ON INTERRELATIONSHIPS BETWEEN FUZZY METRIC STRUCTURES

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Abstract. Considering the increasing interest in fuzzy theory and possible applications, the concept of fuzzy metric space concept has been introduced by several authors from different perspectives. This paper interprets the theory in terms of metrics evaluated on fuzzy numbers and defines a strong Hausdorff topology. We study interrelationships between this theory and other fuzzy theories such as intuitionistic fuzzy metric spaces, Kramosil and Michalek's spaces, Kaleva and Seikkala's spaces, probabilistic metric spaces, probabilistic metric co-spaces, Menger spaces and intuitionistic probabilistic metric spaces, determining their position in the framework of theses different theories.

1. **Introduction**

One of the most interesting research topics in fuzzy topology is to find an appropriate definition of *fuzzy metric space* for its possible applications in several areas. Many authors have considered this problem and have introduced it in different ways [10, 11, 22, 23, 27, 26, 37]. Possibly the first approach to this problem was carried out by Menger [27] who introduced *distribution functions* as distances between points. Using arbitrary triangle functions, Schweizer and Sklar [37] introduced the *probabilistic metric spaces*. The theory of *fuzzy sets*, originally introduced by Zadeh [45], gave a new perspective to this problem, allowing us to apply fuzzy behaviour to model real situations. On the one hand, inspired by this, Kramosil and Michalek [23] generalized the concept of probabilistic metric space to the fuzzy situation and defined a Hausdorff topology on these spaces. In [6], Deng studied the topology of a fuzzy pseudo-metric space. George and Veeramani [11] slightly modified the concept of fuzzy metric space introduced by Kramosil and Michalek, defined a Hausdorff topology and proved some known results including Baire's theorem. On the other hand, using the theory of *fuzzy numbers* [8, 17], Kaleva and Seikkala [22] proposed a class of spaces that set the distance between two points as a nonnegative fuzzy number. Later, Atanassov [3] introduced the concept of *intuitionistic fuzzy set* which is characterized by a membership function and a non-membership function. Recently, much work has been done on this theory by many authors [2, 6, 42, 43]. Based on this concept, Park [30, 33] introduced the *intuitionistic fuzzy metric spaces* as a natural generalization of fuzzy metric spaces due to George and Veeramani.

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However, for our purposes, this paper uses the Castro-Company and Romaguera's version that can be found in [4] (see also [2]).

Due to the increasing interest in this area and motivated by its possible applications ([4, 9, 12, 13, 16, 18, 25, 32, 41, 42, 43]) in this paper we describe a common structure for the previous spaces, allowing us to work in a unified way to deal with very different problems. In addition to the evident connections, it can be observed (see [22]) that every Menger space [27] can be considered as a Kaleva and Seikkala fuzzy metric space, but the converse is not obvious in the general case. Kaleva recently explained in [21] that a fuzzy metric space of a different flavor was introduced by Kramosil and Michalek [23] but an overall view has not been studied. It is necessary to establish connections between the different definitions. This is the main aim of this paper. We have found that what really underlies on common in this type of spaces is a triple formed by a basic set, a *distance* function evaluated on the set of fuzzy numbers and a triangular function that establishes a relationship between *distances* that can be calculated between three points of the space (similar to a triangular inequality). Most of the current notions of *fuzzy metric space* interpret the distance between two points as a distribution function in the real environment. However, in the fuzzy setting, it seems more coherent to use fuzzy numbers for this purpose. Few authors did this in the past. Interpreting a fuzzy number as a couple of distance distribution functions, it is easy to consider a notion of purely fuzzy metric space that is capable of taking advantage of the best of each of the other theories. To do this, we need to do a complete study of triangular functions on fuzzy numbers, relating them to those used in other contexts. In this way, we prove that probabilistic metric spaces, fuzzy metric spaces in the sense of Kramosil and Michalek, intuitionistic fuzzy metric spaces, intuitionistic probabilistic metric spaces and some fuzzy metric spaces in the sense of Kaleva and Seikkala can be included in this new view of the problem.

This paper is organized as follows: Some preliminaries and notations concerning fuzzy metric spaces (from different points of view) are gathered in Section 2. Section 3 describes a way to interpret fuzzy numbers as a pair of distance distribution functions that provide them with a metric. In Section 4, results established in previous sections, will be used to obtain the principal objective of this paper, that is, to introduce a concept of fuzzy metric space that can include other structures defined previously. This concept is used to study the relationships with other concepts previously established. Finally, Section 5 includes some concluding remarks and prospects for further work.

2. **Preliminaries**

In the sequel $\mathbb{R}$ will denote the set of real numbers and $\overline{\mathbb{R}} = [-\infty, \infty]$ the extended real line (for simplicity, $+\infty$ will be denoted as ∞).

Let $(\Lambda, \leq)$ be a partially ordered nonempty set. A *triangle function on* Λ (or a t_Λ*-norm*) is a map $\tau : \Lambda \times \Lambda \to \Lambda$ that is associative, commutative, nondecreasing in both arguments (that is, $\tau(\lambda_1, \lambda_3) \leq \tau(\lambda_2, \lambda_4)$ whenever $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Lambda$ with $\lambda_1 \leq \lambda_2$ and $\lambda_3 \leq \lambda_4$) and has an element $\lambda_0 \in \Lambda$ as identity (i.e., $\tau(\lambda, \lambda_0) = \lambda$ for all $\lambda \in \Lambda$).

A *t-norm* is a triangle function $* : [0,1]^2 \longrightarrow [0,1]$ that has 1 as identity, and a *t-conorm* is a triangle function $\diamond : [0,1]^2 \longrightarrow [0,1]$ that has 0 as identity. If $*$ is a (continuous) t-norm, $a *' b = 1 - [(1-a) * (1-b)]$ for all $a, b \in [0,1]$ defines a (continuous) t-conorm, $*'$, called the *t-conorm associated to* $*$. Conversely, if $\diamond$ is a (continuous) t-conorm, its (continuous) t-norm associated, $\diamond'$, is defined as $a \diamond' b = 1 - [(1-a) \diamond (1-b)]$ for all $a, b \in [0,1]$.

A *distance distribution function* (*d.d.f.*) (which was studied in detail in [37]) is a nondecreasing function $f : [0,\infty] \to [0,1]$ that is left continuous on $]0,\infty[$ with $f(0) = 0$ and $f(\infty) = 1$. This definition is extended to $f : \overline{\mathbb{R}} \to [0,1]$ considering $f(t) = 0$ if $t < 0$. The set of all d.d.f. is denoted by Δ^+. Examples of d.d.f. are the *step functions*, ϵ_a, defined as follows for any $a \in [0,\infty[$:

$$\epsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a, \end{cases} \quad \text{and, similarly,} \quad \epsilon_\infty(t) = \begin{cases} 0, & \text{if } t \in [0,\infty[, \\ 1, & \text{if } t = \infty. \end{cases}$$

The set Δ^+ is partially ordered by the relation $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in]0,\infty[$. In this order, $\epsilon_\infty \leq f \leq \epsilon_0$ for all $f \in \Delta^+$. Furthermore, the set Δ^+ has the following metric. Let $f, g \in \Delta^+$ and let $\delta \in [0,1]$. Let $[f,g;\delta]$ denote the condition $g(x) \leq f(x+\delta) + \delta$ for all $x \in]0, 1/\delta[$. For any $f, g \in \Delta^+$ the *modified Lévy metric between* f *and* g is given by

$$d_L(f,g) = \inf\left(\left\{ \delta > 0 \Big/ [f,g;\delta] \text{ and } [g,f;\delta] \text{ hold} \right\}\right).$$

The map $d_L : (\Delta^+)^2 \to [0,1]$ is a metric on Δ^+ and (Δ^+, d_L) is a compact and complete metric space (see [37]).

