

## EXTENSION OF FUZZY CONTRACTION MAPPINGS

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**ABSTRACT.** In a fuzzy metric space  $(X, M, *)$ , where  $*$  is a continuous  $t$ -norm, a locally fuzzy contraction mapping is defined. It is proved that any locally fuzzy contraction mapping is a global fuzzy contractive. Also, if  $f$  satisfies the locally fuzzy contractivity condition then it satisfies the global fuzzy contractivity condition.

### 1. Introduction

After Zadeh pioneering's paper [19] which introduces the theory of fuzzy sets, hundreds of examples have been supplied where the nature of uncertainty in the behavior of a given system possesses fuzzy rather than stochastic nature (see [1], [8], [10], [13] and [16]). Nonstationary fuzzy systems described by fuzzy processes look as their natural extension into the time domain. From different viewpoints, they were carefully studied.

Fuzzy contraction mappings play a very essential role in the proving of fixed point theorem in fuzzy metric spaces. Recently, many authors have introduced the concept of fuzzy metric space in different ways ([3, 4] and [9]) and have proved the fuzzy Banach's contraction theorem [6], [11]-[18]. It is natural to ask whether the theorem (referred to as fuzzy Banach's contraction principal) could be modified so as to be valid when the global fuzzy contraction is assumed to hold for sufficiently close points only. In order to do this, it is interesting to define the locally fuzzy contraction mapping and to find out that any locally fuzzy contraction mapping is a globally fuzzy contractive map. Before answering this question, we recall some concepts and results that will be required in the sequel.

**Definition 1.1.** [15] A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -norm if  $([0, 1], *)$  is a topological commutative monoid with unit 1 such that  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ , and  $a, b, c, d \in [0, 1]$ .

**Definition 1.2.** [5] The 3-tuple  $(X, M, *)$  is said to be a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times ]0, \infty[$  satisfying the following conditions:

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for all  $x, y, z \in X$  and  $t, s > 0$ ,

- (I)  $M(x, y, t) > 0$ ,
- (II)  $M(x, y, t) = 1$  iff  $x = y$ ,
- (III)  $M(x, y, t) = M(y, x, t)$ ,
- (IV)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
- (V)  $M(x, y, \cdot) : ]0, \infty[ \rightarrow ]0, 1]$  is continuous.

If  $(X, M, *)$  is a fuzzy metric space, we will say that  $(M, *)$  is a fuzzy metric on  $X$ .

**Lemma 1.3.** [6]  $M(x, y, *)$  is nondecreasing for all  $x, y \in X$ .

In order to introduce a Hausdorff topology on the fuzzy metric space, in [5], the authors gave the following definitions.

**Definition 1.4.** [5] Let  $(X, M, *)$  be a fuzzy metric space. The open ball  $B(x, r, t)$  for  $t > 0$  with center  $x \in X$  and radius  $r, 0 < r < 1$ , is defined as  $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ . The topology generated by the system  $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$  is called the topology induced by the fuzzy metric  $M$ .

**Example 1.5.** [5] In a metric space  $(X, d)$  the 3-tuple  $(X, M_d, *)$  where  $M_d(x, y, t) = t/t + d(x, y)$  and  $a * b = ab$ , is a fuzzy metric space.

**Definition 1.6.** The above  $M_d$  is called the standard fuzzy metric induced by  $d$ .

The topologies induced by the standard fuzzy metric and the corresponding metric are the same.

**Theorem 1.7.** [5] A sequence  $(x_n)$  in a fuzzy metric space  $(X, M, *)$  converges to  $x$  if and only if  $M(x_n, x, t) \rightarrow 1$  as  $n \rightarrow \infty$ , for all  $t > 0$ .

**Definition 1.8.** [5] A sequence  $(x_n)$  in a fuzzy metric space  $(X, M, *)$  is a Cauchy sequence iff for each  $\epsilon \in (0, 1)$  and each  $t > 0$ , there exists  $n_0 \in \mathcal{N}$  such that  $M(x_n, x_m, t) > 1 - \epsilon$  for all  $n, m \geq n_0$ .

A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

**Definition 1.9.** [7] Let  $(X, M, *)$  be a fuzzy metric space we say that the mapping  $f : X \rightarrow X$  is fuzzy contractive if there exists  $k \in (0, 1)$  such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right),$$

for each  $x, y \in X$  and  $t > 0$ , ( $k$  is called the fuzzy contractive constant of  $f$ ).

