ALGEBRAICALLY-TOPOLOGICAL SYSTEMS AND ATTACHMENTS

A. FRASCELLA, C. GUIDO AND S. A. SOLOVYOV

Abstract. The paper continues the study of the authors on relationships between *topological systems* of S. Vickers and *attachments* of C. Guido. We extend topological systems to *algebraically-topological systems*. A particular instance of the latter, called *attachment system*, incorporates the notion of attachment, thus, making it categorically redundant in mathematics. We show that attachment systems are equipped with an internal topology, which is similar to the topology induced by locales. In particular, we provide an attachment system analogue of the well-known categorical equivalence between sober topological spaces and spatial locales.

1. Introduction

The paper makes another step towards developing a fruitful theory of the new topological setting, introduced by S. Solovyov in [55] under the name *categorically-algebraic (catalg) topology*. The new framework was motivated by the huge diversification of the current approaches to (lattice-valued) topology and the lack of procedures or methods to translate results in one setting to similar or equivalent results in another setting. It appears that catalg topology successfully copes with both deficiencies, incorporating the most important (lattice-valued) topological settings and providing easy ways of moving from one setting to another [56]. Moreover, the "quasi-pointless" framework of *topological systems* of S. Vickers [67], introduced as a common framework for both topological spaces and their underlying algebraic structures – locales, also has its place in the theory [64, 66]. The catalg viewpoint on topological systems opens a new horizon for the concept, whose initial theory is essentially restricted to [67]. In particular, it is possible to fully embed a large number of categories of topological structures into suitably related categories of systems. The latter categories appear to be (essentially) algebraic over their grounds, and that distinguishes them from other categories for topology, which usually are topological over their grounds. This raises an interesting meta-mathematical point of doing (classical) topology inside algebra, that differs from the well-known algebraic approach of "purely" pointless topology of P. T. Johnstone [33], based in

Received: September 2011; Revised: March 2012; Accepted: May 2012

Key words and phrases: Algebraically-topological system, Attachment system, Categorically-algebraic topology, Dual attachment pair, Localic algebra, Localification of systems, (Variety-based) pointless topology, Spatialization of systems, Topological theory morphism, Variety.

This research was partially supported by the ESF Project of the University of Latvia No. 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008.

a particular functor **Top** → **Loc**, which is not a full embedding. Several aspects of this new algebraic setting for doing (lattice-valued) topology have already been touched in the fuzzy literature [12, 54].

Recently, the theory of systems got an additional impetus in the concept of *attachment*, introduced by C. Guido in [23] to extend the standard membership relation "$\in$" between points and sets, and motivated by P.-M. Pu, Y.-M. Liu [43] and their notion of *quasi-coincidence* between a fuzzy point and a fuzzy set. Every (*spatial*) attachment in a complete lattice L induces a particular functor (embedding) L-**Top** → **Top**, opening a possibility of doing lattice-valued topology inside classical topology. Exploration of the topic has already begun in [24], providing an additional justification for several lattice-valued topological concepts. Such a justification is important for the better development of many-valued topology, whose concepts, based on the *Principle of Fuzzification* of J. A. Goguen [21], often rely on the intuition of their authors. A good example in this respect is the notion of lattice-valued topology itself, which could be done either in the classical setting of C. L. Chang [7], or in the *stratified* setting of R. Lowen [39]. The fuzzy literature still abounds with discussions on the necessity of each of the above-mentioned approaches (see, e.g., [50]).

In [23], C. Guido noticed that the above (embedding) functor L-**Top** → **Top** factors through the *spatialization procedure functor* for topological systems of S. Vickers, which makes a topological space from a topological system. Employing the methods of catalg topology, S. Solovyov explored this factorization in full detail in [61], showing that both the functor of C. Guido and the functor of S. Vickers give rise to the *hypergraph functor* of lattice-valued topology [27]. The study motivated the question on relationships between topological systems and attachments. In particular, some fuzzy researchers tried to find a common framework for both concepts. A proposed initial solution was provided by the authors of this paper in [19], which claimed non-existence of a unifying setting due to the following crucial difference in the nature of the concepts: a topological system contains an internal topology, easily accessed through the above-mentioned spatialization procedure, whereas an attachment provides a morphism of topological theories, resulting in a functor between the categories of the respective topological structures. It is the main purpose of this paper to refute the claim and show that the notion of topological system is more general than that of attachment and, suitably extended, is capable of its incorporation. In this manuscript, the extension is called *(catalg) algebraically-topological systems*. We should underline, however, immediately that later on, A. Frascella alone took a different standpoint in [18], suggesting the opposite (the concept of attachment extends that of topological system), but failing to provide enough evidence for her claim. To address the claim of A. Frascella, in [25], C. Guido and S. Solovyov provided an additional argument in favor of the extension of attachments through suitably generalized topological systems.

The crucial change, introduced in this manuscript to the system framework, adds more algebra to the already mentioned catalg setting, and this is reflected in the name of the new concept. Briefly speaking, a topological system gives rise to a topological space, whereas an algebraically-topological one provides a topological algebra in the sense of, e.g., *topological group* of L. Pontrjagin [42]. The advantage of the new system approach of this manuscript is (at least) threefold. Firstly, the new framework provides a common approach to both topological algebras and the algebraic structures underlying their topologies (the case of the algebraic structures underlying the spaces themselves, e.g., groups in the above-mentioned setting of L. Pontrjagin, is currently under consideration). In particular, the framework extends the most important aspects of topological systems, i.e., both the spatialization (a space from a system) and the localification (an algebra from a system) procedures, with their respective (co-)adjoint functors in the opposite direction. These developments provide a way to switch freely between topological and algebraic viewpoints (cf., e.g., the theory of *natural dualities* of [8], which translates algebraic problems into dual, topological problems and vice versa). Secondly, the new system framework induces a multitude of morphisms of related topological theories, resulting in functors, which extend that of C. Guido and the above-mentioned hypergraph functor. Thirdly, the objects of particular subcategories, called *attachment systems*, of the category of algebraically-topological systems, include both attachments of [61] and dual attachments of [19] (thus, incorporating the theory of *dual attachment pairs*, started in [19] to provide a suitable duality machinery for the attachment framework), thereby suggesting categorical redundance of the concept of attachment in mathematics. This point of categorical redundancy simplifies the available fuzzy research tools, helping to get rid of numerous duplications (a good example here is the paper of J. Gutiérrez García and S. E. Rodabaugh [26], which successfully demonstrates categorical redundancy of interval-valued sets, interval-valued "intuitionistic" sets, and "intuitionistic" fuzzy sets and fuzzy topologies).

As an additional result, we obtain that attachment systems do have an internal topology and the respective spatialization procedure. The fact, simple as it may seem, was never mentioned by the authors of this manuscript in their earlier papers on the topic. Its consequences are far-reaching, e.g., having the well-known functor $\mathbf{Loc} \to \mathbf{Top}$ [33] in hand, one can define its attachment system analogue, which, in the simplest case of [23] (attachment systems based in frames), sheds light on relationships between pointless topology of P. T. Johnstone (going back to *locales* of J. Isbell [31], originating from *local lattices* of C. Ehresmann [17] and J. Bénabou [6] as well as *frames* of C. H. Dowker and D. Papert (Strauss) [14, 15, 16]) and attachment system topology of this paper. For example, at the end of the manuscript, we show an attachment system counterpart of the famous equivalence between the categories of *spatial locales* and *sober topological spaces* [33]. Taken together, the obtained results suggest calling the attachment system approach *variety-based pointless topology*, which provides another viewpoint on lattice-valued pointless topology started by B. Hutton [29, 30]. It will be the topic of our next papers to investigate this topology in full detail.

The manuscript is based on both category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [2, 40, 41]. For algebraic notions we recommend [9, 41]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e.g., with adjoint situations.

2. Categorically-algebraic Topology and Its Related Notions

2.1. **Algebraic and Categorical Preliminaries.** We begin with recalling those algebraic and categorical preliminaries, which are helpful in understanding this paper. An experienced reader can easily skip the matter, consulting the section for the subsequent notations of the authors only.

Part of the foundation of our approach is the notion of *algebra*, which is to be thought of as a set with a family of operations defined on it, satisfying certain identities. The theory of universal algebra of, e.g., [9] calls a class of finitary algebras (induced by a set of finitary operations) closed under the formation of homomorphic images, subalgebras and direct products a *variety*. Motivated by the algebraic structures of lattice-valued topology (where set-theoretic unions are replaced by arbitrary joins), we consider infinitary algebraic theories, extending the approach of varieties to cover our needs.

Definition 2.1. Let $\boldsymbol{\Omega} = (n_\lambda)_{\lambda\in\Lambda}$ be a (possibly proper) class of cardinal numbers. An $\boldsymbol{\Omega}$*-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ*-ary primitive operations* on A). An $\boldsymbol{\Omega}$*-homomorphism* $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that the diagram

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{\varphi^{n_\lambda}} & B^{n_\lambda} \\ {\scriptstyle\omega_\lambda^A}\downarrow & & \downarrow{\scriptstyle\omega_\lambda^B} \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

commutes for every $\lambda \in \Lambda$. $\mathbf{Alg}(\boldsymbol{\Omega})$ stands for the construct of $\boldsymbol{\Omega}$-algebras and $\boldsymbol{\Omega}$-homomorphisms.

For every concrete category in this paper, there is an underlying functor $|-|$ to the respective ground category, which is mentioned explicitly in each case.

Definition 2.2. Let $\mathcal{M}$ and $\mathcal{E}$ be the classes of $\boldsymbol{\Omega}$-homomorphisms with injective and surjective underlying maps, respectively. A *variety of* $\boldsymbol{\Omega}$*-algebras* is a full subcategory of $\mathbf{Alg}(\boldsymbol{\Omega})$ closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients. The objects of a variety are called *algebras*, whereas its morphisms are called *homomorphisms*.

Definition 2.3. Given a variety $\mathbf{A}$, a *reduct* of $\mathbf{A}$ is a pair $(\|-\|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\boldsymbol{\Omega}_{\mathbf{B}} \subseteq \boldsymbol{\Omega}_{\mathbf{A}}$ and $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ is a concrete functor.

The following constructs are examples of varieties: $\mathbf{CSLat}(\bigvee)$ of $\bigvee$*-semilattices* (partially ordered sets having arbitrary joins or $\bigvee$), **SQuant** of *semi-quantales* ($\bigvee$-semilattices, equipped with a binary operation $\otimes$ called *multiplication*), **Quant** of *quantales* (semi-quantales, whose multiplication is associative and distributes across $\bigvee$ from both sides), **UQuant** of *unital quantales* (quantales, whose multiplication has the unit 1), **Frm** of *frames* (unital quantales, whose multiplication coincides with the meet operation), **DmFrm** of *DeMorgan frames* (frames, equipped with an order-reversing involution), **CBAlg** of *complete Boolean algebras* (DeMorgan frames, whose involution provides the complement) [33, 36, 48, 51, 52]. Taken in the reverse order, the categories provide a sequence of reducts. Additionally, another important reduct of **Frm** is the variety **SFrm** (also denoted **CSLF** [44]) of *semi-frames* (unital semi-quantales, whose multiplication coincides with the meet operation) [48, 51].

From now on, varieties are denoted **A**, **B**, **E**, with **C** reserved for their subcategories, and subcategories of their dual categories. Following the notations of the fuzzy community (differing from the category-theoretic ones), the categorical dual of a variety **A** is denoted **LoA**, whose objects are called *localic algebras*, and whose morphisms are called *localic homomorphisms*. For the dual of **Frm**, we will use the already accepted notation **Loc** [5, 33] (adding "**S**" in front in case of semi-frames). Given a homomorphism φ of a variety **A**, the corresponding localic one is denoted φ^{op} and vice versa. Every algebra A of a variety **A** provides the subcategory $\mathbf{S}_A$ of **LoA**, whose only morphism is the identity $A \xrightarrow{1_A} A$.

Given an **A**-algebra A, every set X gives rise to the *powerset* **A**-*algebra* A^X, whose elements are maps $X \xrightarrow{\alpha} A$, and whose algebraic structure is defined point-wise, involving the structure of A. For an element $a \in A$, $\underline{a}$ denotes the constant map $X \to A$ with the value a.

2.2. **Categorically-algebraic Topology.** The required preliminaries in hand, this section serves as a short introduction into the theory of *categorically-algebraic (catalg) topology* (for a thorough discussion see [58]). At the bottom of the catalg approach lies a particular generalization of several aspects of the standard topological framework, developed from the category-theoretic standpoint. Taking the case of classical topological spaces as an example, there are three main cornerstones in their theory (to provide more intuition for the forthcoming concepts, in the following, we do not distinguish between sets and theirs respective characteristic maps).

(1) The starting *backward powerset operator*, which can be represented as a functor $\mathbf{Set} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoCBAlg}$ from the category **Set** of sets and maps to the dual category of the variety **CBAlg** of complete Boolean algebras, which assigns to every set X its powerset $\mathbf{2}^X$ ($\mathbf{2}$ is the two-element complete Boolean algebra), and to every map $X \xrightarrow{f} Y$ its respective powerset extension $\mathbf{2}^Y \xrightarrow{f^{\leftarrow}} \mathbf{2}^X$, which in its turn is defined by the formula $f^{\leftarrow}(\alpha) = \alpha \circ f$ (recall our notations with respect to powerset algebras).
(2) The induced *topological theory*, which describes the underlying algebraic structure of the topology of topological spaces, and which essentially is the composition of the backward powerset operator and the dual of the forgetful functor $\mathbf{CBAlg} \xrightarrow{\|-\|} \mathbf{Frm}$ to the variety of frames.
(3) The resulting category **Top** of *topological spaces* and *continuous maps*, the objects of which are pairs (X, τ), where X is a set and τ (*topology*) is a subframe of $\|\mathbf{2}^X\|$; and whose morphisms $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are maps $X \xrightarrow{f} Y$ such that $f^{\leftarrow}(\alpha) \in \tau$ for every $\alpha \in \sigma$ (*continuity*).