Analogously, a *nondistance distribution function* is a non-increasing function $\tilde{r} : [0,\infty] \to [0,1]$ that is left continuous on $]0,\infty[$ with $\tilde{r}(0) = 1$ and $\tilde{r}(\infty) = 0$. This definition is extended to $\tilde{r} : \overline{\mathbb{R}} \to [0,1]$ considering $\tilde{r}(t) = 1$ if $t < 0$. The set of all nondistance distribution functions is denoted by ∇^+ and $\tilde{\epsilon}_0$ is defined by $\tilde{\epsilon}_0(t) = 1$ if $t \leq 0$ and $\tilde{\epsilon}_0(t) = 0$ if $t > 0$.

A *triangle function* on Δ^+ must have ϵ_0 as identity and ϵ_∞ as null element, and a triangle function on ∇^+ must have $\tilde{\epsilon}_0$ as identity. If $\tilde{\tau} : \nabla^+ \times \nabla^+ \to \nabla^+$ is a triangle function on ∇^+, then $\tilde{\tau}'(f,g) = 1 - \tilde{\tau}(1-f, 1-g)$ is a triangle function on Δ^+, and conversely. The concept of triangle function on Δ^+ is well known. It was studied by Schweizer and Sklar in [37] and it is just a t_{Δ^+}-norm in the terminology of [5]. An interesting class of triangle functions on Δ^+ is defined as follows. Let $*$ be a continuous t-norm and let $f, g \in \Delta^+$, then τ_*, defined as:

$$\tau_*(f,g)(u) = \sup\left(\left\{ f(t) * g(s) \Big/ t + s = u,\ t, s \geq 0 \right\}\right)$$

for each $u \in]0,\infty[$, is a triangle function on Δ^+ that verifies (see [37])

$$\tau_*(\epsilon_a, \epsilon_b) = \epsilon_{a+b} \quad \text{for all } a, b \in]0,\infty[. \tag{1}$$

A *fuzzy set on* $\mathbb{R}$ is a map $F : \mathbb{R} \to [0,1]$. A *fuzzy number on* $\mathbb{R}$ is a fuzzy set F on $\mathbb{R}$ that verifies:

D1: Normality: there exists a real number $x_0 \in \mathbb{R}$ such that $F(x_0) = 1$.

D2: For all $\alpha \in]0,1]$, the set $F_{[\alpha]} = \{x \in \mathbb{R} : F(x) \geq \alpha\}$ is a closed subinterval of $\mathbb{R}$.

The family of all fuzzy numbers F satisfying $F(x_0) = 1$ will be denoted by $\mathcal{F}(x_0)$. In the sequel, we assume without loss of generality that every fuzzy number F verifies $F(0) = 1$, that is, we will consider $\mathcal{F} = \mathcal{F}(0)$.

A fuzzy number $F \in \mathcal{F}$ is nondecreasing in $]-\infty, 0]$ and is nonincreasing in $[0,\infty[$. Furthermore, $F|_{]-\infty,0[}$ is right continuous and $F|_{]0,\infty[}$ is left continuous. The set $\mathcal{F}$ is partially ordered by the relation $F \leq G$ iff $F(x) \leq G(x)$ for all $x \in \mathbb{R}$. Note that considering $F(\pm\infty) = 0$, any fuzzy number F can be extended to $\overline{\mathbb{R}}$. Furthermore, considering $\mathbf{1} : \overline{\mathbb{R}} \to [0,1]$ as 1 if $x \in \mathbb{R}$ and 0 if $x = \pm\infty$ and $\overline{r} : \overline{\mathbb{R}} \to [0,1]$ as 1, if $x = r$, and 0, if $x \neq r$, it is clear that $\overline{r} \leq F \leq \mathbf{1}$ for all $F \in \mathcal{F}(r)$. Finally note that a fuzzy number $F : \mathbb{R} \to [0,1]$ is called *nonnegative* if $F(t) = 0$ for all $t < 0$ (see [22]) . The set of all nonnegative fuzzy numbers is denoted by $\mathcal{G}$.

Since the aim of this paper is to establish connections between fuzzy metric structures, firstly a review of these definitions is considered using the following notation to avoid confusion: *KS-spaces* denote fuzzy metric spaces in the sense of Kaleva and Seikkala and *KM-spaces* denote fuzzy metric spaces in the sense of Kramosil and Michalek.

Henceforth, X always denotes a nonempty set. As usual in other papers, subscripts are used to indicate the arguments of the function. For example, $d(x,y)$ will be denoted by d_{xy}.

- A *probabilistic metric space* (briefly, a PM-space; see [37]) is a triple (X, d, τ) where $d : X \times X \to \Delta^+$ is a map and τ is a triangle function on Δ^+ satisfying the following properties:
  P1: For $x, y \in X$, $d_{xy} = \epsilon_0$ iff $x = y$.
  P2: For all $x, y \in X$, $d_{yx} = d_{xy}$.
  P3: *Triangle inequality*: for all $x, y, z \in X$, $d_{xz} \geq \tau(d_{xy}, d_{yz})$.
- A *probabilistic metric co-space* (briefly, a PM$_c$-space) is a triple $(X, \tilde{d}, \tilde{\tau})$ where $\tilde{d} : X \times X \to \nabla^+$ is a map and $\tilde{\tau} : \nabla^+ \times \nabla^+ \to \nabla^+$ is a triangle function on ∇^+ satisfying the following properties:
  C1: For $x, y \in X$, $\tilde{d}_{xy} = \tilde{\epsilon}_0$ iff $x = y$.
  C2: For all $x, y \in X$, $\tilde{d}_{yx} = \tilde{d}_{xy}$.
  C3: *Cotriangle inequality*: for all $x, y, z \in X$, $\tilde{d}_{xz} \leq \tilde{\tau}(\tilde{d}_{xy}, \tilde{d}_{yz})$.
- A 5-tuple $(X, d, \tau, \tilde{d}, \tilde{\tau})$ is said to be an *intuitionistic probabilistic metric space* (briefly, an IPM-space) if (X, d, τ) is a PM-space, $(X, \tilde{d}, \tilde{\tau})$ is a PM$_c$-space and $d_{xy}(t) + \tilde{d}_{xy}(t) \leq 1$ for all $x, y \in X$ and all $t \in]0, \infty[$.
- A *fuzzy metric space* in the sense of Kaleva and Seikkala [22] (briefly, a KS-space) is a quadruple (X, d, L, R) where $d : X \times X \to \mathcal{G}$ is a mapping (a *fuzzy metric*), $L, R : [0,1]^2 \to [0,1]$ are symmetric and nondecreasing mappings satisfying $L(0,0) = 0$, $R(1,1) = 1$ and the following conditions:
  (i): $d_{xy} = \overline{0}$ iff $x = y$.
  (ii): $d_{xy} = d_{yx}$ for all $x, y \in X$.
  (iii): for all $x, y, z \in X$, denoting $[d_{xy}]_\alpha = \{t \in \mathbb{R} : d_{xy}(t) \geq \alpha\} = \left[\lambda^\alpha_{xy}, \rho^\alpha_{xy}\right]$ for $0 < \alpha \leq 1$:

(1): $d_{xy}(s+t) \geq L(d_{xz}(s), d_{zy}(t))$ whenever $s \leq \lambda^1_{xz}$, $t \leq \lambda^1_{zy}$ and $s+t \leq \lambda^1_{xy}$.

(2): $d_{xy}(s+t) \leq R(d_{xz}(s), d_{zy}(t))$ whenever $s \geq \lambda^1_{xz}$, $t \geq \lambda^1_{zy}$ and $s+t \geq \lambda^1_{xy}$.

