

## OPTIMAL CONTROL OF FUZZY LINEAR CONTROLLED SYSTEM WITH FUZZY INITIAL CONDITIONS

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**ABSTRACT.** In this article we found the solution of fuzzy linear controlled system with fuzzy initial conditions by using  $\alpha$ -cuts and presentation of numbers in a more compact form by moving to the field of complex numbers. Next, a fuzzy optimal control problem for a fuzzy system is considered to optimize the expected value of a fuzzy objective function. Based on Pontryagin Maximum Principle, a constructive equation for the problem is presented. In the last section, three examples are used to show that the method is effective to solve fuzzy and fuzzy optimal linear controlled systems.

### 1. Introduction

Optimal control theory is developed to find optimal ways to control dynamic systems. Uncertainty is inherent in most real-world systems. Fuzziness is a kind of uncertainty in real world problems. So, some dynamic control systems may be described by fuzzy differential equations and fuzzy control. The decision makers must select an optimal decision among all possible ones to reach the result. Such optimal control problems, called fuzzy optimal control problems.

Many authors have studied several concepts of fuzzy systems. Zhu [21] applied Bellman's optimal principle to obtain the principle of optimality for fuzzy optimal control problems. Diamond and Kloeden [4] showed the existence of the fuzzy optimal control for the system  $\dot{\tilde{x}}(t) = a(t) \odot \tilde{x}(t) \oplus \tilde{u}(t)$ ,  $\tilde{x}(0) = \tilde{x}_0$ , where the set of admissible pairs such as  $p = [\tilde{x}(\cdot), \tilde{u}(\cdot)]$  are nonempty compact interval-valued functions on  $E^1$ . Park et.al [12] found the sufficient conditions of nonlocal controllability for the semilinear fuzzy integrodifferential equations.

In [6] Filev and Angelove, formulated the problem of fuzzy optimal control of nonlinear system and solved this problem on the basis of fuzzy mathematical programming. They considered a particular case of fuzzy optimal control with fuzzy objective function and crisp transversality conditions.

Z. Qin [15] considered the time-homogeneous fuzzy optimal control problems, with discounted objective function. Zhu [20] introduced a method to solve fuzzy optimal control problem by using dynamic programming.

In [2], the initial value problem related to the fuzzy differential equation  $\tilde{y}'(t) = a(t) \odot \tilde{y}(t) \oplus \tilde{b}(t)$  was studied, obtaining the expression of (1)-differentiable solutions for the case  $a > 0$  and (2)-differentiable solutions for  $a < 0$ . In [8], authors complete

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this study, showing new solutions for the problem which are not included in [2]. Namely, in Theorems 3.2 and 3.4, they give the expression of (1)-differentiable solutions for  $a < 0$  and (2)-differentiable solutions for  $a > 0$ . In that paper, the authors consider first order linear fuzzy differential equations under the generalized differentiability concept and present the solutions of this problem in the general case. Nieto et.al in [10], obtain explicitly the solution for a linear fuzzy differential equation subject to impulsive behavior and periodic boundary value conditions by proving that the same Greens function obtained for the study of the corresponding ordinary differential equation is valid.

In [9] by using the generalized differentiability authors, found new solutions for some fuzzy two-point boundary value problems for which considering only the Hukuhara differentiability, a solution did not exist. Indeed, , they could find solutions for a larger class of fuzzy boundary value problems by generalized differentiability rather than using the Hukuhara differentiability.

in [7] Georgiou et.al considered  $n$ th-order fuzzy differential equations with initial value conditions. They proved the existence and uniqueness of solution for nonlinearities satisfying a Lipschitz condition. They applied the obtained results to the particular case of linear fuzzy problems.

In [11], Nieto et.al using strongly generalized differentiability, they generalized some numerical methods presented for solving FDEs. The original initial value problem is replaced by two parametric ordinary differential systems which are then solved numerically using classical algorithms.

Applications of optimal control problems involve the control of dynamic systems, that evolve over time either continuous-time systems or discrete time systems. The notion of continuous dynamical system is a basic concept in system theory: it is the formalization of a natural phenomenon in a set of variables defined as the state and a set of deterministic differential equations, defined as the model and a set of variables defined as the control.

In [19] authors presented a new solution for fuzzy differential equation with fuzzy initial conditions by using  $\alpha$ -cut. In [5] a modified  $k$ -step method with one non-step point was presented to solve the fuzzy initial value problem . In [1] Allahviranloo and Kermani considered different method to solve fuzzy partial differential equations such as fuzzy hyperbolic and fuzzy parabolic equations. In [16] a model of an optimal control problem with chance constraints is introduced.

In this paper we extend this technique to solve fuzzy control system governed by fuzzy differential equation, and at the end, by using Pontryagin Maximum Principle (PMP) (see Theorem 4.1 of [14]), the optimal fuzzy controlled system is solved.

## 2. Fundamental Theorem for the Linear Systems

In this section we will consider a fuzzy linear control problem with fuzzy initial conditions. We begin the section by considering the following fuzzy linear dynamical system:

$$\begin{cases} \dot{\tilde{x}}(t) = A \odot \tilde{x}(t) \\ \tilde{x}(t_0) = \tilde{x}_0 \end{cases} \quad (1)$$

where  $\tilde{x}$  is a fuzzy function,  $\tilde{x}_0$  is a fuzzy initial condition,  $A = [a_{ij}]_{n \times m}$ ,  $a_{ij} \in \mathfrak{R}$  and  $\dot{\tilde{x}}(t) = \frac{d\tilde{x}}{dt} = [\frac{d\tilde{x}_1}{dt}, \dots, \frac{d\tilde{x}_n}{dt}]^T$ . To consider (2), one needs the following definition and lemmas.

**Definition 2.1.** Denote  $E^1 = \{u \mid u \text{ satisfies (i) - (iv)}\}$  where:

- (i)  $u$  is normal, i.e. there exists  $x \in \mathfrak{R}$ , such that  $u(x) = 1$ .
- (ii)  $u$  is fuzzy convex, i.e.  
 $\forall x, y \in \mathfrak{R}$  and  $\lambda \in [0, 1]$ ,  $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ .
- (iii)  $u$  is upper semicontinuous.
- (iv)  $cl\{s \in \mathfrak{R} \mid u(s) > 0\}$ , is compact in  $\mathfrak{R}$ .