After a small deliberation, the reader will see that the classical many-valued topology of, e.g., [28, 46] introduces just three changes in the above-mentioned topological setting. Firstly, instead of the variety **CBAlg** of complete Boolean algebras, it takes another variety **A** (e.g., the variety **DmUQuant** of DeMorgan unital quantales). Secondly, instead of the two-element Boolean algebra **2**, it takes either a fixed **A**-algebra A (fixed-basis approach of [28]), or a subcategory **C** of **LoA** (variable-basis approach of [46]). Thirdly, instead of the variety **Frm** of frames, it takes another variety **B** (e.g., the variety **USQuant** of unital semi-quantales). Different approaches to many-valued topology stem from the difference in the choice of the variety **A**, the subcategory **C** of **LoA** and the reduct **B** of **A**. More sophisticated settings for many-valued topology like, e.g., topological theories of S. E. Rodabaugh [48], start from some category **X** instead of the category **Set**. In order to provide a common framework for every such setting, we decided to introduce catalg topology, which follows the above three steps, respecting, however, possible changes in their structure.

To begin with, the reader should recall poslat powerset theories of S. E. Rodabaugh [48, 51] and topological theories of J. Adámek *et al.* [2], the extension of which gave the starting point for the new setting.

For the first step, we notice (partially reiterating the already said) that every set map $X \xrightarrow{f} Y$ provides two operators, namely, *image operator* $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, $f^{\rightarrow}(S) = \{f(x) \,|\, x \in S\}$ and *preimage operator* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, $f^{\leftarrow}(T) = \{x \,|\, f(x) \in T\}$. The latter one can be extended to a more general setting as follows (a generalization of the former is described explicitly in [65]; see also [69]).

Definition 2.4. A *catalg backward powerset theory* (*cabp-theory*) for a variety **A** in a category **X** (*ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$.

The following example illustrates the concept, extending the standard fixed- and variable-basis approaches of lattice-valued topology [28, 46] (recall that **Set** is the category of sets and maps).

Example 2.5. Given a variety **A**, every subcategory **C** of **LoA** induces a functor $\mathbf{Set}\times\mathbf{C} \xrightarrow{\mathcal{S}=(-)^{\leftarrow}} \mathbf{LoA}$, $((X,A) \xrightarrow{(f,\varphi)} (Y,B))^{\leftarrow} = A^X \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} B^Y$, $(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$. The case $\mathbf{C} = \mathbf{S}_A$ is denoted $\mathcal{S}_A = (-)^{\leftarrow}_A$ and is called a *fixed-basis approach*; all other cases are referred to as a *variable-basis approach*. The functor $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}_{\mathbf{2}}} \mathbf{LoCBAlg}$ ($\mathbf{2} = \{\bot, \top\}$) gives the above-mentioned preimage operator, whereas the functors $\mathbf{Set} \times \mathbf{S}_{\mathbb{I}} \xrightarrow{\mathcal{Z}=(-)^{\leftarrow}_{\mathbb{I}}} \mathbf{DmLoc}$ ($\mathbb{I}$ is the unit interval $[0,1]$), $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}=(-)^{\leftarrow}_L} \mathbf{LoUQuant}$ and $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}_1=(-)^{\leftarrow}} \mathbf{SQuant}$, $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}_2=(-)^{\leftarrow}} \mathbf{SLoc}$, $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}_3=(-)^{\leftarrow}} \mathbf{Loc}$ give the operators of L. A. Zadeh [68], J. A. Goguen [22] and S. E. Rodabaugh [48], [44], [13], respectively.

The next step introduces topological theories, based on powerset theories, which ultimately give rise to catalg topological structures (recall the notion of *product of categories* from [2]).

Definition 2.6. Let $\mathbf{X}$ be a category and let $\mathcal{T}_I = ((\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i, (\| - \|_i, \mathbf{B}_i)))_{i \in I}$ be a set-indexed family such that for every $i \in I$, $\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i$ is a cabp-theory in $\mathbf{X}$ and $(\| - \|_i, \mathbf{B}_i)$ is a reduct of $\mathbf{A}_i$. A *composite catalg topological theory* (*ccat-theory*) *in* $\mathbf{X}$ *induced by* $\mathcal{T}_I$ is the functor $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoB}_i$, given by the equality $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoB}_i \xrightarrow{\Gamma_j} \mathbf{LoB}_j = \mathbf{X} \xrightarrow{P_j} \mathbf{LoA}_j \xrightarrow{\| - \|_j^{op}} \mathbf{LoB}_j$ for every $j \in I$, where Γ_j is the jth projection functor. A ccat-theory induced by a singleton family is denoted T.

The reader should notice that the concept of topological theory in the meaning of Definition 2.6 is not used in the fuzzy community. Indeed, following the idea of S. E. Rodabaugh (see, e.g., [48]), the current studies on categorical fuzzy topology pass directly from powerset theories to their respective categories of topological structures. Studying various examples though, we saw a clear need for one more level of abstraction. More precisely, we observed that the algebraic structure employed by powerset theories is usually richer than the one used in the definition of topology. For instance, given a set X, the powerset $\mathcal{P}(X)$ of X is a complete Boolean algebra, whereas a topology τ on X is just a frame, which, however, occasionally employs certain algebraic operations outside its scope, e.g., set complementation. The case of closure spaces of, e.g., [3, 4] provides another good example (recall that a *closure space* is a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a family of subsets of X, which is closed under arbitrary intersections and the empty union). These two (and many more, if needed) examples make us "temporarily forget" a part of the unused algebraic structure, as a consequence – the concept of topological theory and its employed reduct.

Definition 2.7. Let T_I be a ccat-theory in a category $\mathbf{X}$. $\mathbf{CTop}(T_I)$ is the concrete category over $\mathbf{X}$, whose objects (*composite catalg topological spaces* or T_I*-spaces*) are pairs $(X, (\tau_i)_{i \in I})$, where X is an $\mathbf{X}$-object and τ_i is a subalgebra of $T_i(X)$ for every $i \in I$ ($(\tau_i)_{i \in I}$ is called a *composite catalg topology* or T_I*-topology* on X), and whose morphisms $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ for every $i \in I$ (*composite catalg continuity* or T_I*-continuity*). The category $\mathbf{CTop}(T)$ is denoted $\mathbf{Top}(T)$.

The following example illustrates Definition 2.7, justifying the fruitfulness of the proposed catalg approach.

Example 2.8. The case of the ground category $\mathbf{X} = \mathbf{Set} \times \mathbf{C}$ is called a *variety-based topology*. In particular, the category $\mathbf{Top}((\mathcal{S}_Q, \mathbf{B}))$ (recall cabp-theories of Example 2.5) is denoted $Q_{\mathbf{B}}$-$\mathbf{Top}$, providing the framework for *fixed-basis variety-based topology*, whereas the category $\mathbf{Top}((\mathcal{S}, \mathbf{B}))$ is denoted $(\mathbf{C}, \mathbf{B})$-$\mathbf{Top}$ (the case $\mathbf{A} = \mathbf{B}$ is shortened to $\mathbf{C}$-$\mathbf{Top}$), which is the framework for *variable-basis variety-based topology*. More specific, $\mathbf{Top}((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the category $\mathbf{Top}$ of topological spaces, whereas $\mathbf{CTop}(((\mathcal{P}, \mathbf{Frm}))_{i \in \{1,2\}})$ is isomorphic to the category $\mathbf{BiTop}$ of bitopological spaces and bicontinous maps of J. C. Kelly [35]. Moreover, $\mathbf{Top}((\mathcal{Z}, \mathbf{Frm}))$ is isomorphic to the category $\mathbb{I}$-$\mathbf{Top}$ of fixed-basis fuzzy topological spaces of C. L. Chang [7], $\mathbf{Top}((\mathcal{G}, \mathbf{UQuant}))$ is isomorphic to the category L-**Top** of fixed-basis L-fuzzy topological spaces of J. A. Goguen [22], whereas $\mathbf{Top}((\mathcal{R}_1, \mathbf{USQuant}))$, $\mathbf{Top}((\mathcal{R}_2, \mathbf{SFrm}))$, $\mathbf{Top}((\mathcal{R}_3, \mathbf{Frm}))$ are isomorphic to the categories $\mathbf{C\text{-}Top}_i$, $i \in \{1, 2, 3\}$ for variable-basis poslat topology of S. E. Rodabaugh [48], [44], [13].

Example 2.8 backs our claim from the introductory section that the catalg approach provides a common framework for many (lattice-valued) topological settings (notice, however, that we always rely on the classical definition of topology on a given set, i.e., as a certain family of its subsets, leaving other approaches, e.g., through *convergence structures* of G. Jäger [32], temporarily out; moreover, at the moment, we do not touch any parallel development like, e.g., the theory of *lattice-valued bornology* of M. Abel and A. Šostak [1]). The next proposition shows that it also opens convenient means of interaction between different topological settings.

Proposition 2.9. *Let* $\mathbf{X}_i \xrightarrow{T_i} \mathbf{LoB}_i$ *for* $i \in \{1, 2\}$ *be two topological theories, let* $\mathbf{X}_1 \xrightarrow{F} \mathbf{X}_2$ *and* $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ *be functors, and let* $T_2F \xrightarrow{\eta} \Phi^{op}T_1$ *be a natural transformation. There exists a functor* $\mathbf{Top}(T_1) \xrightarrow{H_\eta} \mathbf{Top}(T_2)$ *defined by* $H_\eta((X, \tau) \xrightarrow{f} (Y, \sigma)) = (FX, (\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow}(\Phi\tau)) \xrightarrow{Ff} (FY, (\eta_Y^{op} \circ \Phi e_\sigma)^{\rightarrow}(\Phi\sigma))$*, where* $\tau \xhookrightarrow{e_\tau} T_1X$ *and* $\sigma \xhookrightarrow{e_\sigma} T_1Y$ *are the respective embeddings.*

Proof. We only have to verify continuity of Ff, which can be shown as follows:

$$\begin{aligned}
&(((T_2Ff)^{op})^{\rightarrow} \circ (\eta_Y^{op} \circ \Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&(((T_2Ff)^{op})^{\rightarrow} \circ (\eta_Y^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&(((\eta_Y \circ T_2Ff)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) \overset{(\dagger)}{=} \\
&(((\Phi^{op}T_1f \circ \eta_X)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op})^{\rightarrow} \circ ((\Phi^{op}T_1f)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op})^{\rightarrow} \circ (\Phi(T_1f)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op})^{\rightarrow} \circ (\Phi(T_1f)^{op} \circ \Phi e_\sigma)^{\rightarrow})(\Phi\sigma) \overset{(\dagger\dagger)}{=} \\
&((\eta_X^{op})^{\rightarrow} \circ (\Phi e_\tau \circ \Phi\overline{(T_1f)^{op}})^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow} \circ (\Phi\overline{(T_1f)^{op}})^{\rightarrow})(\Phi\sigma) \overset{(\dagger\dagger\dagger)}{\subseteq} \\
&(\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow}(\Phi\tau),
\end{aligned}$$

where (†) uses commutativity of the diagram

$$\begin{array}{ccc}
T_2FX & \xrightarrow{\eta_X} & \Phi^{op}T_1X \\
{\scriptstyle T_2Ff}\downarrow & & \downarrow{\scriptstyle \Phi^{op}T_1f} \\
T_2FY & \xrightarrow[\eta_Y]{} & \Phi^{op}T_1Y,
\end{array}$$

(††) relies on the application of the functor Φ to the commutative diagram

$$\begin{array}{ccc} \sigma & \xrightarrow{\overline{(T_1f)^{op}}=(T_1f)^{op}|_{\sigma}^{\tau}} & \tau \\ {\scriptstyle e_\sigma}\big\downarrow & & \big\downarrow{\scriptstyle e_\tau} \\ T_1Y & \xrightarrow[(T_1f)^{op}]{} & T_1X, \end{array}$$

whereas († † †) employs the $\mathbf{B}_2$-homomorphism $\Phi\sigma \xrightarrow{\Phi\overline{(T_1f)^{op}}} \Phi\tau$. □

Proposition 2.9 has never appeared before, being a part of our current attempt to develop a topological analogue of *algebraic theories* of F. Lawvere [38]. The main idea is to reduce the use of particular topological structures to the minimum, operating in terms of their generating topological theories instead. Anticipating the developments, we call the natural transformation $T_2F \xrightarrow{\eta} \Phi^{op}T_1$ of Proposition 2.9 a *morphism of topological theories* and use the notation $T_1 \xrightarrow{\eta} T_2$ (an experienced reader will feel our intention to define a new category here; that, however, will be postponed until our next papers).

To use them later in the paper, we show the existence of products of catalg topological spaces. We start by generalizing the classical result of general topology that continuity of a map can be checked on the elements of a subbase (already extended to poslat topology by S. E. Rodabaugh [46]). For the sake of simplicity, we consider singleton topological theories, the case of the composite ones being its easy extension.

Definition 2.10. Let A be an $\mathbf{A}$-algebra, let $S \subseteq A$, and $\boldsymbol{\Omega} \subseteq \boldsymbol{\Omega}_{\mathbf{A}}$. The smallest $\boldsymbol{\Omega}$-subreduct (closure under $\boldsymbol{\Omega}$-operations is required) of A containing S is denoted $\langle S\rangle_{\boldsymbol{\Omega}}$ (or $\langle S\rangle$ if $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathbf{A}}$). Let $\mathbf{X} \xrightarrow{T} \mathbf{LoB}$ be a cat-theory, let (X, τ) be a T-space, let $S \subseteq T(X)$, and let $\boldsymbol{\Omega} \subseteq \boldsymbol{\Omega}_{\mathbf{B}}$. S is called an $\boldsymbol{\Omega}$*-base* of τ provided that $\tau = \langle S\rangle_{\boldsymbol{\Omega}}$. $\boldsymbol{\Omega}_{\mathbf{B}}$-bases are called *subbases*.