- An *intuitionistic fuzzy metric space* [4], (briefly, IFM-space) is a 5-tuple $(X, M, N, *, \diamond)$ where $*$ is a continuous t-norm, $\diamond$ a continuous t-conorm and $M, N : X \times X \times [0, \infty[\to [0, 1]$ are fuzzy sets such that, for all $x, y, z \in X$:
  (1) $M_{xy}(t) + N_{xy}(t) \leq 1$ for all $t \in [0, \infty[$.
  (2) $M_{xy}(0) = 0$.
  (3) $M_{xy}(t) = M_{yx}(t)$ for all $t \in [0, \infty[$.
  (4) $M_{xy}(t) = 1$ for all $t \in]0, \infty[$ if, and only if, $x = y$.
  (5) $M_{xz}(t+s) \geq M_{xy}(t) * M_{yz}(s)$ for all $t, s \in [0, \infty[$.
  (6) The fuzzy set $M_{xy} : [0, \infty[\to [0, 1]$ is left continuous.
  (7) $N_{xy}(0) = 1$.
  (8) $N_{xy}(t) = N_{yx}(t)$ for all $t \in [0, \infty[$.
  (9) $N_{xy}(t) = 0$ for all $t \in]0, \infty[$ if, and only if, $x = y$.
  (10) $N_{xz}(t+s) \leq N_{xy}(t) \diamond N_{yz}(s)$ for all $t, s \in [0, \infty[$.
  (11) The fuzzy set $N_{xy} : [0, \infty[\to [0, 1]$ is left continuous.
- A *fuzzy metric space* in the sense of Kramosil and Michalek [23] (briefly, a KM-space) is a triple $(X, M, *)$ where $*$ is a continuous t-norm and $M : X \times X \times [0, \infty[\to [0, 1]$ is a fuzzy set verifying properties (2)–(6) of a IFM-space.
- A *fuzzy metric space* in the sense of George and Veeramani [11] (briefly, a GV-space) is a triple $(X, M, *)$ where $*$ is a continuous t-norm and $M : X \times X \times]0, \infty[\to [0, 1]$ is a fuzzy set verifying properties (3), (4) and (5) of IFM-spaces and replacing properties (2) and (6) by the following ones: for all $x, y, z \in X$,
  (2) $M_{xy}(t) > 0$ for all $t > 0$.
  (6) The fuzzy set $M_{xy} :]0, \infty[\to [0, 1]$ is continuous.

Note that a *Menger space* is a probabilistic metric space where $\tau = \tau_*$ (and $*$ is a continuous t-norm) and every KM-space $(X, M, *)$ is an IFM-space of the form $(X, M, 1 - M, *, *')$. On the other hand, every GV-space is a KM-space (sometimes called *strong fuzzy metric space*). Indeed, GV-spaces are a slightly modification of KM-spaces in order to get better properties. They considered balls $B(x, r, t) = \{y \in X : M_{xy}(t) > 1 - r\}$ for $x \in X$, $r \in]0, 1[$ and $t > 0$, and proved that they are open sets in the topology:

$$\tau = \{A \subseteq X : x \in A \Leftrightarrow (\exists r \in]0, 1[\,,\ \exists t > 0 \text{ such that } B(x, r, t) \subseteq A)\},$$

which is Hausdorff first countable. The appropriate notions of Cauchy sequence and completeness were also modified in order to make $\mathbb{R}$ a complete FM-space. A Baire's theorem was proved in this paper using that the closed balls $B[x, r, t] = \{y \in X : M_{xy}(t) \geq 1 - r\}$ are closed sets in τ.

There are many notions of fuzzy metric space that are different from the previous ones. Considering only nonascending functions $\mathbb{R} \to [0, 1]$, the intervalar arithmetic on α-cuts and positive fuzzy real numbers, Morsi [28] gave in 1988 a version of a fuzzy metric space that is comparable, in modern terminology, to $(X, N, \diamond)$, and introduced the concept of *fuzzy pseudo-metric topology* on a fuzzy pseudo-metric space. This topology, essentially defined as we have just mentioned, is a particular case of (L, M)*-fuzzy topological spaces*, which have been lately studied in [38, 39, 24] from an algebraic point of view. Indeed, KM and GV fuzzy metrics are also (L, M)-fuzzy metrics.

Recently, Saadati el al. [36] have introduced a *modified* version of IFM-spaces considering a triple $(X, \mathcal{M}_{MN}, \mathcal{T})$, being $\mathcal{M}_{MN}(x, y, t) = (M_{xy}(t), N_{xy}(t))$ and $\mathcal{T}((x_1, y_1), (x_2, y_2)) = (x_1 * x_2, y_1 \diamond y_2)$, where $*$ is a continuous t-norm and $\diamond$ is a continuous t-conorm. However, this new concept (*mIFM-spaces*) is a slight modification of the axioms of IFM-space in the same way as GV-spaces are a slight modification of KM-spaces (essentially, in order to get better properties; for example, in fixed point theory [19]).

It is also possible to consider a similar theory where the codomain is a complete lattice $\mathcal{L} = (L, \leq_L)$ rather than the interval $[0, 1]$, and similar properties can be obtained following the same techniques. These spaces are known as $\mathcal{L}$*-fuzzy metric spaces* [1] and they are efficient settings in which we can deduce fixed point theorems [35, 32, 25].

Applications of FM-spaces and IFM-spaces are numerous. For instance, in [31, 34] the authors apply an intuitionistic fuzzy quasi-metric version of a fixed point theorem to obtain the existence of solution for a recurrence equation associated with the analysis of Quicksort algorithms. In [4], the authors used IFM-spaces in order to predict access histories working on variations of the fuzzy construction. In [15] the authors proposed a fuzzy metric that simultaneously takes into account two different distance criteria between color image pixels and used it to filter noisy images, obtaining promising results. Intuitionistic fuzzy sets, combining with aggregation functions, play a key role in decision making [44]. An intuitionistic fuzzy multi-criteria group decision making method with grey relational analysis is proposed in [47] in order to solve personnel selection problem, in which both subjective and objective assessments rather than just subjective decisions are making, and where linguistic variables are considered. A novel intuitionistic fuzzy clustering method for Geo-Demographic Analysis is developed in [40]. Finally, there exist applications to pattern recognition and medical diagnosis [48].

3. A Canonical Decomposition of Fuzzy Numbers

In this section a way to interpret fuzzy numbers (satisfying the normality condition in zero) as a pair of d.d.f. is considered. This decomposition let us translate the Lévy metric between d.d.f. to the set of fuzzy numbers. Furthermore the relation between triangular functions for fuzzy numbers and d.d.f. is established.

Definition 3.1. Let F be a fuzzy set on $\mathbb{R}$. Define $F^-, F^+ : \overline{\mathbb{R}} \to [0, 1]$ as follows:

$$F^{\pm}(x) = \begin{cases} 0, & \text{if} \quad x < 0, \\ 1 - F(\pm x), & \text{if} \quad 0 \leq x < \infty, \\ 1, & \text{if} \quad x = \infty; \end{cases}$$

The following lemma is a consequence that the map $x \mapsto -x$ is an inverse bijection between $]-\infty, 0[$ and $]0, \infty[$.

Lemma 3.2. *A fuzzy set F on $\mathbb{R}$ is a fuzzy number of $\mathcal{F}$ iff F^- and F^+ are d.d.f. In this case, for all $x \in \mathbb{R}$,*

$$F(x) = \begin{cases} 1 - F^-(-x), & \text{if} \quad x < 0, \\ 1 - F^+(x), & \text{if} \quad x \geq 0. \end{cases}$$

A similar decomposition was already obtained by Zhang in [46]. By Lemma 3.2, each fuzzy number can be determined by a pair of d.d.f. The following theorem establishes that this relation is bijective. It proves that there exists a natural metric topology in $\mathcal{F}$ that generalizes those introduced by Zhang and that makes this space isometric to $\Delta^+ \times \Delta^+$ with the product topology introduced by *Lévy's metric* (slightly modified by Schweizer and Sklar [37]). Note that the order in Δ^+ induces an order in $(\Delta^+)^2$ given by $(f_1, f_2) \leq (g_1, g_2)$ iff $f_1 \leq g_1$ and $f_2 \leq g_2$.

Theorem 3.3. *The map $\Phi : \mathcal{F} \to (\Delta^+)^2$ given by $\Phi(F) = (F^-, F^+)$ for all $F \in \mathcal{F}$, is an inverse order bijection.*

Proof. By Lemma 3.2, Φ is well defined, since $F^-, F^+ \in \Delta^+$, and it is injective because F^- and F^+ determines F. To prove that Φ is surjective, let $f, g \in \Delta^+$ be arbitrary d.d.f. and define F on $\overline{\mathbb{R}}$ as follows:

$$F(x) = \begin{cases} 1 - f(-x), & \text{if } x < 0, \\ 1 - g(x), & \text{if } x \geq 0. \end{cases}$$

With the same arguments as lemma 3.2, it can be concluded that $F \in \mathcal{F}$ and $\Phi(F) = (f, g)$. Now suppose that $F, G \in \mathcal{F}$. Hence $F \leq G$ is equivalent to $G^+ \leq F^+$ and $G^- \leq F^-$, that is, $(G^-, G^+) \leq (F^-, F^+)$, and this is equivalent to $\Phi(F) \geq \Phi(G)$. □

The same reasoning as the previous theorem let us conclude the following result.