The  $\alpha$ -level set of a fuzzy number  $u \in E^1$ ,  $0 \leq \alpha \leq 1$ , denoted by  $u_\alpha$ , is defined as:

$$u_\alpha = \begin{cases} \{s \in \mathfrak{R} \mid u(s) \geq \alpha\} & 0 < \alpha \leq 1 \\ cl\{s \in \mathfrak{R} \mid u(s) > 0\} & \alpha = 0 \end{cases}$$

If  $u \in E^1$ , then  $u$  is fuzzy convex, so  $u_\alpha$  is closed and bounded in  $\mathfrak{R}$ , i.e.  $u_\alpha = [\underline{u}_\alpha, \bar{u}_\alpha]$ , where  $\underline{u}_\alpha = \inf\{x \in \mathfrak{R} : u(x) \geq \alpha\} > -\infty$  and  $\bar{u}_\alpha = \sup\{x \in \mathfrak{R} : u(x) \geq \alpha\} < \infty$ . For more detail see [4].

**Lemma 2.2.** Denote  $I = [0, 1]$ . Assumed that  $a : I \rightarrow \mathfrak{R}$  and  $b : I \rightarrow \mathfrak{R}$  satisfy the following conditions:

- (i)  $a : I \rightarrow \mathfrak{R}$  is a bounded non-decreasing function,
- (ii)  $b : I \rightarrow \mathfrak{R}$  is a bounded non-increasing function,
- (iii)  $a(1) \leq b(1)$ ,
- (iv) for  $0 < k \leq 1$ ,  $\lim_{\alpha \rightarrow k^-} a(\alpha) = a(k)$  and  $\lim_{\alpha \rightarrow k^-} b(\alpha) = b(k)$ ,
- (v)  $\lim_{\alpha \rightarrow 0^+} a(\alpha) = a(0)$  and  $\lim_{\alpha \rightarrow 0^+} b(\alpha) = b(0)$ ,

then  $\eta : \mathfrak{R} \rightarrow I$  defined by  $\eta(x) = \sup\{\alpha \mid a(\alpha) \leq x \leq b(\alpha)\}$  is a fuzzy number with parametrization given by  $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ . Moreover, if  $\eta : \mathfrak{R} \rightarrow I$  is a fuzzy number with parametrization given by  $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$  then functions  $a(\alpha)$  and  $b(\alpha)$  satisfy the conditions (i) – (v) in Lemma 2.2.

*Proof.* See [4]. □

Now, we back to the fuzzy dynamical system (1). Lemma 2.3 points out the way of finding the solution of (1).

**Lemma 2.3.** Assume each entry of the vector  $x$  in (1) be a fuzzy number at the time instant  $t$  where

$$x_\alpha^k(t) = [\underline{x}_\alpha^k(t), \bar{x}_\alpha^k(t)], \quad k = 1, 2, \dots, n, \quad (2)$$

then, the evaluation of the system (1) can be described by  $2n$  differential equations for the endpoints of the intervals (2). The equations for the endpoints of the intervals are as follows:

$$\begin{cases} \dot{\underline{x}}_\alpha^k(t) = \min\{(Au)^k : u^j \in [\underline{x}_\alpha^j(t), \bar{x}_\alpha^j(t)]\}, \\ \dot{\bar{x}}_\alpha^k(t) = \max\{(Au)^k : u^j \in [\underline{x}_\alpha^j(t), \bar{x}_\alpha^j(t)]\} \\ \underline{x}_\alpha(t_0) = \underline{x}_{\alpha 0} \\ \bar{x}_\alpha(t_0) = \bar{x}_{\alpha 0} \end{cases} \quad (3)$$

where  $(Au)^k := \sum_{j=1}^n a_{kj}u^j$  is the  $k$ th row of  $Au$ .

*Proof.* See [18], [13], [17]. □

Since the vector field in (1) is linear, the following rule applies in (3):

$$\dot{\underline{x}}_{\alpha}^k(t) = \sum_{j=1}^n a_{kj}u^j$$

where

$$u^j = \begin{cases} \underline{x}_{\alpha}^j(t) & a_{kj} \geq 0 \\ \bar{x}_{\alpha}^j(t) & a_{kj} < 0 \end{cases}$$

and

$$\dot{\bar{x}}_{\alpha}^k(t) = \sum_{j=1}^n a_{kj}v^j,$$

where

$$v^j = \begin{cases} \bar{x}_{\alpha}^j(t) & a_{kj} \geq 0 \\ \underline{x}_{\alpha}^j(t) & a_{kj} < 0. \end{cases}$$

Now we give a characterization for an important class of fuzzy controlled system. Consider the following fuzzy linear controlled system with fuzzy boundary conditions:

$$\begin{cases} \dot{\tilde{x}}(t) = A \odot \tilde{x}(t) \oplus C \odot \tilde{u}(t) \\ \tilde{x}(t_0) = \tilde{x}_0, \tilde{x}(t_f) = \tilde{x}_f, \end{cases} \quad (4)$$

the task is to carry the controlled system from the initial point  $\tilde{x}(t_0) = \tilde{x}_0$  to final target  $\tilde{x}(t_f) = \tilde{x}_f$ , by using suitable fuzzy control function  $\tilde{u}$ . As indicated in [13], it is possible to represent a fuzzy number in a more compact form by moving to the field of complex numbers. Define new complex variables as follows:

$$x_{\alpha}^k = \underline{x}_{\alpha}^k(t) + i\bar{x}_{\alpha}^k(t), \quad k = 1, \dots, n. \quad (5)$$

Then, the following theorem gives the solution:

**Theorem 2.4.** Let  $A$  and  $C$  be  $n \times n$  and  $n \times m$  matrices respectively. Then for a given  $\tilde{x}_0$ , the fuzzy controlled system

$$\begin{cases} \dot{\tilde{x}}(t) = A \odot \tilde{x}(t) \oplus C \odot \tilde{u}(t) \\ \tilde{x}(t_0) = \tilde{x}_0 \\ \tilde{x}(t_f) = \tilde{x}_f, \end{cases} \quad (6)$$

has the solution as:

$$\begin{cases} \dot{\underline{x}}_{\alpha}(t) + i\dot{\bar{x}}_{\alpha}(t) = B(\underline{x}_{\alpha}(t) + i\bar{x}_{\alpha}(t)) + D(\underline{u}_{\alpha}(t) + i\bar{u}_{\alpha}(t)) \\ \underline{x}_{\alpha}(t_0) + i\bar{x}_{\alpha}(t_0) = \underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0} \\ \underline{x}_{\alpha}(t_f) + i\bar{x}_{\alpha}(t_f) = \underline{x}_{\alpha f} + i\bar{x}_{\alpha f} \end{cases} \quad (7)$$

where the elements of matrices  $B$  and  $D$  are determined from those of  $A$  and  $C$  as follows:

$$b_{ij} = \begin{cases} ea_{ij} & a_{ij} \geq 0 \\ ga_{ij} & a_{ij} < 0 \end{cases} \quad d_{ij} = \begin{cases} ec_{ij}, & c_{ij} \geq 0 \\ gc_{ij} & c_{ij} < 0 \end{cases} \quad (8)$$

where for every  $a + bi \in \mathbb{C}$  (complex numbers field),

$$\begin{aligned} e : a + bi &\rightarrow a + bi \\ g : a + bi &\rightarrow b + ai \end{aligned} \quad (9)$$

