R)$. The *product* of $\mu, \nu \in FHI(R)$ is defined by

$$(\mu.\nu)(z) = \bigvee_{z \in x.y} (\mu(x) \wedge \nu(y)).$$

Proposition 4.6. [3] *Let $\mu \in FHI(R)$. If $\bigwedge_{z \in x-y} \mu(z) = \mu(0)$, for $x, y \in R$, then $\mu(x) = \mu(y)$.*

Definition 4.7. Let $(R, +, .)$ be a hyperring and let μ be a fuzzy subset of R. We define μ_* as follows:

$$\mu_* = \{x \in R | \mu(x) = \mu(0)\}.$$

If μ is a fuzzy hyperideal of R, then μ_* is a hyperideal of R.

Definition 4.8. Let R be a hyperring and let μ be a fuzzy subset of R. Set

$$< \mu >= \bigcap \{\nu \in FHI(R) \ \mid \ \mu \subseteq \nu\},$$

then $< \mu >$ is called *fuzzy hyperideal generated* by μ.

Lemma 4.9. *Let* $\{\mu_i\}_{i\in I} \subseteq FHI(R)$, *then* $\bigcap_{i\in I} \mu_i \in FHI(R)$.

Proof. For each $i \in I$ we have

$$\bigwedge_{z\in x+y} \mu_i(z) \geq \mu_i(x) \wedge \mu_i(y) \geq \bigcap_{i\in I} \mu_i(x) \wedge \bigcap_{i\in I} \mu_i(y)$$

So

$$\bigwedge_{i\in I} (\bigwedge_{z\in x+y} \mu_i(z)) \geq \bigcap_{i\in I} \mu_i(x) \wedge \bigcap_{i\in I} \mu_i(y) \Longrightarrow \bigwedge_{z\in x+y} (\bigwedge_{i\in I} \mu_i(z)) \geq \bigcap_{i\in I} \mu_i(x) \wedge \bigcap_{i\in I} \mu_i(y).$$

Therefore

$$\bigwedge_{z\in x+y} (\bigcap_{i\in I} \mu_i)(z) \geq (\bigcap_{i\in I})(x) \wedge (\bigcap_{i\in I} \mu_i)(y).$$

In the same way we can prove the other parts. □

Corollary 4.10. *Let* μ *be a fuzzy subset of* R. *Then* $< \mu > \in FHI(R)$.

Theorem 4.11. *Let* R *be a hyperring and let* $A \subseteq R$ *and* $c \in [0,1]$. *Then* $c_{<A>} = < c_A >$.

Proof. Since $A \subseteq < A >$, so $c_A \subseteq c_{<A>}$ and then

$$< c_A > \subseteq c_{<A>} \tag{1}$$

Now let $\mu \in FHI(R)$ such that $c_A \subseteq \mu$, so for each $a \in A$, we have $c \leq \mu(a)$. Let $y \in < A >$, then $y \in \sum_{i=1}^{n} r_i a_i$ for $r_i \in R$ and $a_i \in A$ and $n \in \mathbb{N}$. Therefore, for $i = 1, .., n$ there exist $t_i \in r_i a_i$, such that $y \in \sum_{i=1}^{n} t_i$. Now

$$\begin{aligned} \mu(y) \geq \bigwedge_{i=1}^{n} \mu(t_i) &\geq \bigwedge_{i=1}^{n} (\mu(r_i) \vee \mu(a_i)) \\ &\geq \bigwedge_{i=1}^{n} (\mu(r_i) \vee c) \\ &= (\bigwedge_{i=1}^{n} (\mu(r_i)) \vee c \\ &\geq c = c_{<A>}(y) \end{aligned}$$

and if $y \notin < A >$, then $0 = c_{<A>}(y) \leq \mu(y)$. Hence $c_{<A>} \subseteq \mu$ and then

$$c_{<A>} \subseteq \bigcap \{\mu | c_A \subseteq \mu\} = < c_A > \tag{2}$$

By $(1), (2)$ we have $c_{<A>} = < c_A >$. □

Corollary 4.12. $c_{<x>} = < c_x >$ *for* $c \in [0,1]$ *and* $x \in R$.