Example 2.11. $\mathbf{C\text{-}Top}_1$ (recall Example 2.8) provides the well-known $\{\bigvee\}$-bases and $\{\bigvee, \otimes, 1\}$-bases (or subbases) of poslat topology [49]. **Top** gives the classical definition of base (where the elements of the respective topology are unions of the elements of the base) and subbase (where the elements of the respective topology are unions of finite intersections of the elements of the subbase).

The next lemma (see [59, 60] for the proof) shows a relation between (pre)image operators and subreducts.

Lemma 2.12. *Let $A_1 \xrightarrow{\varphi} A_2$ be an $\mathbf{A}$-homomorphism and let $\boldsymbol{\Omega} \subseteq \boldsymbol{\Omega}_{\mathbf{A}}$.*

(1) *For every $\boldsymbol{\Omega}$-subreduct B of A_2, $\varphi^{\leftarrow}(B)$ is an $\boldsymbol{\Omega}$-subreduct of A_1.*
(2) *For every subset $S \subseteq A_1$, $\varphi^{\rightarrow}(\langle S\rangle_{\boldsymbol{\Omega}}) = \langle \varphi^{\rightarrow}(S)\rangle_{\boldsymbol{\Omega}}$.*

The proof of the following lemma can be found in [60] (or conducted by the reader as an easy exercise).

Lemma 2.13. *Let T_I be a ccat-theory in* $\mathbf{X}$ *and let* $(X, (\tau_i)_{i\in I})$, $(Y, (\sigma_i)_{i\in I})$ *be* T_I*-spaces such that* $\sigma_i = \langle S_i \rangle_{\boldsymbol{\Omega}_i}$ *for every* $i \in I$. *An* $\mathbf{X}$*-morphism* $X \xrightarrow{f} Y$ *is* T_I*-continuous iff* $((T_i f)^{op})^{\rightarrow}(S_i) \subseteq \tau_i$ *for every* $i \in I$.

As a consequence of Lemma 2.13, one gets the existence of concrete products of catalg topological spaces.

Lemma 2.14. *Let* T_I *be a ccat-theory in a category* $\mathbf{X}$. *If* $\mathbf{X}$ *has products, then the category* $\mathbf{CTop}(T_I)$ *has concrete products.*

Proof. Given a set-indexed family $((X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$ of T_I-spaces, the respective product can be obtained as $((\prod_{k\in J} X_k, (\prod_{k\in J} \tau_{k_i})_{i\in I}) \xrightarrow{\pi_j} (X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$, where $(\prod_{k\in J} X_k \xrightarrow{\pi_j} X_j)_{j\in J}$ is an $\mathbf{X}$-product of $(X_j)_{j\in J}$ (concreteness) and $\prod_{k\in J} \tau_{k_i} = \langle \bigcup_{j\in J} ((T_i \pi_j)^{op})^{\rightarrow}(\tau_{j_i}) \rangle$ for every $i \in I$. □

Corollary 2.15. *The category* $Q_{\mathbf{B}}$-**Top** *has concrete products, whereas the category* $(\mathbf{C}, \mathbf{B})$-**Top** *has concrete products provided that the category* $\mathbf{C}$ *has products.*

Notice that products in $\mathbf{C}$ correspond to coproducts in its respective subcategory of $\mathbf{A}$, something that neither $\mathbf{A}$ nor its subcategories need to possess, in general (for example, **Frm** does have coproducts [33], whereas the variety **CLat** of *complete lattices* does not [2]).

2.3. **Categorically-algebraic Topological Systems.** With the catalg topology in hand, we recall basic elements of one of the most important of its developments, namely, the theory of *catalg topological systems* (see [58, 62, 64, 66] for a full discussion of its various aspects). To begin with, we recall the original definition of S. Vickers [67] (it is interesting to notice that the theory of topological systems was somehow underestimated by researchers, until their lattice-valued, or, more generally, categorically-algebraic extension appeared).

Definition 2.16. A *topological system* is a triple $(X, A, \models)$, where X is a set, A is a locale, and $\models$ is a binary relation on $X \times |A|$ (called *satisfaction relation*), which fulfills the following two conditions for every $x \in X$:

1. given a finite subset $S \subseteq A$, $x \models \bigwedge S$ iff $x \models s$ for every $s \in S$;
2. given an arbitrary subset $S \subseteq A$, $x \models \bigvee S$ iff $x \models s$ for some $s \in S$.

A *topological system morphism* (also referred to as a *continuous map* in [67]) $(X_1, A_1, \models_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \models_2)$ is a $\mathbf{Set} \times \mathbf{Loc}$-morphism $(X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)$, which satisfies for every element $x \in X_1$ and every element $a \in A_2$ the property $x \models_1 \varphi^{op}(a)$ iff $f(x) \models_2 a$. **TopSys** is the category of topological systems and continuous maps, which is concrete over the product category $\mathbf{Set} \times \mathbf{Loc}$.

There exists another definition of topological systems provided by S. Vickers himself [67] (and extended for their lattice-valued generalization by J. T. Denniston *et al.* [10]). More precisely, given a topological system $(X, A, \models)$, the satisfaction relation $\models$ can be equivalently described as a frame homomorphism $A \xrightarrow{\kappa} \|\mathcal{P}(X)\|$

(notice the reduct to frames), which is defined by $\kappa(a) = \{x \in X \mid x \models a\}$. Moreover, a **Set** $\times$ **Loc**-morphism $|(X_1, A_1, \models_1)| \xrightarrow{(f,\varphi)} |(X_2, A_2, \models_2)|$ is continuous (in the sense of Definition 2.16) iff the diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\varphi^{op}} & A_1 \\ {\scriptstyle \kappa_2}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ \|\mathcal{P}(X_2)\| & \xrightarrow[f^{\leftarrow}]{} & \|\mathcal{P}(X_1)\| \end{array}$$

commutes. This new definition of systems allows their straightforward integration into our catalg framework, employing the well-developed machinery of *comma categories* (see, e.g., [2, 40]). To provide more intuition for the reader, we consider singleton topological theories (the case of the composite ones being then clear).

Definition 2.17. Given a cat-theory $\mathbf{X} \xrightarrow{T} \mathbf{LoB}$, $\mathbf{TopSys}(T)$ is the comma category $(T \downarrow 1_{\mathbf{LoB}})$, concrete over the product category $\mathbf{X} \times \mathbf{LoB}$, the objects of which are called *catalg topological systems* or *T-systems*, and whose morphisms are called *catalg continuous morphisms* or *T-continuous morphisms*.

The following example illustrates Definition 2.17, justifying our introduction of the new concept.

Example 2.18. The case of the ground category $\mathbf{X} = \mathbf{Set} \times \mathbf{C}$ is called *variety-based approach*. In particular, $\mathbf{TopSys}((\mathcal{S}_Q, \mathbf{B}))$ provides the category $Q_{\mathbf{B}}$-**TopSys**, which is the framework for *fixed-basis variety-based topological systems*, whereas $\mathbf{TopSys}((\mathcal{S}, \mathbf{B}))$ gives the category $(\mathbf{C}, \mathbf{B})$-**TopSys** (the case $\mathbf{A} = \mathbf{B}$ is shortened to **C-TopSys**), which is the framework for *variable-basis variety-based topological systems*. More specific, $\mathbf{TopSys}((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the category **TopSys** of topological systems of S. Vickers [67], whereas $\mathbf{TopSys}((\mathcal{P}, \mathbf{Set}))$ is isomorphic to the ground category for the categories of *interchange systems* of J. T. Denniston, A. Melton and S. E. Rodabaugh [11]. $\mathbf{TopSys}((\mathcal{R}_3, \mathbf{Frm}))$ with $\mathbf{C} = \mathbf{Loc}$ is isomorphic to the category **Loc-TopSys** of lattice-valued topological systems of J. T. Denniston *et al.* [10].

One of the main results of the theory of systems is the possibility of representing the category $\mathbf{Top}(T)$ as a full subcategory (with convenient properties) of the category $\mathbf{TopSys}(T)$. Following our remark from Introduction on the nature of both categories ($\mathbf{Top}(T)$ is topological over its ground category, whereas $\mathbf{TopSys}(T)$ is (essentially) algebraic), one obtains the so-called "embedding of topology into algebra" [54], which recently has raised an interest among researchers [12].

Theorem 2.19.

(1) *There exists a full embedding* $\mathbf{Top}(T) \overset{\mathrm{E_T}}{\hookrightarrow} \mathbf{TopSys}(T)$, *which is given by* $\mathrm{E_T}((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, \iota_1^{op}, \tau_1) \xrightarrow{(f,\varphi)} (X_2, \iota_2^{op}, \tau_2)$, *where* ι_i *is the inclusion* $\tau_i \hookrightarrow T(X_i)$ *and* φ^{op} *is the restriction* $\tau_2 \xrightarrow{(Tf)^{op}|_{\tau_2}^{\tau_1}} \tau_1$.

(2) *There is a functor* $\mathbf{TopSys}(T) \xrightarrow{\mathrm{Spat}} \mathbf{Top}(T)$ *with* $\mathrm{Spat}((X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)) = (X_1, (\kappa_1^{op})^{\rightarrow}(B_1)) \xrightarrow{f} (X_2, (\kappa_2^{op})^{\rightarrow}(B_2))$.

(3) Spat *is a right-adjoint-left-inverse to* $\mathrm{E_T}$, *the respective co-universal arrows being regular monomorphisms.*

The functor Spat provides an extension of the system spatialization procedure of S. Vickers [67].

Corollary 2.20. $\mathbf{Top}(T)$ *is isomorphic to a full (regular mono)-coreflective subcategory of* $\mathbf{TopSys}(T)$.

Having briefly introduced the general theory of catalg topology, from now on, we will restrict ourselves to its variable-basis variety-based subcase described in Examples 2.8, 2.18.

3. Algebraically-topological Systems

This section introduces *algebraically-topological systems* developed in a variety-based setting as a generalization of the topological systems of S. Vickers. The main motivation for the new concept was our wish to find a common framework for both topological systems and attachments of C. Guido. Strikingly enough, it appears that the new notion has a natural topological justification, which originates from the concept of *topological algebra*, stemming from, e.g., *topological groups* of L. Pontrjagin [42]. To boost the system intuition of the reader, below we introduce the category of their motivating structures, which is based in the category $(\mathbf{C}, \mathbf{B})$-$\mathbf{Top}$ of Example 2.8 (recall the respective varietal notations), or, using the language of *enriched category theory* of G. M. Kelly [34], is an enrichment of $(\mathbf{C}, \mathbf{B})$-$\mathbf{Top}$ in a variety of algebras. The reader is advised to recall Corollary 2.15.

Definition 3.1. Given a category $(\mathbf{C}, \mathbf{B})$-$\mathbf{Top}$ and a variety $\mathbf{E}$, $(\mathbf{E}, \mathbf{C}, \mathbf{B})$-$\mathbf{TopAlg}$ is the category, concrete over the product category $\mathbf{E} \times \mathbf{C}$, whose objects ($\mathbf{C}$-*topological* $\mathbf{E}$-*algebras*) are triples (E, C, τ) such that E is an $\mathbf{E}$-algebra, $(|E|, C, \tau)$ is a $(\mathbf{C}, \mathbf{B})$-*space* (an object of $(\mathbf{C}, \mathbf{B})$-$\mathbf{Top}$), and every primitive E-operation $E^{n_\lambda} \xrightarrow{\omega_\lambda^E} E$ provides a $C_{\mathbf{B}}$-*continuous map* (a morphism of the category $C_{\mathbf{B}}$-$\mathbf{Top}$ of Example 2.8) $(|E|, \tau)^{n_\lambda} \xrightarrow{\omega_\lambda^E} (|E|, \tau)$ (notice that the product $(|E|, \tau)^{n_\lambda}$ is taken in $C_{\mathbf{B}}$-$\mathbf{Top}$). Morphisms $(E_1, C_1, \tau_1) \xrightarrow{(\varphi,\psi)} (E_2, C_2, \tau_2)$ are $\mathbf{E} \times \mathbf{C}$-morphisms $(E_1, C_1) \xrightarrow{(\varphi,\psi)} (E_2, C_2)$ such that $(|E_1|, C_1, \tau_1) \xrightarrow{(|\varphi|,\psi)} (|E_2|, C_2, \tau_2)$ is $(\mathbf{C}, \mathbf{B})$-*continuous* (a morphism of the category $(\mathbf{C}, \mathbf{B})$-$\mathbf{Top}$ of Example 2.8).

The following provides a simple (actually, motivating) example of the new notion.

Example 3.2. The category $(\mathbf{Grp}, \mathbf{S_2}, \mathbf{Frm})$-$\mathbf{TopAlg}$, where $\mathbf{A} = \mathbf{CBAlg}$, is isomorphic to the category of topological groups of L. Pontrjagin [42], void of any separation condition on the spaces.

In the next step, we introduce the category of variety-based algebraically-topological systems. The new category is based in the category $(\mathbf{C}, \mathbf{B})$-$\mathbf{TopSys}$ of variety-based topological systems of Example 2.18 (recall the respective notations), or, more precisely, is an enrichment of $(\mathbf{C}, \mathbf{B})$-$\mathbf{TopSys}$ in a variety of algebras.