Theorem 3.4. *Let $f, g : [0, \infty] \to [0, 1]$ be two maps such that $f(0) = 0$ and define $F : \overline{\mathbb{R}} \to [0, 1]$ as:*

$$F(x) = \begin{cases} 1 - f(-x), & \text{if} \quad x < 0, \\ 1 - g(x), & \text{if} \quad x \geq 0. \end{cases}$$

Then $F \in \mathcal{F}$ iff $f, g \in \Delta^+$. In this case, $\Phi(F) = (f, g)$.

Kaleva [20] firstly defined three kinds of convergences in fuzzy number space, and studied the relationship between these convergences. Many authors also discussed convergence of sequences of fuzzy numbers (see [49] and references therein) with respect to metrics based on Hausdorff metric. Now, with the previous bijection in mind, we propose a new kind of convergence.

As (Δ^+, d_L) is a metric space and $\Phi : \mathcal{F} \to (\Delta^+)^2$ is a bijection, a *Lévy metric* d_L on $\mathcal{F}$ can be induced considering $d_L(F, G) = \max(d_L(F^-, G^-), d_L(F^+, G^+))$ for all $F, G \in \mathcal{F}$.

Corollary 3.5. *With the induced metric on $\mathcal{F}$ by the* Lévy *distance on Δ^+, $\mathcal{F}$ is a compact and complete metric space.*

It is not difficult to prove that if $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ is a sequence of fuzzy numbers and $F \in \mathcal{F}$, then $\{d_L(F_n, F)\}_{n\in\mathbb{N}} \to 0$ iff $\{F_n\}_{n\in\mathbb{N}}$ *converges weakly* to F, i.e., $\{F_n(x)\}_{n\in\mathbb{N}}$ converges to $F(x)$ at each continuity point of F (see [37]).

As Φ is an inverse order bijection between $\mathcal{F}$ and $(\Delta^+)^2$, and ϵ_0 is the identity of any triangle function on Δ^+, the fuzzy number $\overline{0}$ must be the identity of any triangle function υ on $\mathcal{F}$ and $\mathbf{1}$ will be its null element, that is, $\upsilon : \mathcal{F}^2 \to \mathcal{F}$, $\upsilon\left(\overline{0}, F\right) = F$, $\upsilon(\mathbf{1}, F) = \mathbf{1}$, for all $F \in \mathcal{F}$. For example, the sum of fuzzy numbers is a triangle function on $\mathcal{F}$. In the terminology of [5], a triangular function on $\mathcal{F}$ is just a $\mathrm{t}_{\mathcal{F}}$-conorm.

The following lemma is an easy, but tedious, algebraic exercise.

Lemma 3.6. *There exists a one-to-one correspondence between triangle functions υ on $\mathcal{F}$ and triangle functions η on $(\Delta^+)^2$ such that the following diagram conmutes.*

$$\begin{array}{ccc} \mathcal{F}^2 & \xrightarrow{\Phi\times\Phi} & (\Delta^+)^4 \\ \upsilon\big\downarrow & \equiv & \big\downarrow\eta \\ \mathcal{F} & \xrightarrow[\Phi]{} & (\Delta^+)^2 \end{array} \qquad \eta\circ(\Phi\times\Phi) = \Phi\circ\upsilon$$

In this way, υ is a triangle function on $\mathcal{F}$ iff $\Phi\circ\upsilon\circ(\Phi\times\Phi)^{-1}$ is a triangle function on $(\Delta^+)^2$.

If η is a triangle function on $(\Delta^+)^2$, $\upsilon = \Phi^{-1}\circ\eta\circ(\Phi\times\Phi)$ is called its *associated triangle function on $\mathcal{F}$*. If τ and τ' are triangle functions on Δ^+, then $\eta = \tau\times\tau'$, defined as $(\tau\times\tau')(f_1, f_2, f_3, f_4) = (\tau(f_1, f_3), \tau'(f_2, f_4))$ for all $f_1, f_2, f_3, f_4 \in \Delta^+$, is a triangle function on $(\Delta^+)^2$, and $\upsilon = \Phi^{-1}\circ(\tau\times\tau')\circ(\Phi\times\Phi)$ is its associated triangle function on $\mathcal{F}$. In this case,$\upsilon(F, G) = \Phi^{-1}(\tau(F^-, G^-), \tau'(F^+, G^+))$, for all $F, G \in \mathcal{F}$.

4. A Comparative Study of Fuzzy Metric Structures

In this section, which is the focus of this paper, we prove that all definitions provided in preliminaries section can be conveniently included in the following definition.

Definition 4.1. A **fuzzy metric space** (briefly, a *FM-space*) is a triple $(X, \mathbf{F}, \upsilon)$ where X is a set, $\mathbf{F} : X\times X \to \mathcal{F}$ is a map and $\upsilon : \mathcal{F}\times\mathcal{F} \to \mathcal{F}$ is a triangle function satisfying the following properties:

- **F1:** $\mathbf{F}_{xx} = \overline{0}$ for all $x \in X$.
- **F2:** $\mathbf{F}_{xy} \neq \overline{0}$ for all $x, y \in X$ with $x \neq y$.
- **F3:** $\mathbf{F}_{yx} = \mathbf{F}_{xy}$ for all $x, y \in X$.
- **F4:** *Triangle inequality*: $\mathbf{F}_{xz} \leq \upsilon(\mathbf{F}_{xy}, \mathbf{F}_{yz})$ for all $x, y, z \in X$.

We say that a FM-space is *normalised* (respectively, *conormalised*) if $\mathbf{F}_{xy}(-t) \leq \mathbf{F}_{xy}(t)$ (respectively, $\mathbf{F}_{xy}(t) \leq \mathbf{F}_{xy}(-t)$) for all $x, y \in X$ and all $t \in \,]0, \infty[$.

Any metric space $(X,\mathtt{d})$ is a FM-space if we define $\mathbf{F}_{xy}(t)=1$ if $|t|\leq \mathtt{d}_{xy}$ and $\mathbf{F}_{xy}(t)=0$ if $|t|>\mathtt{d}_{xy}$. If τ^{+} and τ^{-} are triangle functions on Δ^{+} satisfying (2) and υ is the triangle function on $\mathcal{F}$ associated with $\tau^{-}\times\tau^{+}$, then $\mathbf{F}^{\pm}_{xz}\geq \tau^{\pm}\left(\mathbf{F}^{\pm}_{xy},\mathbf{F}^{\pm}_{yz}\right)$.

The variety of FM-spaces presented below will be enormous. But for now, let us see that we can consider a FM-space for each d.d.f. different from ϵ_0 and each triangle function on Δ^{+}. Let X be a set and $f\in\Delta^{+}$ any d.d.f. different from ϵ_0. Define $\mathbf{F}_{xy}=\overline{0}$ if $x=y$ and

$$\mathbf{F}_{xy}(t)=\begin{cases}0, & \text{if } t<0,\\ 1-f(t), & \text{if } t\geq 0,\end{cases}\qquad \text{if } x\neq y.$$

Then $(X,\mathbf{F},\upsilon)$ is a FM-space for any triangle function υ on $\mathcal{F}$ associated to a product $\tau\times\tau'$ of triangle functions on Δ^{+}.

To follow, we prove that every probabilistic metric space is a FM-space and we consider the strong topology on a FM-space.

Theorem 4.2. *Let X be a set, τ a triangle function on Δ^{+}, $d:X\times X\to\Delta^{+}$ a map and define:*

$$\forall x,y\in X,\qquad \mathbf{F}_{xy}=\begin{cases}0, & \text{if } t<0,\\ 1-d_{xy}(t), & \text{if } t\geq 0.\end{cases}$$

Then the following assertions are equivalent.