Proof. Straight forward. □

Theorem 4.13. *Let $\mu, \nu \in F(R)$, if $\mu \in FHI(R)$, then $\mu\nu \subseteq \mu$.*

Proof. Suppose that $x \in y.z$ for $y, z \in R$, so

$$\mu(x) \geq \bigwedge_{t \in y.z} \mu(t) \geq \mu(y) \vee \mu(z) \geq \mu(y) \geq \mu(y) \wedge \nu(z).$$

Then for all $y, z \in R$, such that $x \in y.z$ we have

$$\mu(x) \geq \bigvee \{\mu(y) \wedge \nu(z) \;\;|\;\; x \in y.z\} = (\mu\nu)(x),$$

and so $\mu\nu \subseteq \mu$. □

The next result follows immediately from Theorem 4.13.

Corollary 4.14. *If $\mu, \nu \in FHI(R)$, then $\mu\nu \subseteq \mu \cap \nu$.*

Proof. By the previous theorem it is clear. □

5. Prime Fuzzy Hyperideals

Definition 5.1. A fuzzy hyperideal p of R is called a *prime fuzzy hyperideal* if p is non-constant and for hyperideals μ, ν of R if $\mu\nu \subseteq p$, then either $\mu \subseteq p$ or $\nu \subseteq p$.

Lemma 5.2. *Let $x, y \in R$ and $a, b \in [0, 1]$, then $< a_x >< b_y > \subseteq < (a \wedge b)xy >$.*

Proof. By Theorem 4.11 and Corollary 4.12 it is sufficient to show that $a_{<x>}b_{<y>} \subseteq (a \wedge b)_{<xy>}$. Let $z \in R$, then

$$(a_{<x>}.b_{<y>})(z) = \bigvee \{a_{<x>}(r) \wedge b_{<y>}(s) | z \in r.s\}.$$

If there exist $r \in < x >$ and $s \in < y >$ such that $z \in r.s$, then $(a_{<x>}.b_{<y>})(z) = a \wedge b$. Since $r \in < x >$ and $s \in < y >$, $z \in < x > . < y >$. By Lemma 3.5, $< x > . < y > \subseteq < x.y >$, so $z \in < x.y >$, and hence $(a \wedge b)_{<x.y>}(z) = a \wedge b$. If for each $r \in < x >$ and $s \in < y >, z \notin r.s$, then $(a_{<x>}.b_{<y>})(z) = 0$. Hence, $a_{<x>}.b_{<y>} \subseteq (a \wedge b)_{<x.y>}$. This complete the proof. □

Theorem 5.3. *If p is a prime fuzzy hyperideal of R, then p_* is a prime hyperideal of R.*

Proof. Let $x, y \in R$ and $x.y \subseteq p_*$, then $< x.y > \subseteq p_*$. Let $\mu = 1_{<x>}, \nu = 1_{<y>}$. By Corollary 4.12, $\mu =< 1_x >, \nu =< 1_y >$ and $\mu\nu \subseteq < 1_{(x.y)} >$. Again by Theorem 4.11, $< 1_{(x.y)} >= 1_{<x.y>}$. Therefore $\mu.\nu \subseteq 1_{<x.y>} \subseteq 1_{p_*} \subseteq p$. Since p is a prime fuzzy hyperideal either $\mu \subseteq p$ or $\nu \subseteq p$. So either $1_{<x>} \subseteq p$ or $1_{<y>} \subseteq p$, and hence either $< x > \subseteq p_*$ or $< y > \subseteq p_*$ and finally either $x \in p_*$ or $y \in p_*$. This shows that p_* is prime. □

Theorem 5.4. *Suppose that $p \in FHI(R)$. Then p is a prime fuzzy hyperideal of R if and only if $p(0) = 1, p_*$ is a prime hyperideal of R and $p = 1_{p_*} \cup c_R$ for some $c \in (0, 1]$.*