## 2. Extension of Fuzzy Contractive Map

In this section, definitions of locally fuzzy contractive map and locally fuzzy contractive condition are given. It is shown that (under some assumption on the fuzzy metric space) any locally fuzzy contractive map is fuzzy contractive and that the locally fuzzy contractive condition implies the fuzzy contractive condition. In order to do this, if we recall Definition 1.9 as a global fuzzy contractive map, then the locally fuzzy contractive map is as follows:

**Definition 2.1.** Let  $(X, M, *)$  be a fuzzy metric space, we will say the mapping  $f : X \rightarrow X$  is locally fuzzy contractive if for every  $x \in X$  there exists  $0 < \epsilon < 1, k \in (0, 1)$ , which may depend on  $x$ , such that if

$$p, q \in B(x, \epsilon, t) = \{y | M(x, y, t) > 1 - \epsilon\},$$

then

$$\frac{1}{M(f(p), f(q), t)} - 1 \leq k \left( \frac{1}{M(p, q, t)} - 1 \right),$$

for all  $t > 0$ , for all  $p$  and for all  $q$ , ( $k$  is called the fuzzy contractive constant of  $f$ ).  $f$  is said to be  $(\epsilon, k)$ -uniformly locally fuzzy contractive, if it is locally fuzzy contractive and both  $\epsilon$  and  $k$  do not depend on  $x$ .

**Remark 2.2.** A globally fuzzy contractive mapping can be regarded as a  $(1, k)$ -uniformly locally fuzzy contractive mapping.

Now we define the concept of fuzzy metrically convex space.

**Definition 2.3.** A fuzzy metric space  $(X, M, *)$  is said to be metrically convex if for each  $x, y \in X$ , there is a  $z \neq x, y$  for which  $M(x, y, t) = M(x, z, t_0) * M(z, y, t_1)$ , where  $t = t_0 + t_1$  for all  $t_0$  and  $t_1$  in  $R^+$ .

**Theorem 2.4.** If  $(X, M, *)$  is a metrically convex, fuzzy metric space, then every mapping  $f$  from  $X$  into itself which is  $(\epsilon, 1)$ -uniformly fuzzy contractive is also globally fuzzy contractive.

*Proof.* Due to the metrically convexity of the space, we see that if  $x, y \in X$  then there are points  $x = x_0, x_1, \dots, x_n = y$  and  $t_0, t_1, \dots, t_{n-1}$  such that

$$M(x, y, t) = M(x_0, x_1, t_0) * M(x_1, x_2, t_1) * \dots * M(x_{n-1}, x_n, t_{n-1}),$$

where  $t = t_0 + t_1 + \dots + t_{n-1}$  and  $M(x_{i-1}, x_i, t_{i-1}) > 1 - \epsilon$  for  $i = 1, \dots, n - 1$ . Also,

$$M(f(x), f(y), t) \geq M(f(x_0), f(x_1), t_0) * M(f(x_1), f(x_2), t_1) * \dots * M(f(x_{n-1}), f(x_n), t_{n-1}). \quad (1)$$

Also  $f$  is  $(\epsilon, 1)$ -uniformly fuzzy contractive, this means that for  $i = 1, \dots, n - 1$ ,

$$M(x_{i-1}, x_i, t_{i-1}) > 1 - \epsilon \text{ implies } M(f(x_{i-1}), f(x_i), t_{i-1}) \geq M(x_{i-1}, x_i, t_{i-1}).$$

Thus

$$\begin{aligned} & M(f(x_0), f(x_1), t_0) * M(f(x_1), f(x_2), t_1) * \dots * M(f(x_{n-1}), f(x_n), t_{n-1}) \\ & \geq M(x_0, x_1, t_0) * M(x_1, x_2, t_1) * \dots * M(x_{n-1}, x_n, t_{n-1}) \\ & = M(x, y, t). \end{aligned} \quad (2)$$

Finally, (1) and (2) will imply that

$$M(f(x), f(y), t) \geq M(x, y, t).$$

□

**Example 2.5.** Let  $(M, d)$  be metrically convex metric space, the 3-tuple  $(X, M, *)$  where  $M(x, y, t) = e^{-d(x,y)}$  and  $a * b = ab$  is a fuzzy metric space. Note that

$$\begin{aligned} M(x, y, t) * M(y, z, s) &= e^{-d(x,y)} \cdot e^{-d(y,z)} \\ &= e^{-(d(x,y)+d(y,z))} \\ &= e^{-d(x,z)} = M(x, z, t + s) \end{aligned}$$

Then  $(X, M, *)$  is metrically convex fuzzy metric space. Note that if  $f : X \rightarrow X$  is a nonexpansive mapping respect to  $d$ ,  $f$  is  $(\varepsilon, 1)$ -uniformly fuzzy contraction and by Theorem 2.4,  $f$  is globally fuzzy contraction.