*Proof.* By using (5) and the two operators in (9), the system (6) can be written as (7). The solution of (7) is given by:

$$\begin{aligned} \underline{x}_\alpha(t) + i\bar{x}_\alpha(t) &= \exp((t - t_0)B)(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0}) \\ &+ \int_{t_0}^t \exp((t - s)B)D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds, \end{aligned} \quad (10)$$

since by derivation from(10) one have:

$$\begin{aligned} \dot{\underline{x}}_\alpha(t) + i\dot{\bar{x}}_\alpha(t) &= \frac{d}{dt}(e^{(t-t_0)B}(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0})) + \frac{d}{dt}\left(\int_{t_0}^t e^{(t-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds\right) \\ &= \lim_{h \rightarrow 0} \frac{e^{(t+h-t_0)B} - e^{(t-t_0)B}}{h}(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0}) \\ &+ \lim_{h \rightarrow 0} \frac{\int_{t_0}^{t+h} e^{(t+h-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds - \int_{t_0}^t e^{(t-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds}{h} \\ &= e^{(t-t_0)B} \lim_{h \rightarrow 0} \frac{e^{hB} - I}{h}(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0}) \\ &+ \lim_{h \rightarrow 0} \frac{\int_{t_0}^t e^{(t+h-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds - \int_{t_0}^t e^{(t-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds}{h} \\ &+ \lim_{h \rightarrow 0} \frac{\int_t^{t+h} e^{(t+h-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds}{h} = e^{(t-t_0)B} \lim_{h \rightarrow 0} \frac{e^{hB} - I}{h}(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0}) \\ &+ \lim_{h \rightarrow 0} \frac{\int_{t_0}^t e^{(t-s)B}(e^{hB} - I)D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds}{h} \\ &+ \lim_{h \rightarrow 0} \frac{\int_t^{t+h} e^{(t+h-s)B}D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds}{h} \\ &= e^{(t-t_0)B} \lim_{h \rightarrow 0} \lim_{k \rightarrow +\infty} \left(B + \frac{B^2h}{2!} + \dots + \frac{B^k h^{k-1}}{k!}\right)(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0}) \\ &+ \int_{t_0}^t e^{(t-s)B} \lim_{h \rightarrow 0} \lim_{k \rightarrow +\infty} \left(B + \frac{B^2h}{2!} + \dots + \frac{B^k h^{k-1}}{k!}\right)D(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds \\ &+ \lim_{h \rightarrow 0} \frac{e^{(t+h-t)B}D(\underline{u}_\alpha(t) + i\bar{u}_\alpha(t))h}{h} = e^{(t-t_0)B}B(\underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0}) \\ &+ \int_{t_0}^t e^{(t-s)B}BD(\underline{u}_\alpha(s) + i\bar{u}_\alpha(s))ds + D(\underline{u}_\alpha(t) + i\bar{u}_\alpha(t)). \end{aligned}$$

Since  $B$  commutes  $\exp((t - t_0)B)$ , the required result follows:

$$\dot{\underline{x}}_\alpha(t) + i\dot{\bar{x}}_\alpha(t) = B(\underline{x}_\alpha(t) + i\bar{x}_\alpha(t)) + D(\underline{u}_\alpha(t) + i\bar{u}_\alpha(t))$$

□

### 3. Optimal Control of Time Invariant System with Fuzzy Conditions

In this section consider the fuzzy optimal controlled system:

$$\begin{aligned} & \text{Min} \int_{t_0}^{t_f} \tilde{u}^2(t) dt \\ & \text{s.t.} \\ & \dot{\tilde{x}}(t) = A \odot \tilde{x}(t) \oplus C \odot \tilde{u}(t) \\ & \tilde{x}(t_0) = \tilde{x}_0, \tilde{x}(t_f) = \tilde{x}_f \end{aligned} \quad (11)$$

Similar to the fuzzy controlled system (4), the optimal controlled system(11) changes to:

$$\begin{aligned} & \text{Min} \int_{t_0}^{t_f} (\underline{u}_\alpha^2(t) + i\bar{u}_\alpha^2(t)) dt \\ & \text{s.t.} \\ & \dot{\underline{x}}_\alpha(t) + i\dot{\bar{x}}_\alpha(t) = B(\underline{x}_\alpha(t) + i\bar{x}_\alpha(t)) + D(\underline{u}_\alpha(t) + i\bar{u}_\alpha(t)) \\ & \underline{x}_\alpha(t_0) + i\bar{x}_\alpha(t_0) = \underline{x}_{\alpha 0} + i\bar{x}_{\alpha 0} \\ & \underline{x}_\alpha(t_f) + i\bar{x}_\alpha(t_f) = \underline{x}_{\alpha f} + i\bar{x}_{\alpha f} \end{aligned} \quad (12)$$

where the elements of matrices  $B$  and  $D$  are determined by those of  $A$  and  $C$  as follows:

$$b_{ij} = \begin{cases} ea_{ij}, & a_{ij} \geq 0 \\ ga_{ij}, & a_{ij} < 0 \end{cases}$$

and

$$d_{ij} = \begin{cases} ec_{ij}, & c_{ij} \geq 0 \\ gc_{ij}, & c_{ij} < 0 \end{cases}$$

where

$$\begin{aligned} e : a + bi &\rightarrow a + bi \\ g : a + bi &\rightarrow b + ai. \end{aligned}$$

If  $(\underline{x}_\alpha^*, \bar{x}_\alpha^*, \underline{u}_\alpha^*, \bar{u}_\alpha^*)$  be the optimal solution for (12) then the following theorem is hold:

**Theorem 3.1.** Define the Hamiltonian function

$$H(\underline{x}, \bar{x}, \underline{u}, \bar{u}, \psi, t) = -(\underline{u}_\alpha^2(t) + i\bar{u}_\alpha^2(t)) + \psi^T (B(\underline{x}_\alpha(t) + i\bar{x}_\alpha(t)) + D(\underline{u}_\alpha(t) + i\bar{u}_\alpha(t)))$$