Proof. First suppose that p is a prime fuzzy hyperideal. By Theorem 5.3, p_* is a prime hyperideal of R. Now we show that $p(0) = 1$. Suppose that $p(0) < 1$, since p is non-constant, then there exists $x \in R$ such that $p(x) < p(0)$. Let $\mu, \nu \in F^R$ are defined by:

$$\mu(x) = \begin{cases} 1 & x \in p_*; \\ 0 & x \notin p_* \end{cases} \quad \textit{and} \;\; \nu(x) = p(0).$$

So $\mu, \nu \in FHI(R)$ and $\mu.\nu \subseteq p$. But $\mu(0) = 1 > p(0)$ and for $x \in R$, if $x \notin p_*, \nu(x) = p(0) > p(x)$, so $\mu \not\subseteq p$ and $\nu \not\subseteq p$ which is a contradiction. Therefore $p(0) = 1$. Since $1 \in p(R)$ and p is non-constant, $|p(R)| \geq 2$. Let $x, y \in R \setminus p_*$. We shall show that $p(x) = p(y)$. Let $p(x) = c$. Then by Corollary 4.12, $c_{<x>} =< c_x >\subseteq p$. Now $1_{<x>}, c_R \in FHI(R)$ and by Corollary 4.14, we have

$$1_{<x>}.c_R \subseteq 1_{<x>} \cap c_R = c_{<x>} \subseteq p,$$

and $1_{<x>} \not\subseteq p$. Since p is prime fuzzy hyperideal, $c_R \subseteq p$. Thus $p(x) = c = c_R(y) \leq p(y)$. Similarly it can be shown that $p(y) \leq p(x)$. Hence $p(x) = p(y)$. This means that $|p(R)| = 2$. Therefore, $p = 1_{p_*} \cup c_R$,where p_* is prime hyperideal of R and $c \in [0, 1]$.
Conversely, suppose that $p(0) = 1, p_*$ is a prime hyperideal of R and $p = 1_{p_*} \cup c_R$ for $c \in [0, 1]$. Since $p(R) = \{1, c\}$ then p is non-constant. Let for $\mu, \nu \in FHI(R)$, $\mu\nu \subseteq p$ but $\mu \not\subseteq p$ and $\nu \not\subseteq p$. Then there exist $x, y \in R$, such that $\mu(x) > p(x)$ and $\nu(y) > p(y)$. It means that $x, y \in R \setminus p_*$, then $p(x) = p(y) = c$. Thus $\mu(x) > c$ and $\nu(y) > c$. Since $x, y \in R \setminus p_*$ and p_* is a prime hyperideal, $xy \not\subseteq p_*$. Therefore, there exists $t \in xy$, such that $t \in p_*$ and then $p(t) = c$. Now since $t \in xy$ then:

$$c = p(t) \geq (\mu\nu)(t) \geq \mu(x) \wedge \nu(x).$$

Hence $\mu(x) \leq c$ or $\nu(y) \leq c$, which is a contradiction. Thus p is a prime hyperideal of R, as required. □

6. **Topology on** $FHSpec(R)$

Let R be a commutative hyperring with identity. By $FHSpec(R)$ we mean the set of all the prime fuzzy hyperideals of R. Set $X = FHSpec(R)$. For $\mu \in FHI(R)$ we define $V(\mu)$ and $E(\mu)$ as follows:

$$V(\mu) = \{p \in X | \mu \subseteq p\} \quad E(\mu) = X \setminus V(\mu).$$

Lemma 6.1. *Let $\mu \in F(R)$, then $V(< \mu >) = V(\mu)$.*

Proof. Straight forward. □

Let $T = \{E(\mu) | \mu \in FHI(R)\}$. In the next theorem we will show that the pair (X, T) is a topological space.