- **(a):** *(X,d,τ) is a PM-space.*
- **(b):** *$(X,\mathbf{F},\upsilon)$ is a FM-space, where υ is the triangle function on $\mathcal{F}$ associated to $\tau'\times\tau$, for any triangle function τ' on Δ^{+}.*
- **(c):** *There exists a triangle function τ' on Δ^{+} such that $(X,\mathbf{F},\upsilon)$ is a FM-space, where υ is the triangle function on $\mathcal{F}$ associated with $\tau'\times\tau$.*

Proof. Let τ' be any triangle function on Δ^{+} and let υ be the triangle function on $\mathcal{F}$ associated to $\tau'\times\tau$. It is clear that $d_{xy}=\epsilon_0$ iff $\mathbf{F}_{xy}=\overline{0}$, so the condition $d_{xy}=\epsilon_0$ iff $x=y$ is equivalent to $\mathbf{F}_{xy}=\overline{0}$ iff $x=y$. Furthermore, d is symmetric iff $\mathbf{F}$ is symmetric. Observe that $\Phi(\mathbf{F}_{xy})=\left(\mathbf{F}^{-}_{xy},\mathbf{F}^{+}_{xy}\right)=(\epsilon_0,d_{xy})$ for all $x,y\in X$. Let $x,y,z\in X$. It is clear that $\mathbf{F}^{-}_{xz}=\epsilon_0\geq\tau'(\epsilon_0,\epsilon_0)=\tau'\left(\mathbf{F}^{-}_{xy},\mathbf{F}^{-}_{yz}\right)$. Moreover, $d_{xz}\geq\tau(d_{xy},d_{yz})$ iff $\mathbf{F}^{+}_{xz}\geq\tau(\mathbf{F}^{+}_{xz},\mathbf{F}^{+}_{zy})$. Then:

$$\begin{aligned} d_{xz}\geq\tau(d_{xy},d_{yz}) &\Leftrightarrow \Phi(\mathbf{F}_{xy})=\left(\mathbf{F}^{-}_{xy},\mathbf{F}^{+}_{xy}\right)\geq\left(\tau'\left(\mathbf{F}^{-}_{xy},\mathbf{F}^{-}_{yz}\right),\tau(\mathbf{F}^{+}_{xz},\mathbf{F}^{+}_{zy})\right)\Leftrightarrow\\ &\Leftrightarrow \mathbf{F}_{xz}\leq\upsilon(\mathbf{F}_{xy},\mathbf{F}_{yz}).\end{aligned}$$

It can be concluded that (X,d,τ) is a probabilistic metric space iff $(X,\mathbf{F},\upsilon)$ is a FM-space. □

Corollary 4.3. *Every PM-space is a FM-space.*

Taking into account that $\mathbf{F}^{-}_{xy}$ does not play an important role in the proof of Theorem 4.2, we obtain the following result.

Corollary 4.4. *Let $(X, \mathbf{F}, \upsilon)$ be a FM-space such that υ is associated with a product $\tau' \times \tau$ of triangle functions on Δ^+ and such that $\mathbf{F}^+_{xy} = \epsilon_0$ implies that $\mathbf{F}_{xy} = \overline{0}$. Then $(X, d = 1 - \mathbf{F}|_{[0,\infty]}, \tau)$ is a PM-space.*

Modifying the proof of theorem 4.2 we prove the following one.

Theorem 4.5. *Let X be a set, τ and τ' triangle functions on Δ^+, $d, d' : X \times X \to \Delta^+$ two maps and define:*

$$\forall x, y \in X,\ \forall t \in \overline{\mathbb{R}}, \qquad \mathbf{F}_{xy}(t) = \begin{cases} 1 - d_{xy}(-t), & \text{if } t < 0, \\ 1 - d'_{xy}(t), & \text{if } t \geq 0. \end{cases}$$

Then the following assertions are equivalent.

(a): *(X, d, τ) and (X, d', τ') are* PM-*spaces.*

(b): *$(X, \mathbf{F}, \upsilon_{\tau\times\tau'})$ is a* FM-*space.*

In this case, $d = \mathbf{F}^-$ y $d' = \mathbf{F}^+$. Furthermore, a sequence $\{x_n\} \subseteq X$ $\mathbf{F}$-converges to $x \in X$ (respect., is $\mathbf{F}$-Cauchy) iff it d-converges to x and d'-converges to x (respect., is d-Cauchy and d'-Cauchy).

Theorem 4.2 shows that the relationship between probabilistic metric spaces and FM-spaces is very close. When (X, d, τ) is a probabilistic metric space and τ is continuous (with the metric d_L), it is possible to consider the *strong topology on X* (see [37]). Using this idea, it is possible to define a similar topology on every FM-space $(X, \mathbf{F}, \upsilon)$. Indeed, for each $x \in X$ and each $t > 0$, define:

$$N_x(t) = \left\{ y \in X : \mathbf{F}_{xy}(\pm t) < t \right\}.$$

It is clear that if $t > 1$, then $N_x(t) = X$. Furthermore, if $0 < t_1 < t_2$, then $N_x(t_1) \subseteq N_x(t_2)$. Note that for $f \in \Delta^+$ we have that $f(t) > 1 - t$ iff $d_L(f, \epsilon_0) < t$. Then, for all $x, y \in X$,

$$d_L\left(\mathbf{F}_{xy}, \overline{0}\right) < t \ \Leftrightarrow\ \max\left(d_L\left(1 - \mathbf{F}_{xy}(\pm\cdot), \epsilon_0\right)\right) < t \ \Leftrightarrow\ 1 - \mathbf{F}_{xy}(\pm t) > 1 - t \ \Leftrightarrow$$
$$\Leftrightarrow\ y \in N_x(t).$$

Hence, we can describe $N_x(t) = \left\{ y \in X : d_L(\mathbf{F}_{xy}, \overline{0}) < t \right\}$. We can repeat the arguments of theorem 12.1.2, in [37] to prove the following result.

Theorem 4.6. *If $(X, \mathbf{F}, \upsilon)$ is a FM-space such that υ is continuous, then there exists a Hausdorff topology on X such that $\{N_x(1/n)\}_{n\in\mathbb{N}}$ is a countable basis of neighborhoods of each $x \in X$.*

For example, if $(X, \mathtt{d})$ is a metric space, the strong topology on X coincides with the metric topology induced by $\mathtt{d}$, since, in this case, if $0 < t < 1$, $\mathbf{F}_{xy}(\pm t) < t$ iff $\mathtt{d}_{xy} < t$. Therefore, $N_x(t) = \{y \in X : \mathtt{d}_{xy} < t\} = B^{\mathtt{d}}(x, t)$.

Regarding KS-spaces, it is easy to prove that every IFM-space $(X, M, N, *, \diamond)$ is a KS-space (X, d, L, R) where $L \equiv 0$, $R = *'$ and $d : X \times X \to \mathcal{G}$ is defined as $d_{xy}(t) = 0$, if $t < 0$, and $d_{xy}(t) = 1 - M_{xy}(t)$, if $t \geq 0$, for all $x, y \in X$. Next we show that, under some conditions, a Kaleva-Seikkala fuzzy metric space is a FM-space. A KS-space (X, d, L, R) is *simply* if $d : X \times X \to \mathcal{G} \cap \mathcal{F}$, $L = 0$ and R is a continuous t-conorm. It will be denote as KS*-space.