, then the necessary conditions for the quadruple  $(\underline{x}_\alpha^*, \bar{x}_\alpha^*, \underline{u}_\alpha^*, \bar{u}_\alpha^*)$ , the optimal solution of (12) is the existence of a costate vector function  $\psi$  that satisfies the following differential equations:

$$\dot{\psi} = -\frac{\partial H}{\partial x_\alpha}$$

where  $x_\alpha = (\underline{x}_\alpha, \bar{x}_\alpha)$ , and the Hamiltonian function in  $\underline{u}_\alpha^*$  and  $\bar{u}_\alpha^*$  is maximized ,i.e.

$$\frac{\partial H}{\partial \underline{u}_\alpha^*} = 0, \quad \text{and} \quad \frac{\partial H}{\partial \bar{u}_\alpha^*} = 0.$$

*Proof.* Suppose  $f_0(\underline{u}_\alpha, \bar{u}_\alpha) = \underline{u}_\alpha^2(t) + i\bar{u}_\alpha^2(t)$  and  $f_j(\underline{x}_\alpha, \bar{x}_\alpha, \underline{u}_\alpha, \bar{u}_\alpha)$  be the  $j$ th row of  $B(\underline{x}_\alpha(t) + i\bar{x}_\alpha(t)) + D(\underline{u}_\alpha(t) + i\bar{u}_\alpha(t))$  for  $j = 1, 2, \dots, n$ , and  $\mathcal{J} = \int_{t_0}^{t_f} f_0(\underline{u}_\alpha, \bar{u}_\alpha) dt$ . There is no lose of generality in taking  $n = 2$  and  $m = 1$ . Let  $\underline{u}_\alpha^*(t)$  and  $\bar{u}_\alpha^*(t)$  be the optimal controls and  $\underline{x}_{1\alpha}^*(t)$ ,  $\bar{x}_{1\alpha}^*(t)$ ,  $\underline{x}_{2\alpha}^*(t)$  and  $\bar{x}_{2\alpha}^*(t)$  be the corresponding optimal paths. Consider a small variation of  $\underline{u}_\alpha^*(t)$ , as  $\underline{u}_\alpha(t) = \underline{u}_\alpha^*(t) + \delta\underline{u}_\alpha(t)$  and for  $\bar{u}_\alpha^*(t)$ , as  $\bar{u}_\alpha(t) = \bar{u}_\alpha^*(t) + \delta\bar{u}_\alpha(t)$  with corresponding path  $(\underline{x}_{1\alpha}^*(t) + \delta\underline{x}_{1\alpha}(t), \bar{x}_{1\alpha}^*(t) + \delta\bar{x}_{1\alpha}(t), \underline{x}_{2\alpha}^*(t) + \delta\underline{x}_{2\alpha}(t), \bar{x}_{2\alpha}^*(t) + \delta\bar{x}_{2\alpha}(t))$ . This will not arrive at  $\underline{x}_{1\alpha f}$ ,  $\bar{x}_{1\alpha f}$ ,  $\underline{x}_{2\alpha f}$  and  $\bar{x}_{2\alpha f}$  at  $t_f$  but at a slightly different time  $t_f + \delta t$ . The latter conditions give

$$\underline{x}_{j\alpha}(t_f + \delta t) + i\bar{x}_{j\alpha}(t_f + \delta t) = \underline{x}_{j\alpha f} + i\bar{x}_{j\alpha f} \quad j = 1, 2.$$

As usual in variational arguments we are in the first instance interested only in first-order effects, and from the latter conditions we deduce that

$$(\delta\underline{x}_{j\alpha}(t_f) + i\delta\bar{x}_{j\alpha}(t_f)) + (\dot{\underline{x}}_{j\alpha}(t_f)\delta t + i\dot{\bar{x}}_{j\alpha}(t_f)\delta t) = 0 \quad j = 1, 2.$$

If we now use the state equations we obtain

$$\delta\underline{x}_{j\alpha}(t_f) + i\delta\bar{x}_{j\alpha}(t_f) = -f_j(t_f)\delta t,$$

where  $f_j(t_f) = f_j(\underline{x}_{1\alpha}(t_f), \bar{x}_{1\alpha}(t_f), \underline{x}_{2\alpha}(t_f), \bar{x}_{2\alpha}(t_f), \underline{u}_\alpha(t_f), \bar{u}_\alpha(t_f))$ . The consequent change  $\Delta\mathcal{J}$  in  $\mathcal{J}$  is

$$\begin{aligned} \Delta\mathcal{J} &= \int_{t_0}^{t_f + \delta t} f_0(\underline{u}_\alpha^* + \delta\underline{u}_\alpha, \bar{u}_\alpha^* + \delta\bar{u}_\alpha) dt - \int_{t_0}^{t_f} f_0(\underline{u}_\alpha^*, \bar{u}_\alpha^*) dt \\ &= \int_{t_0}^{t_f} \left\{ \frac{\partial f_0}{\partial \underline{u}_\alpha} \delta\underline{u}_\alpha + \frac{\partial f_0}{\partial \bar{u}_\alpha} \delta\bar{u}_\alpha \right\} dt + f_0(t_f)\delta t + O((\delta\underline{u}_\alpha^*)^2) + O((\delta\bar{u}_\alpha^*)^2), \end{aligned}$$

where  $f_0(t_f)$  is the value of  $f_0$  at  $t = t_f$  and the derivatives in the integrand are evaluated by the optimal trajectory. Let  $\delta\mathcal{J}$  denote the first variation. If  $\underline{u}_\alpha^*$  and  $\bar{u}_\alpha^*$  are optimal, it is necessary that the first variation  $\delta\mathcal{J}$  be zero, so

$$\int_{t_0}^{t_f} \left\{ \frac{\partial f_0}{\partial \underline{u}_\alpha} \delta\underline{u}_\alpha + \frac{\partial f_0}{\partial \bar{u}_\alpha} \delta\bar{u}_\alpha \right\} dt + f_0(t_f)\delta t = 0$$

, on the optimal paths for all variations.