Definition 3.3. Given a category $(\mathbf{C},\mathbf{B})$-**TopSys** and a reduct $(\|-\|,\mathbf{E})$ of $\mathbf{A}$, $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys** is the category, concrete over the product category $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$, whose objects (*variety-based algebraically-topological systems* or $(\mathbf{E},\mathbf{C},\mathbf{B})$-*algebraically-topological systems*) are tuples $D=(\Delta\, D,\Sigma\, D,\Omega\, D,\models)$ such that $(\Delta\, D,\Sigma\, D,\Omega\, D)$ is an object of $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$ and $\Delta\, D\times\Omega\, D\xrightarrow{\models}\Sigma\, D$ is a map (*variety-based satisfaction relation* or $(\mathbf{E},\mathbf{C},\mathbf{B})$-*satisfaction relation*), which is an $(\mathbf{E},\mathbf{B})$-*bimorphism*, i.e., $\Omega\, D\xrightarrow{\models(e,-)}\|\Sigma\, D\|_{\mathbf{B}}$ is a $\mathbf{B}$-homomorphism for every $e\in\Delta\, D$, and $\Delta\, D\xrightarrow{\models(-,b)}\|\Sigma\, D\|_{\mathbf{E}}$ is an $\mathbf{E}$-homomorphism for every $b\in\Omega\, D$. Morphisms of the category $D_1\xrightarrow{f=(\Delta\, f,(\Sigma\, f)^{op},(\Omega\, f)^{op})}D_2$ are $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$-morphisms $(\Delta\, D_1,\Sigma\, D_1,\Omega\, D_1)\xrightarrow{f}(\Delta\, D_2,\Sigma\, D_2,\Omega\, D_2)$ such that for every $e\in\Delta\, D_1$ and every $b\in\Omega\, D_2$, $\models_1(e,\Omega\, f(b))=\Sigma\, f\circ\models_2(\Delta\, f(e),b)$. In the case of $\mathbf{A}=\mathbf{B}$, the notation for the category is shortened to $(\mathbf{E},\mathbf{C})$-**AlgTopSys**.

Example 3.4. The category $(\mathbf{Set},\mathbf{C},\mathbf{B})$-**AlgTopSys** is precisely the category $(\mathbf{C},\mathbf{B})$-**TopSys** of variable-basis variety-based topological systems.

Example 3.4 shows that we have extended the above-mentioned variety-based topological systems, bringing more symmetry in their definition. Indeed, the satisfaction relation of the latter is a $(\mathbf{Set},\mathbf{B})$-bimorphism. It looks natural to replace **Set** with a variety $\mathbf{E}$, demanding the corresponding homomorphism property.

The notations of the new category stem from S. Solovyov [62], partly extending those of S. Vickers [67] (the "Δ" is our invention). The reader should notice that there exists no relation between the underlying varieties $\mathbf{E}$ and $\mathbf{A}$ in the framework of topological algebras, whereas $\mathbf{E}$ is a reduct of $\mathbf{A}$ in the setting of algebraically-topological systems.

As will appear later in the paper, it is worthwhile to introduce a modified category of algebraically-topological systems as follows (an attentive reader will see that the only change made in the new category concerns the direction of the Σ-component of morphisms).

Definition 3.5. Given a category $(\mathbf{C},\mathbf{B})$-**TopSys** and a reduct $(\|-\|,\mathbf{E})$ of $\mathbf{A}$, $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}_m$ is the category, concrete over the product category $\mathbf{E}\times\mathbf{LoC}\times\mathbf{LoB}$, whose objects (*modified* $(\mathbf{E},\mathbf{C},\mathbf{B})$-*algebraically-topological systems*) are tuples $D=(\Delta\, D,\Sigma\, D,\Omega\, D,\models)$, where $(\Delta\, D,\Sigma\, D,\Omega\, D)$ is an $\mathbf{E}\times\mathbf{LoC}\times\mathbf{LoB}$-object and $\Delta\, D\times\Omega\, D\xrightarrow{\models}\Sigma\, D$ is an $(\mathbf{E},\mathbf{B})$-bimorphism. Morphisms of the category $D_1\xrightarrow{f=(\Delta\, f,\Sigma\, f,(\Omega\, f)^{op})}D_2$ are $\mathbf{E}\times\mathbf{LoC}\times\mathbf{LoB}$-morphisms $(\Delta\, D_1,\Sigma\, D_1,\Omega\, D_1)\xrightarrow{f}(\Delta\, D_2,\Sigma\, D_2,\Omega\, D_2)$ such that for every $e\in\Delta\, D_1$ and every $b\in\Omega\, D_2$, $\Sigma\, f\circ\models_1(e,\Omega\, f(b))=\models_2(\Delta\, f(e),b)$. If the case of $\mathbf{A}=\mathbf{B}$, the category is denoted $(\mathbf{E},\mathbf{C})$-$\mathbf{AlgTopSys}_m$.

The next two subsections clarify the reason of introducing two different categories of systems. They also show the motivating category $(\mathbf{E},\mathbf{C},\mathbf{B})$-**TopAlg** in action.

3.1. **Spatialization of Algebraically-topological Systems.** In the previous section on catalg preliminaries, we mentioned that catalg topological systems provide a convenient extension of catalg topological spaces (Theorem 2.19). This subsection partly extends the result to their variety-based algebraic counterpart.

The system spatialization procedure (the functor of the second item of Theorem 2.19, whose name is the abbreviation for “spatialization” and whose main job is to obtain a “space” from a system, or to “spatialize” a system) allows a direct generalization to the new setting.

Proposition 3.6. *There exists a functor*

$$(\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys} \xrightarrow{\mathrm{ASpat}} (\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{TopAlg},$$

which is defined by

$$\mathrm{ASpat}(D_1 \xrightarrow{f} D_2) = (\Delta\, D_1, \Sigma\, D_1, \tau_1) \xrightarrow{(\Delta\, f, (\Sigma\, f)^{op})} (\Delta\, D_2, \Sigma\, D_2, \tau_2),$$

where $\tau_i = \{\models_i(-,b) \,|\, b \in \Omega\, D_i\}$.

Proof. The functor follows the pattern of [62, Lemma 9] and, therefore, the only thing to show is its correctness on objects, i.e., given an algebraically-topological system D, we check that every primitive D-operation is continuous (the reader should recall the notations for powerset theories of Example 2.5). For every $\lambda \in \Lambda_{\mathbf{E}}$, every $b \in \Omega\, D$ and every $\langle e_i\rangle_{n_\lambda} \in (\Delta\, D)^{n_\lambda}$,

$$\begin{aligned}
&((\omega_\lambda^{\Delta\, D})^{\leftarrow}_{\Sigma\, D}(\models(-,b)))(\langle e_i\rangle_{n_\lambda}) = \\
&\models(-,b)\circ\omega_\lambda^{\Delta\, D}(\langle e_i\rangle_{n_\lambda}) = \\
&\models(\omega_\lambda^{\Delta\, D}(\langle e_i\rangle_{n_\lambda}), b) = \\
&\omega_\lambda^{\|\Sigma\, D\|_{\mathbf{E}}}(\langle \models(e_i,b)\rangle_{n_\lambda}) = \\
&\omega_\lambda^{\|\Sigma\, D\|_{\mathbf{E}}}(\langle \models(-,b)\circ\pi_j(\langle e_i\rangle_{n_\lambda})\rangle_{n_\lambda}) = \\
&\omega_\lambda^{\|\Sigma\, D\|_{\mathbf{E}}}(\langle ((\pi_j)^{\leftarrow}_{\Sigma\, D}(\models(-,b)))(\langle e_i\rangle_{n_\lambda})\rangle_{n_\lambda}) = \\
&(\omega_\lambda^{(\|\Sigma\, D\|_{\mathbf{E}})^{|(\Delta\, D)^{n_\lambda}|}}(\langle (\pi_j)^{\leftarrow}_{\Sigma\, D}(\models(-,b))\rangle_{n_\lambda}))(\langle e_i\rangle_{n_\lambda}),
\end{aligned}$$

where $D^{n_\lambda} \xrightarrow{\pi_j} D$ is the jth projection homomorphism. As a result, one obtains $(\omega_\lambda^{\Delta\, D})^{\leftarrow}_{\Sigma\, D}(\models(-,b)) = \omega_\lambda^{(\|\Sigma\, D\|_{\mathbf{E}})^{|(\Delta\, D)^{n_\lambda}|}}(\langle (\pi_j)^{\leftarrow}_{\Sigma\, D}(\models(-,b))\rangle_{n_\lambda}) \in \sigma_{(\Delta\, D,\tau)^{n_\lambda}}$. □

Unlike the spatialization functor, the respective analogue of the embedding one requires more effort. We start with an additional definition, stemming from the theory of *catalg dualities*, developed by S. Solovyov in [57, 60] as an extension of the theory of *natural dualities* of [8].

Definition 3.7. Let $\mathbf{B}$, $\mathbf{E}$ be varieties of algebras and let $(\|-\|, \mathbf{E})$ be a reduct of $\mathbf{B}$. A $\mathbf{B}$-algebra B is called $\mathbf{E}$-*entropic* provided that every primitive $\mathbf{B}$-operation on B is an $\mathbf{E}$-homomorphism. $\mathbf{B}$ is called $\mathbf{E}$-*entropic* provided that every $\mathbf{B}$-algebra is $\mathbf{E}$-entropic.

The following provides a simple illustration of the concept.

Example 3.8. If **DLat** is the variety of *distributive lattices* (not necessarily bounded) and $\mathbf{SLat}(\Theta)$, $\Theta \in \{\vee, \wedge\}$ is the variety of Θ-semilattices, then **DLat** is $\mathbf{SLat}(\Theta)$-entropic. On the other hand, **Frm** is not $\mathbf{CSLat}(\bigvee)$-entropic since given a non-singleton frame L, the empty $\bigwedge$ is not $\bigvee$-preserving.

The terminology of Definition 3.7 stems from the theory of *modes* (idempotent, entropic algebras) [53]. With the help of the new notion, we are finally ready to provide a machinery of transforming topological algebras into algebraically-topological systems.

Proposition 3.9. *Every* $\mathbf{E}$*-entropic variety* $\mathbf{B}$ *gives rise to a faithful functor* $(\mathbf{E},\mathbf{C},\mathbf{B})$-**TopAlg** $\xrightarrow{\mathrm{AE_T}}$ $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys***, which is defined by the formula* $\mathrm{AE_T}((E_1,C_1,\tau_1) \xrightarrow{(\varphi,\psi)} (E_2,C_2,\tau_2)) = (E_1,C_1,\tilde{\tau}_1,\models_1) \xrightarrow{(\varphi,\psi,\phi^{op})} (E_2,C_2,\tilde{\tau}_2,\models_2)$*, where* $\tilde{\tau}_i = \tau_i \bigcap \mathbf{E}(E_i,\|C_i\|_{\mathbf{E}})$*,* $\models_i(e,\alpha) = \alpha(e)$ *and* $\phi = (|\varphi|,\psi)^{\leftarrow}|^{\tilde{\tau}_1}_{\tilde{\tau}_2}$*.*

Proof. It should be underlined at once that the respective embedding of Theorem 2.19 (described explicitly for the variety-based case in [62, Lemma 8]) takes a topological space (X,C,τ) to the system $(X,C,\tau,\models)$ without any truncation of the topology τ and, therefore, the new setting, however being similar to the old one, requires more verification.

Given a space (E,C,τ), we begin by showing that $\tilde{\tau}$ is a $\mathbf{B}$-algebra. It is enough to check that $\mathbf{E}(E,\|C\|_{\mathbf{E}})$ is a $\mathbf{B}$-subalgebra of $(\|C\|_{\mathbf{B}})^{|E|}$. Given $\lambda \in \Lambda_{\mathbf{B}}$, $\langle \alpha_i \rangle_{n_\lambda} \in (\mathbf{E}(E,\|C\|_{\mathbf{E}}))^{n_\lambda}$ and $\lambda' \in \Lambda_{\mathbf{E}}$, $\langle e_j \rangle_{n_{\lambda'}} \in E^{n_{\lambda'}}$,

$$\begin{aligned}
(\omega_\lambda^{(\|C\|_{\mathbf{B}})^{|E|}}(\langle \alpha_i \rangle_{n_\lambda}))(\omega_{\lambda'}^{E}(\langle e_j \rangle_{n_{\lambda'}})) &= \\
\omega_\lambda^{\|C\|_{\mathbf{B}}}(\langle \alpha_i(\omega_{\lambda'}^{E}(\langle e_j \rangle_{n_{\lambda'}}))\rangle_{n_\lambda}) &= \\
\omega_\lambda^{\|C\|_{\mathbf{B}}}(\langle \omega_{\lambda'}^{\|C\|_{\mathbf{E}}}(\langle \alpha_i(e_j)\rangle_{n_{\lambda'}})\rangle_{n_\lambda}) &\overset{(\dagger)}{=} \\
\omega_{\lambda'}^{\|C\|_{\mathbf{E}}}(\langle \omega_{\lambda}^{\|C\|_{\mathbf{B}}}(\langle \alpha_i(e_j)\rangle_{n_\lambda})\rangle_{n_{\lambda'}}) &= \\
\omega_{\lambda'}^{\|C\|_{\mathbf{E}}}(\langle (\omega_\lambda^{(\|C\|_{\mathbf{B}})^{|E|}}(\langle \alpha_i \rangle_{n_\lambda}))(e_j)\rangle_{n_{\lambda'}}),&
\end{aligned}$$

where $(\dagger)$ relies on the assumption that $\mathbf{B}$ is $\mathbf{E}$-entropic.

On the next step, we show that the map $E \times \tilde{\tau} \xrightarrow{\models} C$ provides an $(\mathbf{E},\mathbf{B})$-bimorphism. Given $\alpha \in \tilde{\tau}$, $\models(-,\alpha) = \alpha \in \mathbf{E}(E,\|C\|_{\mathbf{E}})$. The case of $e \in E$ is similar to that of [62, Lemma 8], i.e., for every $\lambda \in \Lambda_{\mathbf{B}}$ and every $\langle \alpha_i \rangle_{n_\lambda} \in \tilde{\tau}^{n_\lambda}$, $\models(e,\omega_\lambda^{\tilde{\tau}}(\langle \alpha_i \rangle_{n_\lambda})) = (\omega_\lambda^{\tilde{\tau}}(\langle \alpha_i \rangle_{n_\lambda}))(e) = \omega_\lambda^{\|C\|_{\mathbf{B}}}(\langle \alpha_i(e)\rangle_{n_\lambda}) = \omega_\lambda^{\|C\|_{\mathbf{B}}}(\langle \models(e,\alpha_i)\rangle_{n_\lambda})$.