**Corollary 2.6.** *If  $(X, M, *)$  is a convex, complete fuzzy metric space, then every mapping  $f$  from  $X$  into itself which is  $(\epsilon, k)$ -uniformly fuzzy contractive is also globally fuzzy contractive with the same  $k$ .*

*Proof.* If  $f$  is  $(\epsilon, k)$ -uniformly fuzzy contractive then it is  $(\epsilon, 1)$ -uniformly fuzzy contractive, and Theorem 2.4 ends the proof.  $\square$

**Definition 2.7.** Let  $(X, M, *)$  be a fuzzy metric space, we say that the mapping  $f : X \rightarrow X$  satisfies the global fuzzy contractivity condition if there exists  $\alpha \in (0, 1)$ , such that

$$M(f(p), f(q), \alpha t) \geq M(p, q, t),$$

for each  $p, q \in X$  and  $t > 0$ .

**Definition 2.8.** Let  $(X, M, *)$  be a fuzzy metric space, we say that the mapping  $f : X \rightarrow X$  satisfies the locally fuzzy contractivity condition if for every  $x \in X$  there exists  $0 < \epsilon < 1, \alpha \in (0, 1)$ , which may depend on  $x$ , such that if

$$p, q \in B(x, \epsilon, t) = \{y | M(x, y, t) > 1 - \epsilon\},$$

then

$$M(f(p), f(q), \alpha t) \geq M(p, q, t),$$

for  $t > 0$ .

**Definition 2.9.** Let  $(X, M, *)$  be a fuzzy metric space, a mapping  $f$  of  $X$  into itself is said to satisfy  $(\epsilon, k)$ -uniformly locally fuzzy contractivity condition if it satisfies the local fuzzy contractivity condition and both  $\epsilon$  and  $k$  do not depend on  $x$ .

**Theorem 2.10.** *Let  $(X, M, *)$  be a convex and fuzzy metric space and  $f$  be a map from  $X$  into  $X$ . If  $f$  satisfies  $(\epsilon, \alpha)$ -uniformly locally fuzzy contractivity condition, then it satisfies the global fuzzy contractivity condition.*

*Proof.* Due to the convexity of the fuzzy metric space, if  $x, y \in X$  then there are points  $x = x_0, x_1, \dots, x_n = y$  and  $t_0, t_1, \dots, t_{n-1}$  such that

$$M(x, y, t) = M(x_0, x_1, t_0) * M(x_1, x_2, t_1) * \dots * M(x_{n-1}, x_n, t_{n-1}),$$

where  $t = t_0 + t_1 + \dots + t_{n-1}$  and  $M(x_{i-1}, x_i, t_{i-1}) > 1 - \epsilon$  for  $i = 1, \dots, n - 1$ . Also,

$$\begin{aligned} M(f(x), f(y), \alpha t) &\geq M(f(x_0), f(x_1), \alpha t_0) * M(f(x_1), f(x_2), \alpha t_1) \\ &\quad * \dots * M(f(x_{n-1}), f(x_n), \alpha t_{n-1}). \end{aligned} \quad (3)$$

Also  $f$  satisfies the locally fuzzy contractivity condition, thus for  $i = 1, \dots, n-1$ ,

$$M(x_{i-1}, x_i, t_{i-1}) > 1 - \epsilon \text{ implies } M(f(x_{i-1}), f(x_i), \alpha t_{i-1}) \geq M(x_{i-1}, x_i, t_{i-1}).$$

Thus

$$\begin{aligned} & M(f(x_0), f(x_1), \alpha t_0) * M(f(x_1), f(x_2), \alpha t_1) * \dots * M(f(x_{n-1}), f(x_n), \alpha t_{n-1}) \\ & \geq M(x_0, x_1, t_0) * M(x_1, x_2, t_1) * \dots * M(x_{n-1}, x_n, t_{n-1}) \\ & = M(x, y, t). \end{aligned} \tag{4}$$

Finally, (3) and (4) will imply that

$$M(f(x), f(y), \alpha t) \geq M(x, y, t).$$

□

### 3. Conclusion

With respect to Theorem 2.4, one can consider every selfmapping which is  $(\epsilon, 1)$ -uniformly locally fuzzy contractive as globally fuzzy contractive (of course, in metrically convex spaces). This is very important, because one can move from local property to global property. Moreover, Theorem 2.10 generalize this idea for  $(\epsilon, \alpha)$ -uniformly locally fuzzy contractivity condition.

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