We simply need to introduce two Lagrange multipliers  $\psi_1(t)$  and  $\psi_2(t)$ . Now consider the pair of integrals

$$\varphi_j = \int_{t_0}^{t_f} \psi_j(t) (\dot{\underline{x}}_{j\alpha} + i\dot{\bar{x}}_{j\alpha} - f_j(\underline{x}_{1\alpha}, \bar{x}_{1\alpha}, \underline{x}_{2\alpha}, \bar{x}_{2\alpha}, \underline{u}_\alpha, \bar{u}_\alpha)) dt \quad j = 1, 2.$$

With respect to [14] one can see that  $\delta\varphi_j = 0$ , since  $\varphi_j = 0$ , for all  $\underline{u}_\alpha$ ,  $\bar{u}_\alpha$ . The calculation is straightforward, giving

$$\begin{aligned} \delta\varphi_j &= \int_{t_0}^{t_f} \psi_j(t) \left\{ -\frac{\partial f_j}{\partial \underline{x}_{1\alpha}} \delta\underline{x}_{1\alpha} - \frac{\partial f_j}{\partial \bar{x}_{1\alpha}} \delta\bar{x}_{1\alpha} - \frac{\partial f_j}{\partial \underline{x}_{2\alpha}} \delta\underline{x}_{2\alpha} - \frac{\partial f_j}{\partial \bar{x}_{2\alpha}} \delta\bar{x}_{2\alpha} \right. \\ &\quad \left. - \frac{\partial f_j}{\partial \underline{u}_\alpha} \delta\underline{u}_\alpha - \frac{\partial f_j}{\partial \bar{u}_\alpha} \delta\bar{u}_\alpha + \frac{d}{dt} (\delta\underline{x}_{j\alpha} + i\delta\bar{x}_{j\alpha}) \right\} dt. \end{aligned}$$

Now

$$\int_{t_0}^{t_f} \psi_j(t) \frac{d}{dt} (\delta \underline{x}_{j\alpha} + i\delta \bar{x}_{j\alpha}) dt = -f_j(t_f) \psi_j(t_f) \delta t - \int_{t_0}^{t_f} \dot{\psi}_j (\delta \underline{x}_{j\alpha} + i\delta \bar{x}_{j\alpha}) dt.$$

The condition that  $\delta \mathcal{J} = 0$  can be replaced by the condition that  $\delta \mathcal{J} + \delta \varphi_1 + \delta \varphi_2 = 0$ . And if we introduce the Hamiltonian function:

$$H = -f_0(\underline{u}_\alpha, \bar{u}_\alpha) + \psi_1 f_1(\underline{x}_{1\alpha}, \bar{x}_{1\alpha}, \underline{x}_{2\alpha}, \bar{x}_{2\alpha}, \underline{u}_\alpha, \bar{u}_\alpha) + \psi_2 f_2(\underline{x}_{1\alpha}, \bar{x}_{1\alpha}, \underline{x}_{2\alpha}, \bar{x}_{2\alpha}, \underline{u}_\alpha, \bar{u}_\alpha),$$

then one can obtain the condition of the Theorem easily.  $\square$

By using Theorem 3.1 and (7) one can solve the fuzzy optimal controlled system (11). In the next section, we apply this technique to solve two numerical examples in detail.

#### 4. Numerical Examples

**Example 4.1.** Consider the following fuzzy linear controlled systems:

$$\begin{cases} \dot{\tilde{x}}_1(t) = -2 \odot \tilde{x}_2(t) - \tilde{u}(t) \\ \dot{\tilde{x}}_2(t) = 2 \odot \tilde{x}_1(t) \\ \tilde{x}_1(0) = \tilde{x}_2(0) = (1, 2, 3) \\ \tilde{x}_1(1) = \tilde{x}_2(1) = (-0.5, 0, 0.5). \end{cases}$$

By using (7) and definitions  $B$  and  $D$ , one can find the following controlled system:

$$\begin{bmatrix} \dot{\tilde{x}}_{1\alpha} \\ \dot{\tilde{x}}_{1\alpha} \\ \dot{\tilde{x}}_{2\alpha} \\ \dot{\tilde{x}}_{2\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{1\alpha} \\ \bar{x}_{1\alpha} \\ \underline{x}_{2\alpha} \\ \bar{x}_{2\alpha} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_\alpha \\ \bar{u}_\alpha \end{bmatrix}$$

The solution of the above controlled system is:

$$\begin{bmatrix} \underline{x}_{1\alpha} \\ \bar{x}_{1\alpha} \\ \underline{x}_{2\alpha} \\ \bar{x}_{2\alpha} \end{bmatrix} = \begin{bmatrix} \frac{\cos 2t + e^{2t+e-2t}}{2} & \frac{\cos 2t - e^{2t+e-2t}}{2} & \frac{-\sin 2t + e^{2t-e-2t}}{2} \\ \frac{\cos 2t - e^{2t+e-2t}}{2} & \frac{\cos 2t + e^{2t+e-2t}}{2} & \frac{-\sin 2t - e^{2t-e-2t}}{2} \\ \frac{\sin 2t + e^{2t-e-2t}}{2} & \frac{\sin 2t - e^{2t-e-2t}}{2} & \frac{\cos 2t + e^{2t+e-2t}}{2} \\ \frac{\sin 2t - e^{2t-e-2t}}{2} & \frac{\sin 2t + e^{2t+e-2t}}{2} & \frac{\cos 2t - e^{2t+e-2t}}{2} \end{bmatrix} \begin{bmatrix} 2\alpha + (1-\alpha) \\ 2\alpha + 3(1-\alpha) \\ 2\alpha + (1-\alpha) \\ 2\alpha + 3(1-\alpha) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{\cos 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} & \frac{\cos 2(t-s) - e^{2(t-s)+e-2(t-s)}}{2} & \frac{-\sin 2(t-s) + e^{2(t-s)-e-2(t-s)}}{2} \\ \frac{\cos 2(t-s) - e^{2(t-s)+e-2(t-s)}}{2} & \frac{\cos 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} & \frac{-\sin 2(t-s) - e^{2(t-s)-e-2(t-s)}}{2} \\ \frac{\sin 2(t-s) + e^{2(t-s)-e-2(t-s)}}{2} & \frac{\sin 2(t-s) - e^{2(t-s)-e-2(t-s)}}{2} & \frac{\cos 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} \\ \frac{\sin 2(t-s) - e^{2(t-s)-e-2(t-s)}}{2} & \frac{\sin 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} & \frac{\cos 2(t-s) - e^{2(t-s)+e-2(t-s)}}{2} \\ \frac{-\sin 2(t-s) + e^{2(t-s)-e-2(t-s)}}{2} & \frac{-\sin 2(t-s) - e^{2(t-s)-e-2(t-s)}}{2} & \frac{\cos 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} \\ \frac{-\sin 2(t-s) - e^{2(t-s)-e-2(t-s)}}{2} & \frac{-\sin 2(t-s) + e^{2(t-s)-e-2(t-s)}}{2} & \frac{\cos 2(t-s) - e^{2(t-s)+e-2(t-s)}}{2} \\ \frac{\cos 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} & \frac{\cos 2(t-s) - e^{2(t-s)+e-2(t-s)}}{2} & \frac{-\sin 2(t-s) + e^{2(t-s)-e-2(t-s)}}{2} \\ \frac{\cos 2(t-s) - e^{2(t-s)+e-2(t-s)}}{2} & \frac{\cos 2(t-s) + e^{2(t-s)+e-2(t-s)}}{2} & \frac{-\sin 2(t-s) - e^{2(t-s)-e-2(t-s)}}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_\alpha(s) \\ \bar{u}_\alpha(s) \end{bmatrix} ds.$$