To show correctness of the definition on morphisms, i.e., on all their components, notice that given $\alpha \in \tilde{\tau}_2$, $\phi(\alpha) = \psi^{op} \circ \alpha \circ \varphi \in \tau_1 \bigcap \mathbf{E}(E_1,\|C_1\|_{\mathbf{E}})$. □

As appears from Proposition 3.9, the new setting strips the functor $\mathrm{AE_T}$ of its two main properties, namely, being a full embedding. It is a challenging question to find the necessary and sufficient conditions for restoring these important features (the reader is advised to do the job). On the other hand, the above two functors still provide an adjoint situation.

Proposition 3.10. *The functor* ASpat *is a right adjoint to the functor* $\mathrm{AE_T}$*. The respective co-universal arrows are regular monomorphisms.*

Proof. The following construction essentially reiterates that of [62, Theorem 3]. For an algebraically-topological system D, define $(\mathrm{AE_T}\,\mathrm{ASpat}\,D = (\Delta\, D, \Sigma\, D, \sigma =$

$\{\models(-,b)\,|\,b\in\Omega\,D\},\widetilde{\models}))\xrightarrow{u=(1_{\Delta\,D},1_{\Sigma\,D},\phi^{op})} D$ by $\phi(b)=\models(-,b)$. Straightforward computations show that u is the required ASpat-co-universal arrow for D, i.e., every system morphism $\mathrm{AE_T}(E,C,\tau)\xrightarrow{f} D$ has a unique topological algebra morphism $(E,C,\tau)\xrightarrow{(\varphi,\psi)}\mathrm{ASpat}\,D$ (given by $\varphi=\Delta\,f,\ \psi=(\Sigma\,f)^{op}$) making the following triangle commute

$$\begin{array}{ccc} \mathrm{AE_T}(E,C,\tau) & & \\ {\scriptstyle \mathrm{AE_T}(\varphi,\psi)}\downarrow & \searrow{\scriptstyle f} & \\ \mathrm{AE_T}\,\mathrm{ASpat}\,D & \xrightarrow[u]{} & D. \end{array}$$

Since ϕ is a coequalizer of the pair of homomorphisms $\mathrm{Ker}\,\phi \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \Omega\,D$, where $\pi_i(b_1,b_2)=b_i$ being the ith projection, u is an equalizer of the system morphisms $D \underset{(1_{\Delta\,D},1_{\Sigma\,D},\pi_2^{op})}{\overset{(1_{\Delta\,D},1_{\Sigma\,D},\pi_1^{op})}{\rightrightarrows}} (\Delta\,D,\Sigma\,D,\mathrm{Ker}\,\phi,\overline{\models})$, in which $\overline{\models}(e,(b_1,b_2))=\models(e,b_1)=\models(e,b_2)$, i.e., u is a regular monomorphism in $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys**. □

An attentive reader has probably already noticed that ASpat is no longer necessarily a left inverse to $\mathrm{AE_T}$, due to the fact that we do not consider the whole topology when coming from spaces to systems.

3.2. **Localification of Algebraically-topological Systems.** The previous subsection considered an extension of the spatialization procedure of S. Vickers. There exists, however, another machinery called *localification of systems* [67]. It is the purpose of this section to provide its analogue in the current setting, extending the respective variety-based generalization of [63]. Following the pattern of [63], the developments are based on the modified category of systems of Definition 3.5.

We start with the generalization of the localification procedure (recall the spatialization functor from the previous subsection; the name of current machinery stems from the classical functor of S. Vickers, which produces a "locale" from a system, or "localifies" a system), extending [63, Lemma 29].

Proposition 3.11. *There exists a functor* $(\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys}_m \xrightarrow{\mathrm{ALoc}} \mathbf{LoC}\times\mathbf{LoB}$, $\mathrm{ALoc}(D_1\xrightarrow{f} D_2)=(\Sigma\,D_1,\Omega\,D_1)\xrightarrow{(\Sigma\,f,(\Omega\,f)^{op})}(\Sigma\,D_2,\Omega\,D_2)$.

The opposite way is more demanding, requiring the notion of entropicity, introduced in Definition 3.7.

Proposition 3.12. *Every* $\mathbf{E}$*-entropic variety* $\mathbf{B}$ *provides a full embedding*

$$\mathbf{LoC}\times\mathbf{LoB} \xhookrightarrow{\mathrm{AE_A}} (\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys}_m,$$

which is defined by the formula

$$\mathrm{AE_A}((C_1,B_1)\xrightarrow{(\varphi,\psi)}(C_2,B_2))=$$
$$(\mathbf{B}(B_1,\|C_1\|_{\mathbf{B}}),C_1,B_1,\models_1)\xrightarrow{((|\psi^{op}|,\varphi^{op})^{\leftarrow},\varphi,\psi)}(\mathbf{B}(B_2,\|C_2\|_{\mathbf{B}}),C_2,B_2,\models_2),$$

where $\models_i(\alpha, b) = \alpha(b)$.

Proof. The construction of the functor is based on [63, Lemma 28], with several additional verifications. Notice that we rely on a significant degree of the modification of the category of systems to define the action of the new functor on morphisms.

Given (C, B) in $\mathbf{LoC}\times\mathbf{LoB}$, we begin by checking that the set $\mathbf{B}(B, \|C\|_{\mathbf{B}})$ is an $\mathbf{E}$-subalgebra of $(\|C\|_{\mathbf{E}})^{|B|}$. Given $\lambda \in \Lambda_{\mathbf{E}}$, $\langle\alpha_i\rangle_{n_\lambda} \in (\mathbf{B}(B, \|C\|_{\mathbf{B}}))^{n_\lambda}$ and $\lambda' \in \Lambda_{\mathbf{B}}$, $\langle b_j\rangle_{n_{\lambda'}} \in B^{n_{\lambda'}}$,

$$\begin{aligned}(\omega_\lambda^{(\|C\|_{\mathbf{E}})^{|B|}}(\langle\alpha_i\rangle_{n_\lambda}))(\omega_{\lambda'}^B(\langle b_j\rangle_{n_{\lambda'}})) &=\\ \omega_\lambda^{\|C\|_{\mathbf{E}}}(\langle\alpha_i(\omega_{\lambda'}^B(\langle b_j\rangle_{n_{\lambda'}}))\rangle_{n_\lambda}) &=\\ \omega_\lambda^{\|C\|_{\mathbf{E}}}(\langle\omega_{\lambda'}^{\|C\|_{\mathbf{B}}}(\langle\alpha_i(b_j)\rangle_{n_{\lambda'}})\rangle_{n_\lambda}) &\overset{(\dagger)}{=}\\ \omega_{\lambda'}^{\|C\|_{\mathbf{B}}}(\langle\omega_\lambda^{\|C\|_{\mathbf{E}}}(\langle\alpha_i(b_j)\rangle_{n_\lambda})\rangle_{n_{\lambda'}}) &=\\ \omega_{\lambda'}^{\|C\|_{\mathbf{B}}}(\langle(\omega_\lambda^{(\|C\|_{\mathbf{E}})^{|B|}}(\langle\alpha_i\rangle_{n_\lambda}))(b_j)\rangle_{n_{\lambda'}}),&\end{aligned}$$

where $(\dagger)$ uses the assumption on entropicity.

On the next step, we show that the map $\mathbf{B}(B, \|C\|_{\mathbf{B}}) \times B \xrightarrow{\models} C$ provides an $(\mathbf{E}, \mathbf{B})$-bimorphism. Given $\alpha \in \mathbf{B}(B, \|C\|_{\mathbf{B}})$, $\models(\alpha, -) = \alpha$. Given $b \in B$, for every $\lambda \in \Lambda_{\mathbf{E}}$ and every $\langle\alpha_i\rangle_{n_\lambda} \in (\mathbf{B}(B, \|C\|_{\mathbf{B}}))^{n_\lambda}$, $\models(\omega_\lambda^{(\|C\|_{\mathbf{E}})^{|B|}}(\langle\alpha_i\rangle_{n_\lambda}), b) = (\omega_\lambda^{(\|C\|_{\mathbf{E}})^{|B|}}(\langle\alpha_i\rangle_{n_\lambda}))(b) = \omega_\lambda^{\|C\|_{\mathbf{E}}}(\langle\alpha_i(b)\rangle_{n_\lambda}) = \omega_\lambda^{\|C\|_{\mathbf{E}}}(\langle\models(\alpha_i, b)\rangle_{n_\lambda})$.

To show correctness on morphisms, i.e., that all components are homomorphisms, notice that for every $\lambda \in \Lambda_{\mathbf{E}}$ and every $\langle\alpha_i\rangle_{n_\lambda} \in (\mathbf{B}(B_1, \|C_1\|_{\mathbf{B}}))^{n_\lambda}$,

$$\begin{aligned}(|\psi^{op}|, \varphi^{op})^{\leftarrow}(\omega_\lambda^{\mathbf{B}(B_1,\|C_1\|_{\mathbf{B}})}(\langle\alpha_i\rangle_{n_\lambda})) &=\\ \varphi\circ(\omega_\lambda^{\mathbf{B}(B_1,\|C_1\|_{\mathbf{B}})}(\langle\alpha_i\rangle_{n_\lambda}))\circ\psi^{op} &=\\ \varphi\circ(\omega_\lambda^{\mathbf{B}(B_2,\|C_1\|_{\mathbf{B}})}(\langle\alpha_i\circ\psi^{op}\rangle_{n_\lambda})) &=\\ \omega_\lambda^{\mathbf{B}(B_2,\|C_2\|_{\mathbf{B}})}(\langle\varphi\circ\alpha_i\circ\psi^{op}\rangle_{n_\lambda}),&\end{aligned}$$

that essentially is a reiteration of the proof of the functorial features of $\mathcal{S} = (-)^{\leftarrow}$ from Example 2.5.

The embedding property is clear. For fullness, notice that given a system morphism $(\mathbf{B}(B_1, \|C_1\|_{\mathbf{B}}), C_1, B_1, \models_1) \xrightarrow{f} (\mathbf{B}(B_2, \|C_2\|_{\mathbf{B}}), C_2, B_2, \models_2)$, for every $\alpha \in \mathbf{B}(B_1, \|C_1\|_{\mathbf{B}})$ and every $b \in B_2$, $(\Delta\, f(\alpha))(b) = \models_2(\Delta\, f(\alpha), b) = \Sigma\, f \circ \models_1(\alpha, \Omega\, f(b)) = \Sigma\, f\circ\alpha\circ\Omega\, f(b) = ((\Omega\, f, (\Sigma\, f)^{op})^{\leftarrow}(\alpha))(b)$. $\square$

The reader should pay attention to the difference from the spatialization setting, where the respective functor looses its essential properties. As a consequence, which deepens the distinction even more, the next proposition shows that ALoc is a left-adjoint-left-inverse to AE_{A}.

Proposition 3.13. ALoc *is a left-adjoint-left-inverse to* AE_{A}.

Proof. The following machinery is essentially that of [63, Theorem 30]. Given an algebraically-topological system D, define $D \xrightarrow{v=(\varphi,1_{\Sigma D},1_{\Omega D})} (\mathrm{AE_A}\,\mathrm{ALoc}\,D = (\mathbf{B}(\Omega D, \|\Sigma D\|_{\mathbf{B}}), \Sigma D, \Omega D, \overline{\models}))$ by $\varphi(e) = \models(e,-)$. Straightforward computations show that v is an $\mathrm{AE_A}$-universal arrow for D, i.e., every system morphism $D \xrightarrow{f} \mathrm{AE_A}(C,B)$ has a unique $\mathbf{LoC}\times\mathbf{LoB}$-morphism $\mathrm{ALoc}\,D \xrightarrow{(\varphi,\psi)} (C,B)$ (given by $\varphi = \Sigma f$, $\psi = (\Omega f)^{op}$) making the triangle

$$\begin{array}{ccc} D & \xrightarrow{\ v\ } & \mathrm{AE_A}\,\mathrm{ALoc}\,D \\ & {\scriptstyle f}\searrow & \Big\downarrow {\scriptstyle \mathrm{AE_A}(\varphi,\psi)} \\ & & \mathrm{AE_A}(C,B) \end{array}$$

commute. The claim on $\mathrm{ALoc}\,\mathrm{AE_A} = 1_{\mathbf{LoC}\times\mathbf{LoB}}$ is clear. □

Corollary 3.14. *The category* $\mathbf{LoC}\times\mathbf{LoB}$ *is isomorphic to a full reflective subcategory of the category* $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}$*, which (in general) is neither mono- nor epi-reflective.*

Comparing the results of the above two subsections, the reader will easily come to the conclusion that the new localification procedure is far better preserved in the algebraically-topological setting than the respective spatialization one. The simple reason is its complete ignorance of the Δ-component of systems, which contains the only change made in the new framework. A challenging question on whether the forgetful functor of the category $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}$ should be also considered as a possible candidate for localification procedure will be postponed until further development of the topic.

3.3. **Fixed-basis Topological Theory Morphisms Induced by Algebraically-topological Systems.** Having clarified basic relationships between algebraically-topological systems and topological algebras, we are going to deepen our investigation of the topic and show yet another connection between new systems and variety-based topology, namely, topological theory morphisms induced by them (the reader should recall Proposition 2.9 and its related concepts and notations). This subsection is restricted to the fixed-basis approach, which relies on a given algebraically-topological system D.

We begin with a lemma, which will be the cornerstone of the subsequent developments. The notations employed stem from S. Solovyov [61] and A. Frascella *et al.* [19] and will explained later on in the paper.