Theorem 6.2. *The pair (X, T) is a topological space.*

Remark 6.3. The resulting topology in the previous theorem is called *Zariski topology* and the topological space (X, T) is called the *prime fuzzy spectrum* of R.

Proof. 1) Let $\mu = 1_R$ and $\nu = 0_R$. Then $V(\mu) = \emptyset$ and $V(\nu) = X$, and hence $E(\mu) = X$ and $E(\nu) = \emptyset$. Therefore, $X, \emptyset \in T$.

2) Suppose that $\mu_1, \mu_2 \in FHI(R)$. We show that

$$E(\mu_1) \cap E(\mu_2) = E(\mu_1 \cap \mu_2).$$

Let $\nu \in E(\mu_1) \cap E(\mu_2)$, then $\mu_1 \not\subseteq \nu$ and $\mu_2 \not\subseteq \nu$. Since ν is a prime fuzzy hyperideal, $\mu_1\mu_2 \not\subseteq \nu$. By Corollary 4.14, $\mu_1\mu_2 \subseteq \mu_1 \cap \mu_2$. Therefore $\mu_1 \cap \mu_2 \not\subseteq \nu$ and hence $\nu \in E(\mu_1 \cap \mu_2)$. Thus

$$E(\mu_1) \cap E(\mu_2) \subseteq E(\mu_1 \cap \mu_2) \tag{1}$$

Now let $\nu \in E(\mu_1 \cap \mu_2)$ then $\mu_1 \cap \mu_2 \not\subseteq \nu$, so $\mu_1 \not\subseteq \nu$ and $\mu_2 \not\subseteq \nu$. Hence $\nu \in E(\mu_1)$ and $\nu \in E(\mu_2)$ and Thus $\nu \in E(\mu_1) \cap E(\mu_2)$, therefore

$$E(\mu_1 \cap \mu_2) \subseteq E(\mu_1) \cap E(\mu_2) \tag{2}$$

From $(1), (2)$ it follows $E(\mu_1 \cap \mu_2) = E(\mu_1) \cap E(\mu_2)$.

3) Suppose that $\{\mu_i | i \in I\}$ is a family of fuzzy hyperideals of R. We show that $\bigcup\limits_{i\in I} E(\mu_i) = E(< \bigcup\limits_{i\in I} \mu_i >)$. Let $\mu \notin \bigcup\limits_{i\in I} E(\mu_i)$, then for each $i \in I, \mu \notin E(\mu_i)$ and so for each $i \in I, \mu_i \subseteq \mu$. Then $\bigcup\limits_{i\in I} \mu_i \subseteq \mu$. Now we have

$$< \bigcup_{i\in I} \mu_i > \subseteq \mu \Longrightarrow \mu \in V(< \bigcup_{i\in I} \mu_i >) \Longrightarrow \mu \notin E(< \bigcup_{i\in I} \mu_i >).$$

It means that

$$E(< \bigcup_{i\in I} \mu_i >) \subseteq \bigcup_{i\in I} E(\mu_i) \tag{1}$$

Let $\mu \notin E(< \bigcup\limits_{i\in I} \mu_i >)$ then $\mu \in V(< \bigcup\limits_{i\in I} \mu_i >)$ and hence $< \bigcup\limits_{i\in I} \mu_i > \subseteq \mu$. For each $i \in I, \mu_i \subseteq \bigcup\limits_{i\in I} \mu_i \subseteq < \bigcup\limits_{i\in I} \mu_i > \subseteq \mu$. So for each $i \in I, \mu_i \subseteq \mu$. Thus

$$\mu \in V(\mu_i), \forall i \in I \Longrightarrow \mu \notin E(\mu_i), \forall i \in I \Longrightarrow \mu \notin \bigcup_{i\in I} E(\mu_i).$$

Thus

$$\bigcup_{i\in} E(\mu_i) \subseteq E(< \bigcup_{i\in I} \mu_i >) \tag{2}$$

(1) and (2) imply that

$$\bigcup_{i\in I} E(\mu_i) = E(< \bigcup_{i\in I} \mu_i >).$$

Thus the pair (X, T) satisfies in axioms of a topological space. Therefore (X, T) is a topological space. $\square$