The control function  $u(\cdot)$  and states  $x_1(\cdot)$  and  $x_2(\cdot)$  are given respectively in Figure 1 for  $\alpha \in [0, 1]$ , where the red lines are the centers and the blue lines are the lower bounds and the green lines are the upper bounds for fuzzy functions.

**Example 4.2.** Consider the following optimal fuzzy controlled system:

$$\begin{aligned} & \text{Min} \int_0^1 \tilde{u}^2(t) dt \\ & \text{s.t.} \\ & \dot{\tilde{x}}_1(t) = -2 \odot \tilde{x}_2(t) \oplus \tilde{u}(t) \\ & \dot{\tilde{x}}_2(t) = 2 \odot \tilde{x}_1(t) \\ & \tilde{x}_1(0) = \tilde{x}_2(0) = (1, 2, 3) \\ & \tilde{x}_1(1) = \tilde{x}_2(1) = (-0.5, 0, 0.5) \end{aligned}$$

Similar the optimal fuzzy controlled system (11), the above optimal controlled system is changed to the following form:

$$\begin{aligned} & \text{Min} \int_0^1 (\underline{u}_\alpha^2(t) + \bar{u}_\alpha^2(t)) dt \\ & \text{s.t.} \\ & \begin{bmatrix} \dot{\underline{x}}_{1\alpha} \\ \dot{\bar{x}}_{1\alpha} \\ \dot{\underline{x}}_{2\alpha} \\ \dot{\bar{x}}_{2\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{1\alpha} \\ \bar{x}_{1\alpha} \\ \underline{x}_{2\alpha} \\ \bar{x}_{2\alpha} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_\alpha \\ \bar{u}_\alpha \end{bmatrix} \\ & \begin{bmatrix} \dot{\underline{x}}_{1\alpha}(0) \\ \dot{\bar{x}}_{1\alpha}(0) \\ \dot{\underline{x}}_{2\alpha}(0) \\ \dot{\bar{x}}_{2\alpha}(0) \end{bmatrix} = \begin{bmatrix} 2\alpha + (1 - \alpha) \\ 2\alpha + 3(1 - \alpha) \\ 2\alpha + (1 - \alpha) \\ 2\alpha + 3(1 - \alpha) \end{bmatrix} \\ & \begin{bmatrix} \dot{\underline{x}}_{1\alpha}(1) \\ \dot{\bar{x}}_{1\alpha}(1) \\ \dot{\underline{x}}_{2\alpha}(1) \\ \dot{\bar{x}}_{2\alpha}(1) \end{bmatrix} = \begin{bmatrix} -0.5(1 - \alpha) \\ 0.5(1 - \alpha) \\ -0.5(1 - \alpha) \\ 0.5(1 - \alpha) \end{bmatrix} \end{aligned}$$

The Hamiltonian function for the above optimal controlled system is:

$$H = -(\underline{u}_\alpha^2(t) + \bar{u}_\alpha^2(t)) + \psi_1(-2\bar{x}_{2\alpha} + \underline{u}_\alpha) + \psi_2(-2\underline{x}_{2\alpha} + \bar{u}_\alpha) + \psi_3(2\underline{x}_{1\alpha}) + \psi_4(2\bar{x}_{1\alpha}).$$

By considering Theorem 3.1,

$$\underline{u}_\alpha = 0.5\psi_1$$

$$\bar{u}_\alpha = 0.5\psi_2$$

and

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \\ \dot{\psi}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

Now by using [19] we will have:

$$\begin{aligned} \underline{u}_\alpha(t) = & \frac{1}{8}[(2\cos 2t + (e^{-2t} + e^{2t}))c_1 + (2\cos 2t - (e^{-2t} + e^{2t}))c_2 \\ & + (-2\sin 2t + (e^{-2t} - e^{2t}))c_3 + (-2\sin 2t - (e^{-2t} - e^{2t}))c_4] \end{aligned} \quad (13)$$

$$\begin{aligned} \bar{u}_\alpha(t) = & \frac{1}{8}[(2\cos 2t - (e^{-2t} + e^{2t}))c_1 + (2\cos 2t + (e^{-2t} + e^{2t}))c_2 \\ & + (-2\sin 2t - (e^{-2t} - e^{2t}))c_3 + (-2\sin 2t + (e^{-2t} - e^{2t}))c_4], \end{aligned} \quad (14)$$

where  $c_i$  for  $i = 1, \dots, 4$  are real numbers. The solution of optimal controlled system is:

$$\begin{aligned} \begin{bmatrix} \underline{x}_{1\alpha} \\ \bar{x}_{1\alpha} \\ \underline{x}_{2\alpha} \\ \bar{x}_{2\alpha} \end{bmatrix} = & \begin{bmatrix} \frac{\cos 2t}{2} + \frac{e^{2t} + e^{-2t}}{4} & \frac{\cos 2t}{2} - \frac{e^{2t} + e^{-2t}}{4} & \frac{-\sin 2t}{2} + \frac{e^{2t} - e^{-2t}}{4} \\ \frac{\cos 2t}{2} - \frac{e^{2t} + e^{-2t}}{4} & \frac{\cos 2t}{2} + \frac{e^{2t} + e^{-2t}}{4} & \frac{-\sin 2t}{2} - \frac{e^{2t} - e^{-2t}}{4} \\ \frac{\sin 2t}{2} + \frac{e^{2t} - e^{-2t}}{4} & \frac{\sin 2t}{2} - \frac{e^{2t} - e^{-2t}}{4} & \frac{\cos 2t}{2} + \frac{e^{2t} + e^{-2t}}{4} \\ \frac{\sin 2t}{2} - \frac{e^{2t} - e^{-2t}}{4} & \frac{\sin 2t}{2} + \frac{e^{2t} - e^{-2t}}{4} & \frac{\cos 2t}{2} - \frac{e^{2t} + e^{-2t}}{4} \\ \frac{-\sin 2t}{2} - \frac{e^{2t} - e^{-2t}}{4} & \frac{-\sin 2t}{2} + \frac{e^{2t} - e^{-2t}}{4} & \frac{\cos 2t}{2} - \frac{e^{2t} + e^{-2t}}{4} \\ \frac{-\sin 2t}{2} + \frac{e^{2t} - e^{-2t}}{4} & \frac{\cos 2t}{2} - \frac{e^{2t} + e^{-2t}}{4} & \frac{\cos 2t}{2} + \frac{e^{2t} + e^{-2t}}{4} \\ \frac{\cos 2t}{2} - \frac{e^{2t} + e^{-2t}}{4} & \frac{\cos 2t}{2} + \frac{e^{2t} + e^{-2t}}{4} & \frac{-\sin 2t}{2} - \frac{e^{2t} - e^{-2t}}{4} \\ \frac{\cos 2t}{2} + \frac{e^{2t} + e^{-2t}}{4} & \frac{-\sin 2t}{2} - \frac{e^{2t} - e^{-2t}}{4} & \frac{-\sin 2t}{2} + \frac{e^{2t} - e^{-2t}}{4} \\ \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} \\ \frac{\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} \\ \frac{-\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{-\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{-\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{-\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \end{bmatrix} \begin{bmatrix} 2\alpha + (1 - \alpha) \\ 2\alpha + 3(1 - \alpha) \\ 2\alpha + (1 - \alpha) \\ 2\alpha + 3(1 - \alpha) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} \\ \frac{\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} \\ \frac{-\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{-\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{-\sin 2(t-s)}{2} - \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{-\sin 2(t-s)}{2} + \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \\ \frac{\cos 2(t-s)}{2} - \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{\cos 2(t-s)}{2} + \frac{e^{2(t-s)} + e^{-2(t-s)}}{4} & \frac{e^{2(t-s)} - e^{-2(t-s)}}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_\alpha(s) \\ \bar{u}_\alpha(s) \end{bmatrix} ds \end{aligned}$$

Now by replacing (13) and (14) in the above equations and using the trapezoidal integration for approximating the integral we can calculate the states of system. The control function  $u(\cdot)$  and states  $x_1(\cdot)$  and  $x_2(\cdot)$ , are respectively given in Figure 2 for  $\alpha \in [0, 1]$  where the red lines are the centers and the blue lines are the lower bounds and the green lines are the upper bounds for fuzzy functions.

**Example 4.3.** Consider the system shown in Figure 3. The variables of interest are noted on the figure and defined as:  $M_1, M_2$  = mass of carts,  $\tilde{p}, \tilde{q}$  = position of carts,  $\tilde{u}$  = external force acting on system,  $k_1, k_2$  = spring constants, and  $b_1, b_2$  = damping coefficients, and  $\dot{\tilde{p}}, \dot{\tilde{q}}$  = velocity of  $M_1$  and  $M_2$ , respectively. We

assume that the carts have negligible rolling friction. We consider any existing rolling friction to be lumped into the damping coefficients,  $b_1$  and  $b_2$ . The task is to control the carts from the initial point  $\tilde{p}(t_0)$ ,  $\tilde{q}(t_0)$ ,  $\dot{\tilde{p}}(t_0)$ , and  $\dot{\tilde{q}}(t_0)$  to final target  $\tilde{p}(t_f)$ ,  $\tilde{q}(t_f)$ ,  $\dot{\tilde{p}}(t_f)$ , and  $\dot{\tilde{q}}(t_f)$ , by using external force acting  $\tilde{u}(\cdot)$ .

We use Newton's second law (sum of the forces equals mass of the object multiplied by its acceleration) to obtain the motion equations. We assume that  $\tilde{x}_1 = \tilde{p}$ ,  $\tilde{x}_2 = \tilde{q}$ ,  $\tilde{x}_3 = \dot{\tilde{p}}$  and  $\tilde{x}_4 = \dot{\tilde{q}}$ , therefore the above fuzzy controlled dynamical system can be modeled by the following fuzzy linear control system ( see [3] for more details):

$$\dot{\tilde{x}} = A \odot \tilde{x} \oplus B \odot \tilde{u}$$

where

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ \tilde{q} \\ \dot{\tilde{p}} \\ \dot{\tilde{q}} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1+k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1+b_2}{M_2} \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix}.$$

Suppose that the two rolling carts have the following parameter values:  $k_1 = 150N/m$ ,  $k_2 = 700N/m$ ,  $b_1 = 15Ns/m$ ,  $b_2 = 30Ns/m$ ,  $M_1 = 5kg$ , and  $M_2 = 20kg$ . The response of the two rolling cart system is shown in Figure 4, when the boundary conditions are  $\tilde{p}(0) = (9, 10, 11)$ ,  $\tilde{q}(0) = (-1, 0, 1)$ , and  $\dot{\tilde{p}}(0) = \dot{\tilde{q}}(0) = (-0.5, 0, 0.5)$ , and  $\tilde{p}(2) = \tilde{q}(2) = \dot{\tilde{p}}(2) = \dot{\tilde{q}}(2) = (-0.5, 0, 0.5)$ .

## 5. Conclusion

We studied fuzzy linear time invariant controlled system and fuzzy optimal controlled system, where in the fuzzy optimal control system, the target is to minimize a functional subject to fuzzy differential equation with fuzzy initial conditions. By applying  $\alpha$ -cuts and presenting numbers in more compact form by moving to the field of complex numbers, the fuzzy controlled system and fuzzy optimal controlled system, extended to a new form involve in lower and upper state and control. Three numerical examples are given to show the effectiveness of the method.