Lemma 3.15. *Let D be a variety-based algebraically-topological system and let X be a set. There exist:*

1. *an* $\mathbf{E}$*-homomorphism* $(\Delta D)^X \xrightarrow{(-)^{\Vdash\Delta}_X} (\|\Sigma D\|_{\mathbf{E}})^{X\times|\Omega D|}$ *with* $\alpha^{\Vdash\Delta}_X(x,b) = \models(\alpha(x),b)$*;*
2. *a* $\mathbf{B}$*-homomorphism* $(\Omega D)^X \xrightarrow{(-)^{\Vdash\Omega}_X} (\|\Sigma D\|_{\mathbf{B}})^{|\Delta D|\times X}$ *with* $\alpha^{\Vdash\Omega}_X(e,x) = \models(e,\alpha(x))$*.*

Proof. We show the first item. The proof of the second one is similar. Given $\lambda \in \Lambda_{\mathbf{E}}$ and $\langle \alpha_i \rangle_{n_\lambda} \in (\Delta\, D)^X$, for every $(x, b) \in X \times |\,\Omega\, D|$,

$$\begin{aligned}
(\omega_\lambda^{(\Delta\, D)^X}(\langle \alpha_i \rangle_{n_\lambda}))_X^{\Vdash\Delta}(x,b) &= \\
\models((\omega_\lambda^{(\Delta\, D)^X}(\langle \alpha_i \rangle_{n_\lambda}))(x), b) &= \\
\models(\omega_\lambda^{\Delta\, D}(\langle \alpha_i(x) \rangle_{n_\lambda}), b) &= \\
\omega_\lambda^{\|\,\Sigma\, D\|_{\mathbf{E}}}(\langle \models(\alpha_i(x), b) \rangle_{n_\lambda}) &= \\
\omega_\lambda^{\|\,\Sigma\, D\|_{\mathbf{E}}}(\langle (\alpha_i)_X^{\Vdash\Delta}(x,b) \rangle_{n_\lambda}) &= \\
(\omega_\lambda^{(\|\,\Sigma\, D\|_{\mathbf{E}})^{X \times |\,\Omega\, D|}}(\langle (\alpha_i)_X^{\Vdash\Delta} \rangle_{n_\lambda}))(x,b)&.
\end{aligned}$$

□

On the next step, we define two additional functors:

(1) $\mathbf{Set} \xrightarrow{(-\times|\,\Omega\, D|)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}}} \mathbf{LoE} = \mathbf{Set} \xrightarrow{-\times|\,\Omega\, D|} \mathbf{Set} \xrightarrow{(-)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}}} \mathbf{LoE}$;

(2) $\mathbf{Set} \xrightarrow{(|\,\Delta\, D|\times-)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{B}}}} \mathbf{LoB} = \mathbf{Set} \xrightarrow{|\,\Delta\, D|\times-} \mathbf{Set} \xrightarrow{(-)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{B}}}} \mathbf{LoB}$.

With the new functors in hand, we can obtain the following important result.

Lemma 3.16. *Every variety-based algebraically-topological system D provides two natural transformations:*

(1) $(-\times|\,\Omega\, D|)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}} \xrightarrow{((-)^{\Vdash\Delta})^{op}} (-)^{\leftarrow}_{\Delta\, D}$;

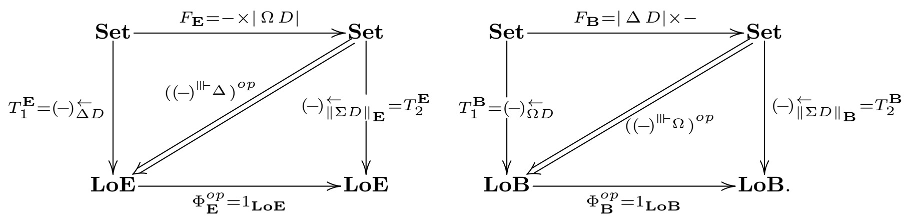
(2) $(|\,\Delta\, D|\times-)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{B}}} \xrightarrow{((-)^{\Vdash\Omega})^{op}} (-)^{\leftarrow}_{\Omega\, D}$.

Proof. We show the first item, the proof of the second one being similar. Given a map $X_1 \xrightarrow{f} X_2$, we have to verify commutativity of the following diagram:

$$\begin{array}{ccc}
(\Delta\, D)^{X_2} & \xrightarrow{(-)^{\Vdash\Delta}_{X_2}} & (\|\,\Sigma\, D\|_{\mathbf{E}})^{X_2\times|\,\Omega\, D|} \\
\downarrow{\scriptstyle f^{\leftarrow}_{\Delta\, D}} & & \downarrow{\scriptstyle (f\times 1_{|\,\Omega\, D|})^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}}} \\
(\Delta\, D)^{X_1} & \xrightarrow[(-)^{\Vdash\Delta}_{X_1}]{} & (\|\,\Sigma\, D\|_{\mathbf{E}})^{X_1\times|\,\Omega\, D|}.
\end{array}$$

Given $\alpha \in (\Delta\, D)^{X_2}$, for every $(x,b) \in X_1 \times |\,\Omega\, D|$, $(((-)^{\Vdash\Delta}_{X_1} \circ f^{\leftarrow}_{\Delta\, D})(\alpha))(x,b) = (\alpha \circ f)^{\Vdash\Delta}_{X_1}(x,b) = \models(\alpha \circ f(x), b) = \alpha^{\Vdash\Delta}_{X_2}(f(x),b) = (\alpha^{\Vdash\Delta}_{X_2} \circ (f \times 1_{|\,\Omega\, D|}))(x,b) = (((f\times 1_{|\,\Omega\, D|})^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}} \circ (-)^{\Vdash\Delta}_{X_2})(\alpha))(x,b)$. □

As a consequence, we obtain two morphisms of topological theories provided by a given system D:

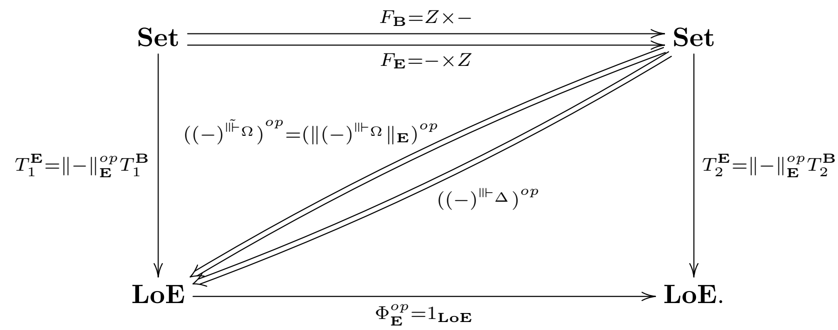


As an easy exercise, the reader is advised to describe the respective categories of fixed-basis variety-based topological spaces and the functors between them, which not only change the basis of the space (from Δ or Ω to Σ), but also modify the respective carrier set (from X to $X \times |\,\Omega\, D|$ or $|\,\Delta\, D| \times X$). To help with the exercise, we consider explicitly one of the above diagrams.

Remark 3.17. The left-hand diagram of the above-mentioned two gives a functor $\mathbf{Top}(T_1^{\mathbf{E}}) \xrightarrow{\mathrm{H}^{\Vvdash\Delta}} \mathbf{Top}(T_2^{\mathbf{E}})$, whose explicit form is given by the following data. The objects of the category $\mathbf{Top}(T_1^{\mathbf{E}})$ are pairs (X, τ), where X is a set and τ is an $\mathbf{E}$-subalgebra of the powerset $\mathbf{E}$-algebra $(\Delta\, D)^X$; and whose morphisms $(X,\tau) \xrightarrow{f} (Y,\sigma)$ are maps $X \xrightarrow{f} Y$ such that $f^{\leftarrow}_{\Delta\, D}(\alpha) \in \tau$ for every $\alpha \in \sigma$. On the other hand, the objects of the category $\mathbf{Top}(T_2^{\mathbf{E}})$ are pairs (X, τ), where X is a set and τ is an $\mathbf{E}$-subalgebra of the powerset $\mathbf{E}$-algebra $\|(\Sigma\, D)^X\|_{\mathbf{E}}$ (notice the $\mathbf{E}$-reduct); and whose morphisms $(X,\tau) \xrightarrow{f} (Y,\sigma)$ are maps $X \xrightarrow{f} Y$ such that $f^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}}(\alpha) \in \tau$ for every $\alpha \in \sigma$. Given a $\mathbf{Top}(T_1^{\mathbf{E}})$-morphism $(X,\tau) \xrightarrow{f} (Y,\sigma)$, $\mathrm{H}^{\Vvdash\Delta}((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X \times |\,\Omega\, D|, \hat{\tau}) \xrightarrow{f} (Y \times |\,\Omega\, D|, \hat{\sigma})$, where $\hat{\tau} = \{\alpha_X^{\Vvdash\Delta} \mid \alpha \in \tau\}$, and similarly for $\hat{\sigma}$.

An attentive reader will immediately notice the striking similarity in the construction of both morphisms. A natural question on their coincidence has at least the following simple answer.

Let $(\|-\|, \mathbf{E})$ be a reduct of $\mathbf{B}$ and let D be an object of $(\mathbf{E}, \mathbf{C}, \mathbf{B})$-**AlgTopSys** such that $\Delta\, D = \|\,\Omega\, D\|_{\mathbf{E}}$. Assuming that $|\,\Delta\, D| = Z = |\,\Omega\, D|$ and merging the above diagrams, we obtain the following one:



Given a set X, define the map $X \times Z \xrightarrow{\ell_X} Z \times X$ by $\ell_X(x,z) = (z,x)$ (the standard bijection) and obtain the $\mathbf{E}$-isomorphism $(\|\,\Sigma\, D\|_{\mathbf{E}})^{Z\times X} \xrightarrow{(\ell_X)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}}} (\|\,\Sigma\, D\|_{\mathbf{E}})^{X \times Z}$. The question on coincidence of the above-mentioned topological theory morphisms can be now answered as follows.

Lemma 3.18. *Let* $(\|-\|, \mathbf{E})$ *be a reduct of* $\mathbf{B}$, *let* D *be an* $(\mathbf{E}, \mathbf{C}, \mathbf{B})$-**AlgTopSys**-*object such that* $\Delta\, D = \|\,\Omega\, D\|_{\mathbf{E}}$ *and let* $|\,\Delta\, D| = Z = |\,\Omega\, D|$. *For every non-empty set* X, $(\Delta\, D)^X \xrightarrow{(-)_X^{\tilde{\Vvdash}\Omega}} (\|\,\Sigma\, D\|_{\mathbf{E}})^{Z\times X} \xrightarrow{(\ell_X)^{\leftarrow}_{\|\,\Sigma\, D\|_{\mathbf{E}}}} (\|\,\Sigma\, D\|_{\mathbf{E}})^{X\times Z} = (\Delta\, D)^X \xrightarrow{(-)_X^{\Vvdash\Delta}} (\|\,\Sigma\, D\|_{\mathbf{E}})^{X\times Z}$ *iff* $\models(z_1, z_2) = \models(z_2, z_1)$ *for every* $z_1, z_2 \in Z$.

Proof. For the sufficiency, notice that given $\alpha \in (\Delta\, D)^X$, for every $(x,z) \in X \times Z$, $(((\ell_X)^{\leftarrow}_{\|\Sigma\, D\|_{\mathbf{E}}} \circ (-)^{\tilde{\Vvdash}\Omega}_X)(\alpha))(x,z) = \alpha^{\Vvdash\Omega}_X \circ \ell_X(x,z) = \alpha^{\Vvdash\Omega}_X(z,x) = \models(z,\alpha(x)) = \models(\alpha(x),z) = \alpha^{\Vvdash\Delta}_X(x,z) = ((-)^{\Vvdash\Delta}_X(\alpha))(x,z)$.

For the necessity, notice first that given $\alpha \in (\Delta\, D)^X$, for every $(x,z) \in X \times Z$, the above computations provide $\models(z,\alpha(x)) = \models(\alpha(x),z)$. Now given $z_1, z_2 \in Z$, use the constant map $X \xrightarrow{\underline{z_2}} \Delta\, D$ (recall the definition from the last lines of Subsection 2.1) and the existence of some element $x_0 \in X$, to obtain that $\models(z_1,z_2) = \models(z_1,\underline{z_2}(x_0)) = \models(\underline{z_2}(x_0),z_1) = \models(z_2,z_1)$. □

For future reference, we call the systems satisfying the conditions of Lemma 3.18 *symmetric*. By the lemma, symmetric systems provide essentially one morphism of topological theories. It is also the case that both morphisms (almost) coincide in the case of $\mathbf{E} = \mathbf{B}$ (recall from Lemma 3.15 that the codomains of $(-)^{\Vvdash\Delta}_X$ and $(-)^{\Vvdash\Omega}_X$ involve an $\mathbf{E}$-reduct and a $\mathbf{B}$-reduct of $\Sigma\, D$, respectively).

3.4. **Variable-basis Topological Theory Morphisms Induced by algebraically-topological Systems.** This subsection continues the topic of the previous one, extending the approach to variable-basis, namely, to the whole categories $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys** and $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}_m$, respectively. Notice that the distinction between the respective topological theory morphisms, generated by the categories in question, never appeared in the previous subsection, since the employed fixed-basis case disguised the difference in their morphisms. It is precisely the variable-basis case, which visibly separates them.

The case of the category $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys** essentially extends the fixed-basis approach provided by the second item of Lemma 3.15. We begin by introducing several new functors:

(1) the functor $\mathbf{Set} \times (\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys} \xrightarrow{T^{ATS}_{\Omega}} \mathbf{LoB}$ is given by the formula $T^{ATS}_{\Omega}((X_1,D_1) \xrightarrow{(f,g)} (X_2,D_2)) = (\Omega\, D_1)^{X_1} \xrightarrow{((f,(\Omega\, g)^{op})^{\leftarrow})^{op}} (\Omega\, D_2)^{X_2}$;

(2) the functor $\mathbf{Set} \times (\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys} \xrightarrow{K^{ATS}_{\Delta\times}} \mathbf{Set} \times \mathbf{LoB}$ is given by $K^{ATS}_{\Delta\times}((X_1,D_1) \xrightarrow{(f,g)} (X_2,D_2)) = (|\,\Delta\, D_1| \times X_1, \|\Sigma\, D_1\|_{\mathbf{B}}) \xrightarrow{(\Delta\, g \times f, (\Sigma\, g)^{op})} (|\,\Delta\, D_2| \times X_2, \|\Sigma\, D_2\|_{\mathbf{B}})$;

(3) $\mathbf{Set} \times \mathbf{LoB} \xrightarrow{T_{\|\Sigma\|_{\mathbf{B}}}} \mathbf{LoB} = \mathbf{Set} \times \mathbf{LoB} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoB}$.