Theorem 6.4. *Let $x, y \in R$ and $a, b \in (0,1]$. Then*

$$E(x_a) \cap E(y_b) = \bigcup_{z \in x.y} (z_{(a \wedge b)}).$$

Proof. Suppose that $\mu \in E(x_a) \cap E(y_b)$ then

$$\mu \in E(x_a) \text{ and } \mu \in E(y_b) \Longrightarrow x_a \not\subseteq \mu \text{ and } y_b \not\subseteq \mu \Longrightarrow a > \mu(x) \text{ and } b > \mu(y).$$

Thus $\mu(x) \neq 1$ and $\mu(y) \neq 1$. Since μ is prime and $|Im(\mu)| = 2$ then $\mu(x) = \mu(y)$ and $x, y \notin \mu_*$. But μ_* is a prime hyperideal, and hence $x.y \not\subseteq \mu_*$, then there exists $t \in x.y$, such that $t \notin \mu_*$, hence $\mu(t) = \mu(x) = \mu(y)$ and we have

$$\mu(t) < a, b \Longrightarrow \mu(t) < a \wedge b \Longrightarrow t_{(a \wedge b)} \not\subseteq \mu \Longrightarrow \mu \in E(t_{(a \wedge b)}) \Longrightarrow \mu \in \bigcup_{t \in x.y} (t_{(a \wedge b)}).$$

Thus

$$E(x_a) \cap E(y_b) \subseteq \bigcup_{t \in x.y} E(t_{(a \wedge b)}) \tag{1}$$

Now suppose that $\mu \in \bigcup_{t \in x.y} E(t_{(a \wedge b)})$, then $t_{(a \wedge b)} \not\subseteq \mu \Longrightarrow \mu(t) < a \wedge b \leq a, b$. But $\mu(t) \neq 1$ and so $t \notin \mu_*$. Now we have

$$a, b > \mu(t) \geq \bigwedge_{t \in x.y} \mu(t) \geq \mu(x) \vee \mu(y) \geq \mu(x), \mu(y).$$

So

$$\mu(x) < a, \mu(y) < b \Longrightarrow x_a \not\subseteq \mu, y_b \not\subseteq \mu \Longrightarrow \mu \in E(x_a), E(y_b) \Longrightarrow \mu \in E(x_a) \cap E(y_b).$$

Therefore,

$$\bigcup_{t \in x.y} E(t_{(a \wedge b)}) \subseteq E(x_a) \cap E(y_b) \tag{2}$$

From (1) and (2) we have

$$\bigcup_{t \in x.y} E(t_{(a \wedge b)}) = E(x_a) \cap E(y_b).$$

□

Theorem 6.5. *The set $B = \{E(x_a) | x \in R, a \in (0,1]\}$ forms a base for (X, T).*

Proof. Let $E(\mu)$ be an open set in T and let $\nu \in E(\mu)$, then $\mu \not\subseteq \nu$ and so for some $x \in R$, $\mu(x) > \nu(x)$. Letting $\alpha = \mu(x)$ then $x_a \not\subseteq \nu$, therefore $\nu \in E(x_a)$. Now we show that $V(\mu) \subseteq V(x_a)$. Let $\theta \in V(\mu)$, then

$$\mu \subseteq \theta \Longrightarrow x_a(x) = \mu(x) \leq \theta(x) \Longrightarrow x_a \subseteq \theta \Longrightarrow \theta \in V(x_a).$$

Hence $V(\mu) \subseteq V(x_a)$ and so $E(x_a) \subseteq E(\mu)$. Thus $\nu \in E(x_a) \subseteq E(\mu)$. It means that B is a base for (X, T). □

Lemma 6.6. *Let $a, b \in (0,1]$ and $a \leq b$. Then $E(x_a) \subseteq E(x_b)$ for $x \in R$.*