Theorem 4.7. *Every KS*-space $(X, d, L = 0, R)$ is a FM-space $(X, \mathbf{F} = d, \upsilon)$, where υ is the triangle function on $\mathcal{F}$ associated to $\tau \times \tau_{R'}$ and τ is any triangle function on Δ^+.*

Proof. As R is a continuous t-conorm, then R' is a continuous t-norm. Let υ be the triangle function on $\mathcal{F}$ associated to $\tau \times \tau_{R'}$, where τ is any triangle function on Δ^+. Define $\mathbf{F}_{xy} = d_{xy} \in \mathcal{F}$ for all $x, y \in X$. The conditions F1, F2 and F3 are trivial. To prove F4, we observe that $\Phi(\mathbf{F}_{xy}) = \left(\mathbf{F}^-_{xy}, \mathbf{F}^+_{xy}\right) = (\epsilon_0, 1 - d_{xy})$ for all $x, y \in X$. Given $s, t \geq 0 = \lambda_1(x, y) = \lambda_1(x, z) = \lambda_1(y, z)$, we have that:

$$\begin{aligned} d_{xy}(s+t) \leq R(d_{xz}(s), d_{zy}(t)) \;\Leftrightarrow\; 1 - d_{xy}(s+t) \geq 1 - R(d_{xz}(s), d_{zy}(t)) \Leftrightarrow \\ \Leftrightarrow\; 1 - d_{xy}(s+t) \geq R'(1 - d_{xz}(s), 1 - d_{zy}(t)) \\ \Leftrightarrow\; \mathbf{F}^+_{xy}(s+t) \geq R'\left(\mathbf{F}^+_{xz}(s), \mathbf{F}^+_{zy}(t)\right). \end{aligned}$$

If $u = t + s$ and taking supreme, we have proved that for any $u \geq 0$:

$$\mathbf{F}^+_{xy}(u) \geq \sup\left(\left\{ R'\left(\mathbf{F}^+_{xz}(s), \mathbf{F}^+_{zy}(t)\right) \;/\; t + s = u,\; t, s \geq 0\right\}\right) = \tau_{R'}\left(\mathbf{F}^+_{xz}, \mathbf{F}^+_{zy}\right)(u).$$

Then $\mathbf{F}^+_{xy} \geq \tau_{R'}\left(\mathbf{F}^+_{xz}, \mathbf{F}^+_{zy}\right)$. As $\mathbf{F}^-_{xy} = \epsilon_0 \geq \tau\left(\mathbf{F}^-_{xz}, \mathbf{F}^-_{zy}\right)$, we have deduced that:

$$\begin{aligned} \Phi(\mathbf{F}_{xy}) &= \left(\mathbf{F}^-_{xy}, \mathbf{F}^+_{xy}\right) \geq \left(\tau\left(\mathbf{F}^-_{xz}, \mathbf{F}^-_{zy}\right), \tau_{R'}\left(\mathbf{F}^+_{xz}, \mathbf{F}^+_{zy}\right)\right) = \\ &= (\tau \times \tau_{R'})\left(\left(\mathbf{F}^-_{xz}, \mathbf{F}^+_{xz}\right), \left(\mathbf{F}^-_{zy}, \mathbf{F}^+_{zy}\right)\right) = \\ &= ((\tau \times \tau_{R'}) \circ (\Phi \times \Phi))(\mathbf{F}_{xz}, \mathbf{F}_{zy}) = (\Phi \circ \upsilon)(\mathbf{F}_{xz}, \mathbf{F}_{zy}). \end{aligned}$$

Taking the inverse order map Φ^{-1}, we obtain $\mathbf{F}_{xy} \leq \upsilon(\mathbf{F}_{xz}, \mathbf{F}_{zy})$, and $(X, \mathbf{F}, \upsilon)$ is a FM-space. □

Corollary 4.8. *Every KS*-space is a FM-space.*

Kaleva and Seikkala [22] and Pap [29] proved that every Menger space is a KS^*-*space*, and discussed that the converse statement is not obvious in the general case (using KS-spaces). As a consequence of Theorems 4.7 and 4.2, we have obtained the converse for KS*-spaces.

Corollary 4.9. *Every KS*-space (X, d, L, R) is a Menger space $(X, d', \tau_{R'})$, where $d'_{xy}(t) = 1 - d_{xy}(t)$, for all $x, y \in X$ and all $t \geq 0$.*

Kramosil and Michalek (see [23]) introduced a definition of fuzzy metric space modifying the axioms used until now and established a new class of fuzzy metric spaces provided with a Hausdorff topology. This definition is indeed similar to the IFM-space by Park [30], but we must understand that the other was earlier. Finally we study the relationship between IFM-spaces and FM-spaces and apply it to KM-spaces.

Theorem 4.10. *Let $(X, M, N, *, \diamond)$ be an IFM-space and define $\mathbf{F}, \mathbf{G} : X \times X \to \mathcal{F}$, for all $x, y \in X$, as:*

$$\mathbf{F}_{xy}(t) = \begin{cases} N_{xy}(-t), & \text{if } t < 0, \\ 1 - M_{xy}(t), & \text{if } t \geq 0; \end{cases} \qquad \mathbf{G}_{xy}(t) = \begin{cases} 1 - M_{xy}(-t), & \text{if } t < 0, \\ N_{xy}(t), & \text{if } t \geq 0. \end{cases}$$

Let $\diamond'$ be the t-norm induced by $\diamond$ and let υ and υ' be the triangular functions on $\mathcal{F}$ induced by $\tau_{\diamond'} \times \tau_*$ and $\tau_* \times \tau_{\diamond'}$, respectively. Then $(X, \mathbf{F}, \upsilon)$ is a normalised FM-space and $(X, \mathbf{G}, \upsilon')$ is a conormalised FM-space.

Proof. Let us prove that $(X, \mathbf{F}, \upsilon)$ is a normalised FM-space. As $*$ and $\diamond'$ are continuous t-norm, then τ_* and $\tau_{\diamond'}$ are triangle functions on Δ^+, and $\upsilon = \Phi^{-1} \circ (\tau_{\diamond'} \times \tau_*) \circ (\Phi \times \Phi)$ is a triangle function on $\mathcal{F}$. Note that for all $F, G \in \mathcal{F}$, $\upsilon(F, G) = \Phi^{-1}(\tau_{\diamond'}(F^-, G^-), \tau_*(F^+, G^+))$. Let $x, y \in X$. As $\mathbf{F}_{xy}(0) = 1 - M_{xy}(0) = 1$, by Theorem 3.4, $\mathbf{F}_{xy}$ is well defined and it is a fuzzy number of $\mathcal{F}$. Then $\mathbf{F} : X \times X \to \mathcal{F}$ is well defined. As

$$\mathbf{F}_{xy} = \overline{0} \quad \Leftrightarrow \quad M_{xy}(t) = 1 \text{ and } N_{xy}(t) = 0, \quad \forall t \in]0, \infty[\quad \Leftrightarrow \quad x = y,$$

the properties F1, F2 and F3 are trivial. To prove the triangle inequality, observe that $\Phi(\mathbf{F}_{xy}) = \left(\mathbf{F}^-_{xy}, \mathbf{F}^+_{xy}\right) = (1 - N_{xy}, M_{xy})$. Let $x, y, z \in X$ and consider the d.d.f. $\tau_{\diamond'}\left(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}\right)$, $\tau_*\left(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}\right) \in \Delta^+$. We compare these d.d.f. on $]0, \infty[$ with $\mathbf{F}^-_{xz}$ and $\mathbf{F}^+_{xz}$, respectively. Let $u \in]0, \infty[$ be a positive real number. If $u = t + s$, where $t, s \in]0, \infty[$, we have that $M_{xz}(u) = M_{xz}(t + s) \geq M_{xy}(t) * M_{yz}(s)$. Therefore,

$$\begin{aligned} \tau_*\left(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}\right)(u) &= \sup\left(\{\, M_{xy}(t) * M_{yz}(s) \ / \ t + s = u,\ t, s \geq 0 \,\}\right) \leq \\ &\leq \sup\left(\{\, M_{xz}(t + s) \ / \ t + s = u,\ t, s \geq 0 \,\}\right) = M_{xz}(u) = \mathbf{F}^+_{xz}(u). \end{aligned}$$