Lemma 3.19. *There exists a natural transformation* $T_{\|\Sigma\|_{\mathbf{B}}} K^{ATS}_{\Delta\times} \xrightarrow{\eta} T^{ATS}_{\Omega}$ *defined by* $T^{ATS}_{\Omega}(X,D) \xrightarrow{\eta^{op}_{(X,D)}} T_{\|\Sigma\|_{\mathbf{B}}} K^{ATS}_{\Delta\times}(X,D) = (\Omega\, D)^X \xrightarrow{(-)^{\Vvdash\Omega}_X} (\|\Sigma\, D\|_{\mathbf{B}})^{|\,\Delta\, D| \times X}$.

Proof. Given a $\mathbf{Set} \times (\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys**-morphism $(X_1,D_1) \xrightarrow{(f,g)} (X_2,D_2)$, we have to verify commutativity of the following diagram:

$$\begin{array}{ccc}
(\Omega\, D_2)^{X_2} & \xrightarrow{(-)^{\Vvdash\Omega}_{X_2}} & (\|\Sigma\, D_2\|_{\mathbf{B}})^{|\,\Delta\, D_2| \times X_2} \\
{\scriptstyle (f,(\Omega\, g)^{op})^{\leftarrow}} \big\downarrow & & \big\downarrow {\scriptstyle (\Delta\, g \times f, (\Sigma\, g)^{op})^{\leftarrow}} \\
(\Omega\, D_1)^{X_1} & \xrightarrow[(-)^{\Vvdash\Omega}_{X_1}]{} & (\|\Sigma\, D_1\|_{\mathbf{B}})^{|\,\Delta\, D_1| \times X_1}.
\end{array}$$

Given $\alpha \in (\Omega\, D_2)^{X_2}$, for every $(e,x) \in |\,\Delta\, D_1| \times X_1$, it follows that $(((-)^{\Vdash\Omega}_{X_1} \circ (f,(\Omega\, g)^{op})^{\leftarrow})(\alpha))(e,x) = (\Omega\, g \circ \alpha \circ f)^{\Vdash\Omega}_{X_1}(e,x) = \models_1(e, \Omega\, g \circ \alpha \circ f(x)) = \Sigma\, g \circ \models_2(\Delta\, g(e), \alpha \circ f(x)) = \Sigma\, g \circ \alpha^{\Vdash\Omega}_{X_2}(\Delta\, g(c), f(x)) = (\Sigma\, g \circ \alpha^{\Vdash\Omega}_{X_2} \circ (\Delta\, g \times f))(e,x) = (((\Delta\, g \times f, (\Sigma\, g)^{op})^{\leftarrow} \circ (-)^{\Vdash\Omega}_{X_2})(\alpha))(e,x)$. □

As a result, we obtain a variable-basis topological theory morphism (notice that the theory T^{ATS}_{Ω} is no longer restricted to the variety-based approach, but uses the full force of the catalg developments):

$$\begin{array}{ccc}
\mathbf{Set} \times (\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys} & \xrightarrow{K^{ATS}_{\Delta\,\times}} & \mathbf{Set} \times \mathbf{LoB} \\
{\scriptstyle T^{ATS}_{\Omega}} \downarrow & \overset{\eta}{\swarrow\!\!\!\!\swarrow} & \downarrow {\scriptstyle T_{\|\,\Sigma\,\|_{\mathbf{B}}}} \\
\mathbf{LoB} & \xrightarrow[\Phi^{op}_{\Omega} = 1_{\mathbf{LoB}}]{} & \mathbf{LoB}.
\end{array}$$

To give the reader more intuition in the new concepts, we describe explicitly the respective categories of topological structures and the induced functor, which transforms a truly catalg space into a variety-based one, changing its respective carrier set. Notice that we deliberately have chosen the functor, which corresponds to the right-hand diagram of the explicitly considered fixed-basis approach (Remark 3.17).

Remark 3.20. The above diagram gives a functor $\mathbf{Top}(T^{ATS}_{\Omega}) \xrightarrow{\mathrm{H}^{\eta}} \mathbf{Top}(T_{\|\,\Sigma\,\|_{\mathbf{B}}})$, whose explicit form is given by the following data. The objects of the category $\mathbf{Top}(T^{ATS}_{\Omega})$ are triples (X, D, τ), where X is a set, D is an $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys**-object (algebraically-topological system), whereas τ is a $\mathbf{B}$-subalgebra of the powerset $\mathbf{B}$-algebra $(\Omega\, D)^X$; and whose morphisms $(X_1, D_1, \tau_1) \xrightarrow{(f,g)} (X_2, D_2, \tau_2)$ are $\mathbf{Set} \times (\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys**-morphisms $(X_1, D_1) \xrightarrow{(f,g)} (X_2, D_2)$ such that $(f,(\Omega\, g)^{op})^{\leftarrow}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$. On the other hand, the objects of the category $\mathbf{Top}(T_{\|\,\Sigma\,\|_{\mathbf{B}}})$ are triples (X, B, τ), where X is a set, B is a $\mathbf{B}$-algebra, whereas τ is a $\mathbf{B}$-subalgebra of the powerset $\mathbf{B}$-algebra B^X; and whose morphisms $(X_1, B_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, B_2, \tau_2)$ are $\mathbf{Set} \times \mathbf{LoB}$-morphisms $(X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)$ such that $(f,\varphi)^{\leftarrow}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$. Given a $\mathbf{Top}(T^{ATS}_{\Omega})$-morphism $(X_1, D_1, \tau_1) \xrightarrow{(f,g)} (X_2, D_2, \tau_2)$, it follows that $\mathrm{H}^{\eta}((X_1, D_1, \tau_1) \xrightarrow{(f,g)} (X_2, D_2, \tau_2)) = (|\,\Delta\, D_1| \times X_1, \|\,\Sigma\, D_1\|_{\mathbf{B}}, \hat{\tau}_1) \xrightarrow{(\Delta\, g \times f, (\Sigma\, g)^{op})} (|\,\Delta\, D_2| \times X_2, \|\,\Sigma\, D_2\|_{\mathbf{B}}, \hat{\tau}_2)$, where $\hat{\tau}_i = \{\alpha^{\Vdash\Omega}_{X_i} \mid \alpha \in \tau_i\}$ (recall Lemma 3.19).

The extension of the first item of Lemma 3.15 brings the modified category of systems in play. Following the pattern, we again begin with the introduction of new functors:

(1) the functor $\mathbf{Set} \times ((\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys}_m)^{op} \xrightarrow{T^{ATS_m}_{\Delta}} \mathbf{LoE}$ is defined by $T^{ATS_m}_{\Delta}((X_1, D_1) \xrightarrow{(f,g^{op})} (X_2, D_2)) = (\Delta\, D_1)^{X_1} \xrightarrow{((f,(\Delta\, g)^{op})^{\leftarrow})^{op}} (\Delta\, D_2)^{X_2}$;

(2) the functor $\mathbf{Set}\times((\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys}_m)^{op} \xrightarrow{K^{ATS_m}_{\times\,\Omega}} \mathbf{Set}\times\mathbf{LoE}$ is given by $K^{ATS_m}_{\times\,\Omega}((X_1,D_1)\xrightarrow{(f,g^{op})}(X_2,D_2)) = (X_1\times|\,\Omega\,D_1|, \|\,\Sigma\,D_1\|_{\mathbf{E}}) \xrightarrow{(f\times\Omega\,g,(\Sigma\,g)^{op})} (X_2\times|\,\Omega\,D_2|, \|\,\Sigma\,D_2\|_{\mathbf{E}})$;

(3) $\mathbf{Set}\times\mathbf{LoE} \xrightarrow{T_{\|\,\Sigma\,\|_{\mathbf{E}}}} \mathbf{LoE} = \mathbf{Set}\times\mathbf{LoE} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoE}$.

Lemma 3.21. *There exists a natural transformation* $T_{\|\,\Sigma\,\|_{\mathbf{E}}} K^{ATS_m}_{\times\,\Omega} \xrightarrow{\varepsilon} T^{ATS_m}_{\Delta}$, $T^{ATS_m}_{\Delta}(X,D) \xrightarrow{\varepsilon^{op}_{(X,D)}} T_{\|\,\Sigma\,\|_{\mathbf{E}}} K^{ATS_m}_{\times\,\Omega}(X,D) = (\Delta\,D)^X \xrightarrow{(-)^{\Vdash\Delta}_X} (\|\,\Sigma\,D\|_{\mathbf{E}})^{X\times|\,\Omega\,D|}$.

Proof. For a $\mathbf{Set}\times((\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys}_m)^{op}$-morphism $(X_1,D_1)\xrightarrow{(f,g^{op})}(X_2,D_2)$, we have to check commutativity of the following diagram:

$$\begin{array}{ccc}
(\Delta\,D_2)^{X_2} & \xrightarrow{(-)^{\Vdash\Delta}_{X_2}} & (\|\,\Sigma\,D_2\|_{\mathbf{E}})^{X_2\times|\,\Omega\,D_2|} \\
{\scriptstyle (f,(\Delta\,g)^{op})^{\leftarrow}}\downarrow & & \downarrow{\scriptstyle (f\times\Omega\,g,(\Sigma\,g)^{op})^{\leftarrow}} \\
(\Delta\,D_1)^{X_1} & \xrightarrow[(-)^{\Vdash\Delta}_{X_1}]{} & (\|\,\Sigma\,D_1\|_{\mathbf{E}})^{X_1\times|\,\Omega\,D_1|}.
\end{array}$$

Given $\alpha\in(\Delta\,D_2)^{X_2}$, for every $(x,b)\in X_1\times|\,\Omega\,D_1|$, it follows that $(((-)^{\Vdash\Delta}_{X_1}\circ(f,(\Delta\,g)^{op})^{\leftarrow})(\alpha))(x,b) = (\Delta\,g\circ\alpha\circ f)^{\Vdash\Delta}_{X_1}(x,b) = \models_1(\Delta\,g\circ\alpha\circ f(x),b) = \Sigma\,g\circ\models_2(\alpha\circ f(x),\Omega\,g(b)) = \Sigma\,g\circ\alpha^{\Vdash\Delta}_{X_2}(f(x),\Omega\,g(b)) = (\Sigma\,g\circ\alpha^{\Vdash\Delta}_{X_2}\circ(f\times\Omega\,g))(x,b) = (((f\times\Omega\,g,(\Sigma\,g)^{op})^{\leftarrow}\circ(-)^{\Vdash\Delta}_{X_2})(\alpha))(x,b)$. ☐

As a consequence, we obtain another variable-basis topological theory morphism (the reader is advised to construct the respective categories of topological structures and the induced functor; also notice the impossibility of the machinery in the case of the category $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}$):

$$\begin{array}{ccc}
\mathbf{Set}\times((\mathbf{E},\mathbf{C},\mathbf{B})\text{-}\mathbf{AlgTopSys}_m)^{op} & \xrightarrow{K^{ATS_m}_{\times\,\Omega}} & \mathbf{Set}\times\mathbf{LoE} \\
{\scriptstyle T^{ATS_m}_{\Delta}}\downarrow & \overset{\varepsilon}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle T_{\|\,\Sigma\,\|_{\mathbf{E}}}} \\
\mathbf{LoE} & \xrightarrow[\Phi^{op}_{\Delta}=1_{\mathbf{LoE}}]{} & \mathbf{LoE}.
\end{array}$$

In the previous subsection, we considered the question on coincidence of the obtained topological theory morphisms. Being simply solved in the fixed-basis case, the variable-basis challenge is more complicated (being highly dependant on the properties of the categories of systems, which should as yet be clarified) and will be investigated as a separate problem in one of our subsequent papers.

Up to now, we have briefly considered elements of the theory of algebraically-topological systems, paying much attention to its relation to the theory of variety-based topology. It appears, however, that there exists another topic of interest with respect to the new theory, namely, its relationships with the concept of *attachment* of C. Guido [23]. It is the purpose of the next section to elucidate the topic.

4. Algebraically-topological Systems and Attachments

The main goal of this section is to show that the theory of attachments of C. Guido [23] and its further variety-based extension of S. Solovyov [61] and A. Frascella *et al.* [19] can be conveniently incorporated in the framework of systems through their algebraically-topological extension, making attachments categorically redundant in mathematics. Our results answer the problem posed by C. Guido on the existence of a common setting for both concepts (attachment and topological system), already touched in [19], but erroneously deemed unfruitful due to significant differences in the nature of both notions.

4.1. Additional Characterizations of Algebraically-topological Systems.

To begin with, this subsection provides additional characterizations of algebraically-topological systems, which will be extremely helpful in relating them to attachments. The reader should keep in mind that while considering a triple of varieties $(\mathbf{A},\mathbf{B},\mathbf{E})$, we always assume that both $\mathbf{B}$ and $\mathbf{E}$ are reducts of $\mathbf{A}$.