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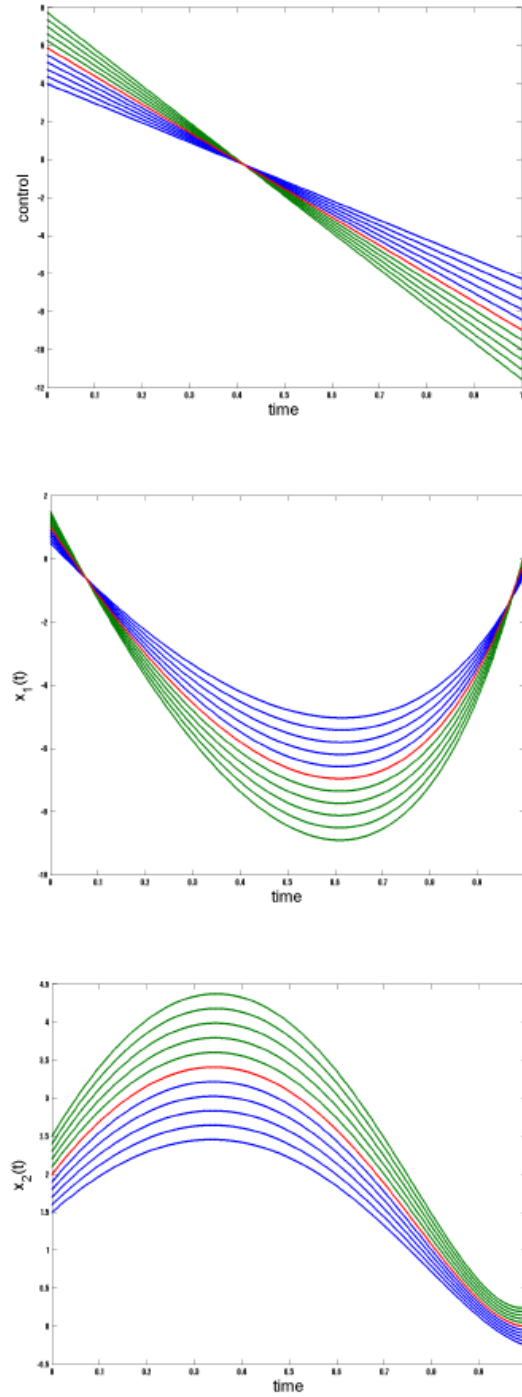


FIGURE 1. The Control and State Functions for Example 4.1

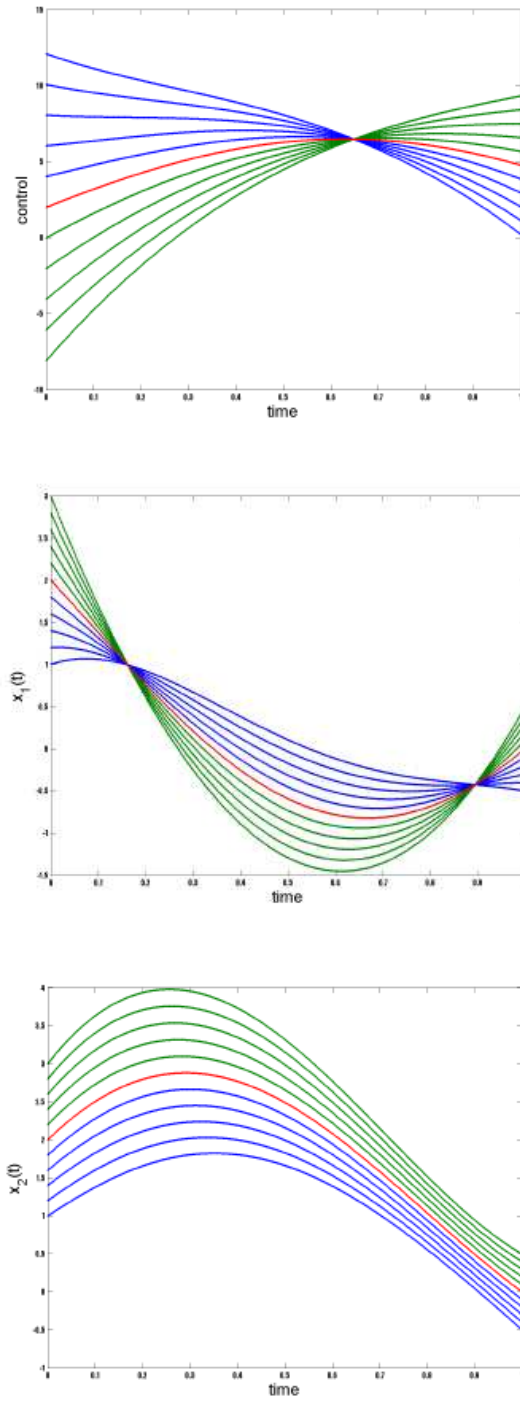


FIGURE 2. The Control and State Functions for Example 4.2

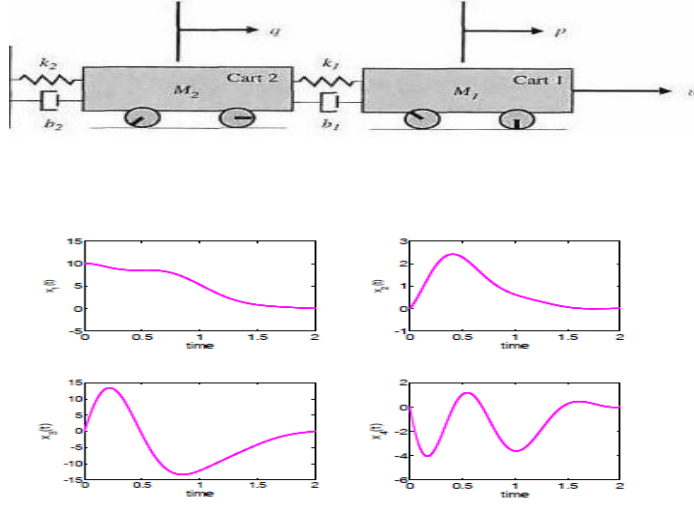


FIGURE 3. Two Rolling Carts are Attached with Springs, Dampers and the Crisp State Functions for Example 4.3

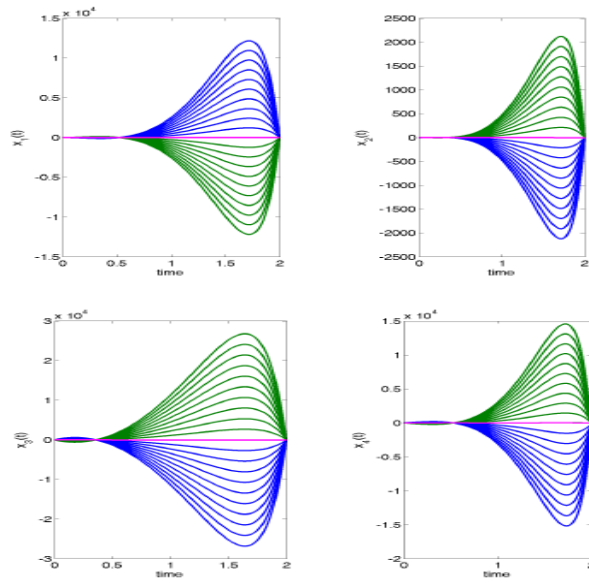


FIGURE 4. The Fuzzy State Functions for Example 4.3

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