Consequently, we deduce $\tau_*\left(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}\right) \leq \mathbf{F}^+_{xz}$. In the same way, if $u = t + s$, we have that $N_{xz}(u) = N_{xz}(t + s) \leq N_{xy}(t) \diamond N_{yz}(s)$, and then:

$$\begin{aligned} &\tau_{\diamond'}\left(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}\right)(u) = \\ &\quad = \sup\left(\left\{\, 1 - \left[\left(1 - \mathbf{F}^-_{xy}(t)\right) \diamond \left(1 - \mathbf{F}^-_{yz}(s)\right)\right] \ / \ t + s = u,\ t, s \geq 0 \,\right\}\right) = \\ &\quad = \sup\left(\{\, 1 - N_{xy}(t) \diamond N_{yz}(s) \ / \ t + s = u,\ t, s \geq 0 \,\}\right) = \\ &\quad = 1 - \inf\left(\{\, N_{xy}(t) \diamond N_{yz}(s) \ / \ t + s = u,\ t, s \geq 0 \,\}\right) \leq \\ &\quad \leq 1 - \inf\left(\{\, N_{xz}(t + s) \ / \ t + s = u,\ t, s \geq 0 \,\}\right) = 1 - N_{xz}(u) = \mathbf{F}^-_{xz}(u). \end{aligned}$$

As $\tau_*\left(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}\right) \leq \mathbf{F}^+_{xz}$, $\tau_{\diamond'}\left(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}\right) \leq \mathbf{F}^-_{xz}$ and Φ^{-1} is an inverse order map, we conclude that:

$$\upsilon(\mathbf{F}_{xy}, \mathbf{F}_{yz}) = \Phi^{-1}\left(\tau_{\diamond'}\left(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}\right), \tau_*\left(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}\right),\right) \geq \Phi^{-1}\left(\mathbf{F}^-_{xz}, \mathbf{F}^+_{xz}\right) = \mathbf{F}_{xz}.$$

Finally, $(X, \mathbf{F}, \upsilon)$ is a normalised FM-space since, for $t > 0$:

$$\begin{aligned} M_{xy}(t) + N_{xy}(t) \leq 1 \quad &\Leftrightarrow \quad N_{xy}(t) \leq 1 - M_{xy}(t) \\ &\Leftrightarrow \quad \mathbf{F}_{xy}(-t) \leq \mathbf{F}_{xy}(t), \text{ for all } t \in]0, \infty[\ . \end{aligned} \tag{2}$$

□

Corollary 4.11. *Every IFM-space $(X, M, N, *, \diamond)$ is a normalised FM-space and is a Menger space (X, M, τ_*).*

The previous corollary is a consequence of applying that $(X, \mathbf{F}, \upsilon)$ is the *FM-space generated by* $(X, M, N, *, \diamond)$ and Theorem 4.2 (an IFM-space can be considered a FM-space, see Theorem 4.10). The following theorem is the converse of Theorem 4.10.

Theorem 4.12. *Let X be a set, $*$ a continuous t-norm, $\diamond$ a continuous t-conorm and υ the triangular function on $\mathcal{F}$ corresponding to $\tau_{\diamond'} \times \tau_*$. Suppose that for every $x, y \in X$ there exist fuzzy sets $M_{xy}, N_{xy} : [0, \infty[\to [0, 1]$ and $\mathbf{F}_{xy} : \mathbb{R} \to [0, 1]$ verifying:*

$$M_{xy}(t) = 1 - \mathbf{F}_{xy}(t) \quad \text{and} \quad N_{xy}(t) = \mathbf{F}_{xy}(-t) \quad \text{for all } t \in [0, \infty[\,.$$

Then the following assertions are equivalent:

- **(a):** $(X, M, N, *, \diamond)$ *is an IFM-space.*
- **(b):** $(X, \mathbf{F}, \upsilon)$ *is a normalised FM-space verifying the following properties:*
  - **(b.1):** *If $x, y \in X$, then $\mathbf{F}^-_{xy} = \epsilon_0$ iff $x = y$ iff $\mathbf{F}^+_{xy} = \epsilon_0$.*
  - **(b.2):** *$\mathbf{F}_{xy}(t) *' \mathbf{F}_{yz}(s) \geq \mathbf{F}_{xz}(t + s)$ for all $x, y, z \in X$ and all $t, s \in [0, \infty[$ (where $*'$ is the t-conorm induced by $*$).*
  - **(b.3):** *$\mathbf{F}_{xy}(-t) \diamond \mathbf{F}_{yz}(-s) \geq \mathbf{F}_{xz}(-t - s)$ for all $x, y, z \in X$ and all $t, s \in [0, \infty[$.*

In this case, $(X, \mathbf{F}, \upsilon)$ is the FM-space generated by $(X, M, N, *, \diamond)$.

Proof. Theorem 4.10 shows that $(a) \Rightarrow (b)$. Conversely, suppose that $(X, \mathbf{F}, \upsilon)$ is a normalised FM-space verifying the properties b.1 to b.3. Conditions 1-11 are easy to prove. We only mention some details. By (**??**), the normalised condition is equivalent to $M_{xy} + N_{xy} \leq 1$. The fuzzy sets $M_{xy}, N_{xy} : [0, \infty[\to [0, 1]$ are left continuous functions because $\mathbf{F}_{xy}$ is a fuzzy number (see Theorem 3.4). Using b.1,

$$x = y \;\Leftrightarrow\; \mathbf{F}^+_{xy} = \mathbf{F}^-_{xy} = \epsilon_0 \;\Leftrightarrow\; M_{xy}(t) = 1 \text{ and } N_{xy}(t) = 0, \text{ for all } t \in \,]0, \infty[\,.$$

By b.2 we have that, for all $x, y, z \in X$ and all $t, s \in [0, \infty[$:

$$\begin{aligned}
&\mathbf{F}_{xy}(t) *' \mathbf{F}_{yz}(s) \geq \mathbf{F}_{xz}(t + s) \;\Leftrightarrow \\
&\qquad \Leftrightarrow\; 1 - [(1 - \mathbf{F}_{xy}(t)) * (1 - \mathbf{F}_{yz}(s))] \geq \mathbf{F}_{xz}(t + s) \;\Leftrightarrow \\
&\qquad \Leftrightarrow 1 - \mathbf{F}_{xz}(t + s) \geq (1 - \mathbf{F}_{xy}(t)) * (1 - \mathbf{F}_{yz}(s)) \;\Leftrightarrow \\
&\qquad \Leftrightarrow\; M_{xz}(t + s) \geq M_{xy}(t) * M_{yz}(s)\,.
\end{aligned}$$

Since $N_{xy}(t) = \mathbf{F}_{xy}(-t)$, the condition b.3 is equivalent to $N_{xy}(t) \diamond N_{yz}(s) \geq N_{xz}(t + s)$ for all $x, y, z \in X$ and all $t, s \in [0, \infty[$. Therefore $(X, M, N, *, \diamond)$ is an IFM-space and $(X, \mathbf{F}, \upsilon)$ is the FM-space generated by $(X, M, N, *, \diamond)$ since:

$$\mathbf{F}_{xy}(t) = \begin{cases} N_{xy}(-t)\,, & \text{if } t < 0, \\ 1 - M_{xy}(t)\,, & \text{if } t \geq 0 \end{cases} \qquad \text{for all } x, y \in X.$$

□

A conormalised version of the previous theorem is also true if M, N and $\mathbf{F}$ are related by: $M_{xy}(t) = 1 - \mathbf{F}_{xy}(-t)$ and $N_{xy}(t) = \mathbf{F}_{xy}(t)$ for all $t \in [0, \infty[$.

Taking into account that every KM-space $(X, M, *)$ is an IFM-space of the form $(X, M, 1 - M, *, *')$, we conclude the following.

Corollary 4.13. *Every KM-space is a normalised FM-space.*

In fact, every KM-space $(X, M, *)$ is an IFM-space of the form $(X, M, N, *, \diamond)$, for all N and all continuous t-conorm $\diamond$ verifying $M + N \leq 1$ and properties 7–11 of an IFM-space.

Theorem 4.14. *If $(X, d, L = 0, R)$ is a KS*-space, then $(X, M = 1 - d|_{[0,\infty[}, N, \tau_{R'}, \diamond)$ is an IFM-space whatever the continuous t-conorm $\diamond$ and N verifying $M + N \leq 1$ and properties 7–11.*

Next we deal with the link between KS*-spaces and KM-spaces.