Proof. Suppose that $\mu \in E(x_a)$ then

$$x_a \not\subseteq \mu \Longrightarrow a > \mu(x).$$

But $b \geq a$ then $b > \mu(x)$ and hence

$$x_b \not\subseteq \mu \Longrightarrow \mu \in E(x_b).$$

Therefore

$$E(x_a) \subseteq E(x_b).$$

□

Lemma 6.7. *Let $K \subseteq (0,1]$ and let $X = \bigcup\{E((x_i)_t)|i \in I, t \in K, x_i \in R\}$. Then $\bigvee\{t|t \in K\} = 1$.*

Proof. The proof is similar to that of [9]. □

Theorem 6.8. *The topological space (X, T) is compact.*

Proof. Since the set $B = \{E(x_a)|x \in R, a \in (0,1]\}$ is a base for this topological space, we assume that the set $\{E((x_i)_t)|i \in I, x_i \in R, t \in K \subseteq (0,1]\}$ is a cover for X. Let $\alpha = \bigvee\{t|t \in K\}$. By Lemma 6.8, $\alpha = 1$ and by Lemma 6.7, the set $\{E((x_i)_1)|i \in I\}$ is a covering for X. Now we have

$$X = \bigcup_{i \in I} E((x_i)_1) = E(<\bigcup_{i \in I}(x_i)_1>) = X \setminus V(<\bigcup_{i \in I}(x_i)_1>).$$

On the other hands we have $V(<\bigcup_{i \in I}(x_i)_1>) = V(\bigcup_{i \in I}(x_i)_1)$, so $X = X \setminus V(\bigcup_{i \in I}(x_i)_1)$, and hence $V(\bigcup_{i \in I}(x_i)_1) = \emptyset$. Let p be any prime hyperideal of R and let

$$\mu(x) = \begin{cases} 1 & x \in p \\ 0 & x \notin p. \end{cases}$$

Clearly, μ is a prime hyperideal of R and $\mu \notin V(\bigcup_{i \in I}(x_i)_1)$, then $\bigcup_{i \in I}(x_i)_1 \not\subseteq \mu$, so there exists $j \in I$, such that $(x_j)_i \not\subseteq \mu$. Therefore $\mu(x_j) < 1$ and hence $x_j \notin p$. Thus there is no any prime hyperideal consisting the set $\{x_i|i \in I\}$ and then there is no hyperideal consisting the set $\{x_i|i \in I\}$, otherwise $I \subseteq m$ for some maximal hyperideal m (by Lemma 3.9) and so by Lemma 3.1, m is prime which is a contradiction. Hence $< \{x_i|i \in I\} >= R$. Since $1_R \in R$. Then $1_R \in \sum_{i=1}^{n} r_i x_i$ for $r_i \in R$ and $n \in \mathbb{N}$. Now we show that $V(\bigcup_{i=1}^{n}(x_i)_1) = \emptyset$. Let $\mu \in V(\bigcup_{i=1}^{n}(x_i)_1)$. Then $\bigcup_{i=1}^{n}(x_i)_1 \subseteq \mu$. So for each $i = 1, ..., n, (x_i)_1 \subseteq \mu$, therefore for each $i = 1, ..., n, 1 \leq \mu(x_i)$ and so $\mu(x_i) = 1$ for each $i = 1, ..., n$. Thus for each $i = 1, ..., n, x_i \in \mu_*$, and hence $\sum_{i=1}^{n} r_i x_i \subseteq \mu_*$, therefore $1_R \in \mu_*$, which is a contradiction. Thus $V(\bigcup_{i=1}^{n}(x_i)_1) = \emptyset$, and so

$$X = X \setminus V(\bigcup_{i=1}^{n}(x_i)_1) = X \setminus V(< \bigcup_{i=1}^{n}(x_i)_1 >) = E(< \bigcup_{i=1}^{n}(x_i)_1 >) = \bigcup_{i=1}^{n} E((x_i)_1).$$

This shows that X is compact. $\square$

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