Theorem 4.15. *Let X be a set, $*$ a continuous t-norm and let $M : X \times X \times [0, \infty[\to [0, 1]$ and $d : X \times X \to \operatorname{map}(\mathbb{R}, [0, 1])$ be two mappings related by:*

$$d_{xy}(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1 - M_{xy}(t), & \text{if } t \geq 0 \end{cases} \qquad \text{for all } x, y \in X \text{ and all } t \in \mathbb{R}.$$

Then $(X, M, *)$ is a KM-space iff $(X, d, L = 0, R = *')$ is a KS-space.*

Proof. Note that, for a KS*-space, $\lambda_1(x, y) = 0$, and for all $x, y, z \in X$ and all $t, s \in [0, \infty[$:

$$\begin{aligned} &M_{xz}(t + s) \geq M_{xy}(t) * M_{yz}(s) \Leftrightarrow 1 - d_{xz}(t + s) \geq (1 - d_{xy}(t)) * (1 - d_{yz}(s)) \Leftrightarrow \\ &\quad \Leftrightarrow d_{xz}(t + s) \leq d_{xy}(t) *' d_{yz}(s). \end{aligned}$$

□

Corollary 4.16. *The concepts of Menger space, simple Kaleva-Seikkala space and Kramosil-Michalek space are equivalent.*

We conclude this paper analyzing the relationship between FM-spaces and IPM-spaces.

Theorem 4.17. *Let X be a set, τ a triangle function on Δ^+, $\tilde{\tau}$ a triangle function on ∇^+ and υ the triangle function on $\mathcal{F}$ asociated to $\tau \times \tilde{\tau}'$. Let $d : X \times X \to \Delta^+$, $\tilde{d} : X \times X \to \nabla^+$ and $\mathbf{F} : X \times X \to \mathcal{F}$ three maps verifying:*

$$\mathbf{F}_{xy}(t) = \begin{cases} 1 - d_{xy}(-t), & \text{if } t < 0, \\ \tilde{d}_{xy}(t), & \text{if } t \geq 0. \end{cases}$$

Then (X, d, τ, d, τ) is an IPM-space iff $(X, \mathbf{F}, \upsilon)$ is a conormalised FM-space such that $\mathbf{F}_{xy}|_{]-\infty,0[} = \overline{0}|_{]-\infty,0[} \Leftrightarrow \mathbf{F}_{xy}|_{]0,\infty[} = \overline{0}|_{]0,\infty[}$.

As a result, every IPM-space is a FM-space.

Proof. It is clear that $\mathbf{F}_{xy} = \overline{0}$ iff $d_{xy} = \epsilon_0$ and $\tilde{d}_{xy} = \tilde{\epsilon}_0$, but this is equivalent to $d_{xy} = \epsilon_0$ or $\tilde{d}_{xy} = \tilde{\epsilon}_0$, so F1 and F2 are equivalent to P1 and C1. It is also clear that $\mathbf{F}$ is a symmetric function iff d and $\tilde{d}$ are also symmetric. Note that, for all $x, y \in X$ and all $t > 0$:

$$d_{xy}(t) = 1 - \mathbf{F}_{xy}(-t) = \mathbf{F}^-_{xy}(t), \quad \tilde{d}_{xy}(t) = \mathbf{F}_{xy}(t) = 1 - \mathbf{F}^+_{xy}(t).$$

This means that:

$$\begin{aligned} &d_{xz} \geq \tau(d_{xy}, d_{yz}) \Leftrightarrow \mathbf{F}^-_{xz} \geq \tau(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}); \\ &\tilde{d}_{xz} \leq \tilde{\tau}(\tilde{d}_{xy}, \tilde{d}_{yz}) \Leftrightarrow 1 - \mathbf{F}^+_{xz} \leq \tilde{\tau}(1 - \mathbf{F}^+_{xy}, 1 - \mathbf{F}^+_{yz}) \Leftrightarrow \\ &\qquad \Leftrightarrow 1 - \tilde{\tau}(1 - \mathbf{F}^+_{xy}, 1 - \mathbf{F}^+_{yz}) \leq \mathbf{F}^+_{xz} \Leftrightarrow \tilde{\tau}'(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}) \leq \mathbf{F}^+_{xz}. \end{aligned}$$

As Φ is an inverse order bijection and υ is the triangle function on $\mathcal{F}$ associated with $\tau \times \tilde{\tau}'$:

$$\mathbf{F}_{xz} \leq \upsilon(\mathbf{F}_{xy}, \mathbf{F}_{yz}) \;\Leftrightarrow\; \Phi(\mathbf{F}_{xz}) \geq \Phi(\upsilon(\mathbf{F}_{xy}, \mathbf{F}_{yz})) \;\Leftrightarrow$$
$$\Leftrightarrow\; (\mathbf{F}^-_{xz}, \mathbf{F}^+_{xz}) \geq (\tau(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}), \tilde{\tau}'(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz})) \;\Leftrightarrow$$
$$\Leftrightarrow \left\{ \begin{array}{l} \mathbf{F}^-_{xz} \geq \tau(\mathbf{F}^-_{xy}, \mathbf{F}^-_{yz}) \text{ and} \\ \mathbf{F}^+_{xz} \geq \tilde{\tau}'(\mathbf{F}^+_{xy}, \mathbf{F}^+_{yz}) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} d_{xz} \geq \tau(d_{xy}, d_{yz}) \text{ and} \\ \tilde{d}_{xz} \leq \tilde{\tau}(\tilde{d}_{xy}, \tilde{d}_{yz}). \end{array} \right.$$

Therefore, F3 is equivalent to P3 and C3. Finally $(X, \mathbf{F}, \upsilon)$ is conormalised iff, for all $x, y \in X$ and all $t \in]0, \infty[$,

$$\mathbf{F}_{xy}(t) \leq \mathbf{F}_{xy}(-t) \Leftrightarrow \tilde{d}_{xy}(t) \leq 1 - d_{xy}(t) \Leftrightarrow d_{xy}(t) + \tilde{d}_{xy}(t) \leq 1.$$

□

5. **Concluding Remarks**

In this paper we study the common structure of different classes of spaces that have been introduced independently by several authors to model real situations. We have shown that each of these spaces can be interpreted in terms of distances evaluated by fuzzy numbers, and that the underlying triangular inequality can be expressed through triangular functions. The class of fuzzy metric spaces introduced in this paper is also provided with a strong Hausdorff topology. This class has been used to study the interrelationships between the fuzzy metric structures considered. Figure 1 gives an overview of the interrelations between the different theories considered.

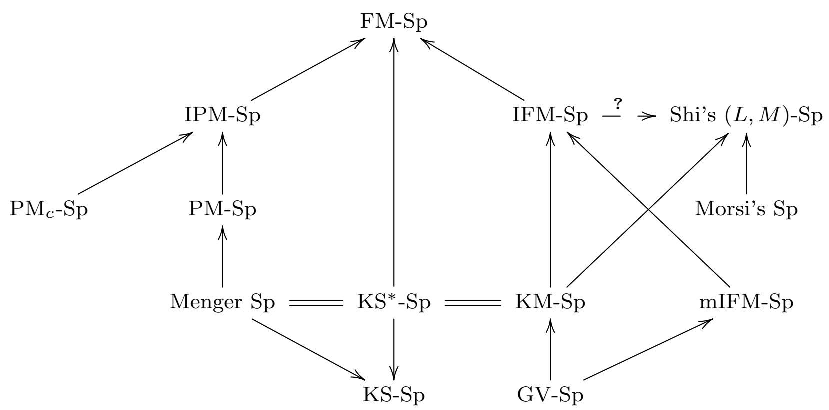

FIGURE 1. Links Between the Different Theories ("Sp"means "spaces")

Further investigation will be required to apply this class of spaces that generate a unified view of different theories. For example, following [5], it would be interesting to study the interrelationship of the FM-spaces introduced in this paper with other modeling imprecision theories and consider the possible applicability to real scenarios. In particular, we think (L, M)-fuzzy metric introduced by Shi is a deeper concept that is closely related to our notion of FM-space when we choose an appropriate lattice $\mathcal{L}$.

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