Proposition 4.1. *Given an* $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$*-object* $(\Delta\, D,\Sigma\, D,\Omega\, D)$*, there exists a map* $\Delta\, D\times\Omega\, D\xrightarrow{\models}\Sigma\, D$ *providing an algebraically-topological system* D *iff the following conditions are fulfilled:*

1. *there exists a* $\mathbf{B}$*-homomorphism* $\Omega\, D\xrightarrow{\Vdash_\Omega}(\|\,\Sigma\, D\|_{\mathbf{B}})^{|\,\Delta\, D|}$*;*
2. *there exists an* $\mathbf{E}$*-homomorphism* $\Delta\, D\xrightarrow{\Vdash_\Delta}(\|\,\Sigma\, D\|_{\mathbf{E}})^{|\,\Omega\, D|}$*;*
3. $(\Vdash_\Omega(b))(e)=(\Vdash_\Delta(e))(b)$ *for every* $e\in\Delta\, D$ *and every* $b\in\Omega\, D$*.*

An $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$*-morphism* $(\Delta\, D_1,\Sigma\, D_1,\Omega\, D_1)\xrightarrow{f}(\Delta\, D_2,\Sigma\, D_2,\Omega\, D_2)$ *is an algebraically-topological system morphism iff* $(\Vdash_{\Omega_1}(\Omega\, f(b)))(e)=(\Sigma\, f\circ\Vdash_{\Omega_2}(b))(\Delta\, f(e))$ *iff* $(\Vdash_{\Delta_1}(e))(\Omega\, f(b))=(\Sigma\, f\circ\Vdash_{\Delta_2}(\Delta\, f(e)))(b)$ *for every* $e\in\Delta\, D_1$ *and every* $b\in\Omega\, D_2$*. An* $\mathbf{E}\times\mathbf{C}'\times\mathbf{LoB}$*-morphism* $(\Delta\, D_1,\Sigma\, D_1,\Omega\, D_1)\xrightarrow{f}(\Delta\, D_2,\Sigma\, D_2,\Omega\, D_2)$ *is a modified algebraically-topological system morphism iff* $(\Sigma\, f\circ\Vdash_{\Omega_1}(\Omega\, f(b)))(e)=(\Vdash_{\Omega_2}(b))(\Delta\, f(e))$ *iff* $(\Sigma\, f\circ\Vdash_{\Delta_1}(e))(\Omega\, f(b))=(\Vdash_{\Delta_2}(\Delta\, f(e)))(b)$ *for every* $e\in\Delta\, D_1$ *and every* $b\in\Omega\, D_2$*.*

Proof. For the necessity, we show the validity of the items in a row. For the first one, define the required map by $\Vdash_\Omega(b)={\models}(-,b)$. To prove that the map is a $\mathbf{B}$-homomorphism, notice that given $\lambda\in\Lambda_{\mathbf{B}}$ and $\langle b_i\rangle_{n_\lambda}\in(\Omega\, D)^{n_\lambda}$, for every $e\in\Delta\, D$, $(\Vdash_\Omega(\omega_\lambda^{\Omega\, D}(\langle b_i\rangle_{n_\lambda})))(e)={\models}(e,\omega_\lambda^{\Omega\, D}(\langle b_i\rangle_{n_\lambda}))=\omega_\lambda^{\|\,\Sigma\, D\|_{\mathbf{B}}}(\langle{\models}(e,b_i)\rangle_{n_\lambda})=\omega_\lambda^{\|\,\Sigma\, D\|_{\mathbf{B}}}(\langle(\Vdash_\Omega(b_i))(e)\rangle_{n_\lambda})=(\omega_\lambda^{(\|\,\Sigma\, D\|_{\mathbf{B}})^{|\,\Delta\, D|}}(\langle\Vdash_\Omega(b_i)\rangle_{n_\lambda}))(e)$. To show the second item, define the required map by $\Vdash_\Delta(e)={\models}(e,-)$ and follow the above procedure. For the last item, notice that given $e\in\Delta\, D$ and $b\in\Omega\, D$, $(\Vdash_\Omega(b))(e)={\models}(e,b)=(\Vdash_\Delta(e))(b)$.

To prove the sufficiency, we define the required map $\Delta\, D\times\Omega\, D\xrightarrow{\models}\Sigma\, D$ by ${\models}(e,b)=(\Vdash_\Omega(b))(e)=(\Vdash_\Delta(e))(b)$. To verify that the map is an $(\mathbf{E},\mathbf{B})$-bimorphism, notice that given $e\in\Delta\, D$, for every $\lambda\in\Lambda_{\mathbf{B}}$ and every $\langle b_i\rangle_{n_\lambda}\in(\Omega\, D)^{n_\lambda}$,

$\models(e,\omega_\lambda^{\Omega D}(\langle b_i\rangle_{n_\lambda})) = (\Vdash_\Omega(\omega_\lambda^{\Omega D}(\langle b_i\rangle_{n_\lambda})))(e) = (\omega_\lambda^{(\|\Sigma D\|_{\mathbf{B}})^{|\Delta D|}}(\langle \Vdash_\Omega(b_i)\rangle_{n_\lambda}))(e) = \omega_\lambda^{\|\Sigma D\|_{\mathbf{B}}}(\langle(\Vdash_\Omega(b_i))(e)\rangle_{n_\lambda}) = \omega_\lambda^{\|\Sigma D\|_{\mathbf{B}}}(\langle\models(e,b_i)\rangle_{n_\lambda})$. For $b\in\Omega D$, use the map $\Vdash_\Delta$ in the above machinery.

For the last claim, notice that the system morphism condition $\models_1(e,\Omega f(b)) = \Sigma f\circ\models_2(\Delta f(e),b)$ can be rewritten as $(\Vdash_{\Omega_1}(\Omega f(b)))(e)=(\Sigma f\circ\Vdash_{\Omega_2}(b))(\Delta f(e))$ or $(\Vdash_{\Delta_1}(e))(\Omega f(b))=(\Sigma f\circ\Vdash_{\Delta_2}(\Delta f(e)))(b)$. □

With the help of Example 3.4, we obtain a characterization of variety-based topological systems (already noticed by J. T. Denniston *et al.* [10] in the particular case of frames).

Corollary 4.2. *Given a* **Set** $\times$ **C** $\times$ **LoB***-object* $(\Delta D,\Sigma D,\Omega D)$*, there exists a map* $\Delta D\times\Omega D\xrightarrow{\models}\Sigma D$ *providing a topological system* D *iff there exists a* **B***-homomorphism* $\Omega D\xrightarrow{\Vdash}(\|\Sigma D\|_{\mathbf{B}})^{|\Delta D|}$.

Proposition 4.3. *Given an* **E** $\times$ **C** $\times$ **LoB***-object* $(\Delta D,\Sigma D,\Omega D)$*, there exists a map* $\Delta D\times\Omega D\xrightarrow{\models}\Sigma D$ *providing an algebraically-topological system* D *iff the following conditions are fulfilled:*

(1) *there exists a map* $\Omega D\xrightarrow{\Vdash_\Omega}\mathbf{E}(\Delta D,\|\Sigma D\|_{\mathbf{E}})$*;*
(2) *there exists a map* $\Delta D\xrightarrow{\Vdash_\Delta}\mathbf{B}(\Omega D,\|\Sigma D\|_{\mathbf{B}})$*;*
(3) $(\Vdash_\Omega(b))(e)=(\Vdash_\Delta(e))(b)$ *for every* $e\in\Delta D$ *and every* $b\in\Omega D$.

Proof. For the necessity, notice that the required maps can be defined by $\Vdash_\Omega(b) = \models(-,b)$ and $\Vdash_\Delta(e)=\models(e,-)$, respectively. For the sufficiency, define the required $(\mathbf{E},\mathbf{B})$-bimorphism $\Delta D\times\Omega D\xrightarrow{\models}\Sigma D$ by $\models(e,b)=(\Vdash_\Omega(b))(e)=(\Vdash_\Delta(e))(b)$. □

Corollary 4.4. *Given a* **Set** $\times$ **C** $\times$ **LoB***-object* $(\Delta D,\Sigma D,\Omega D)$*, there exists a map* $\Delta D\times\Omega D\xrightarrow{\models}\Sigma D$ *providing a topological system* D *iff there exists a map* $\Delta D\xrightarrow{\Vdash}\mathbf{B}(\Omega D,\|\Sigma D\|_{\mathbf{B}})$.

The above propositions allow two alternative definitions of the category of algebraically-topological systems, one of which is provided below (leaving the missing one to the reader).

Definition 4.5. Given a reduct $(\|-\|,\mathbf{E})$ of **A**, $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}^{\Vdash}$ is the category, concrete over the product category $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$, whose objects are quintuples $D=(\Delta D,\Sigma D,\Omega D,\Vdash_\Delta,\Vdash_\Omega)$, comprising an $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$-object $(\Delta D,\Sigma D,\Omega D)$, an **E**-homomorphism $\Delta D\xrightarrow{\Vdash_\Delta}(\|\Sigma D\|_{\mathbf{E}})^{|\Omega D|}$ and a **B**-homomorphism $\Omega D\xrightarrow{\Vdash_\Omega}(\|\Sigma D\|_{\mathbf{B}})^{|\Delta D|}$ such that $(\Vdash_\Omega(b))(e)=(\Vdash_\Delta(e))(b)$ for every $e\in\Delta D$ and every $b\in\Omega D$. Morphisms $D_1\xrightarrow{f=(\Delta f,(\Sigma f)^{op},(\Omega f)^{op})}D_2$ are $\mathbf{E}\times\mathbf{C}\times\mathbf{LoB}$-morphisms $(\Delta D_1,\Sigma D_1,\Omega D_1)\xrightarrow{f}(\Delta D_2,\Sigma D_2,\Omega D_2)$ such that for every $e\in\Delta D_1$ and every $b\in\Omega D_2$, it follows that $(\Vdash_{\Delta_1}(e))(\Omega f(b))=(\Sigma f\circ\Vdash_{\Delta_2}(\Delta f(e)))(b)$ or, alternatively, $(\Vdash_{\Omega_1}(\Omega f(b)))(e)=(\Sigma f\circ\Vdash_{\Omega_2}(b))(\Delta f(e))$.

In view of Proposition 4.1, it follows that the functor $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys} \xrightarrow{F} (\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}^{\Vdash\!\vdash}$, which is defined explicitly by the formula $F(D_1 \xrightarrow{f} D_2) = (\Delta\, D_1, \Sigma\, D_1, \Omega\, D_1, \Vvdash_{\Delta_1}, \Vvdash_{\Omega_1}) \xrightarrow{f} (\Delta\, D_2, \Sigma\, D_2, \Omega\, D_2, \Vvdash_{\Delta_2}, \Vvdash_{\Omega_2})$, where $\Vvdash_{\Delta_i}(e) = \models_i(e,-)$ and $\Vvdash_{\Omega_i}(b) = \models_i(-,b)$, is an isomorphism.

Using the pattern of Definition 4.5, but relying on Proposition 4.3 instead of Proposition 4.1 (namely, using, firstly, the maps $\Omega\, D \xrightarrow{\Vdash_\Omega} \mathbf{E}(\Delta\, D, \|\,\Sigma\, D\|_{\mathbf{E}})$ and $\Delta\, D \xrightarrow{\Vdash_\Delta} \mathbf{B}(\Omega\, D, \|\,\Sigma\, D\|_{\mathbf{B}})$ instead of an $\mathbf{E}$-homomorphism $\Delta\, D \xrightarrow{\Vvdash_\Delta} (\|\,\Sigma\, D\|_{\mathbf{E}})^{|\,\Omega\, D|}$ and a $\mathbf{B}$-homomorphism $\Omega\, D \xrightarrow{\Vvdash_\Omega} (\|\,\Sigma\, D\|_{\mathbf{B}})^{|\,\Delta\, D|}$, and, secondly, using the condition $(\Vdash_\Omega(b))(e) = (\Vdash_\Delta(e))(b)$ instead of the condition $(\Vvdash_\Omega(b))(e) = (\Vvdash_\Delta(e))(b)$), one can define the category $(\mathbf{E},\mathbf{C},\mathbf{B})$-$\mathbf{AlgTopSys}^{\Vdash}$ and the respective isomorphism to the standard category of systems.

4.2. **Attachment Versus Algebraically-topological Systems.** With the characterizations of algebraically-topological systems provided in the previous subsections in hand, we are able now to show the setting to incorporate the concept of attachment of C. Guido [23] and its variety-based modification of S. Solovyov [61] and A. Frascella *et al.* [19].

As a starting point, we consider an analogue of the respective concept of [61], which extended [23] and motivated [19] (the reader is advised to recall the shortened notation for the category of algebraically-topological systems from Definition 3.3).

Definition 4.6. Given a variety $\mathbf{A}$, $\mathbf{AttSA}$ is the full subcategory of the category $(\mathbf{Set},\mathbf{LoA})$-$\mathbf{AlgTopSys}^{\Vdash}$, the objects of which (*variety-based attachment systems* or $\mathbf{A}$-*attachment systems*) are precisely those systems D for which $\Delta\, D = |\,\Omega\, D|$.

Notice that $(\mathbf{Set},\mathbf{LoA})$-$\mathbf{AlgTopSys}^{\Vdash}$ is nothing else than a particular instance of the category of variety-based topological systems of, e.g., [62]. Strikingly enough, it appears that we have restricted ourselves to the already much investigated and studied ground.

To provide more intuition for the concept, we show an explicit definition of the category $\mathbf{AttSA}$.

Definition 4.7. Given a variety $\mathbf{A}$, $\mathbf{AttSA}$ is the category, concrete over the product category $\mathbf{Set}\times\mathbf{LoA}\times\mathbf{LoA}$, whose objects are tuples $D = (|\,\Omega\, D|, \Sigma\, D, \Omega\, D, \Vdash)$, comprising a $\mathbf{Set}\times\mathbf{LoA}\times\mathbf{LoA}$-object $(|\,\Omega\, D|, \Sigma\, D, \Omega\, D)$ and a map $|\,\Omega\, D| \xrightarrow{\Vdash} \mathbf{A}(\Omega\, D, \Sigma\, D)$, and whose morphisms $D_1 \xrightarrow{f=(\Delta\, f,(\Sigma\, f)^{op},(\Omega\, f)^{op})} D_2$ are $\mathbf{Set}\times\mathbf{LoA}\times\mathbf{LoA}$-morphisms $(|\,\Omega\, D_1|, \Sigma\, D_1, \Omega\, D_1) \xrightarrow{f} (|\,\Omega\, D_2|, \Sigma\, D_2, \Omega\, D_2)$ such that for every $a_1 \in \Omega\, D_1$ and every $a_2 \in \Omega\, D_2$, $(\Vdash_1(a_1))(\Omega\, f(a_2)) = (\Sigma\, f \circ \Vdash_2(\Delta\, f(a_1)))(a_2)$.

In the next step, we recall variety-based attachments from [61, Definition 3].

Definition 4.8. Let $\mathbf{A}$ be a variety of algebras and let $\mathbf{A} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ be a functor such that $A^* = |A|$ for every $\mathbf{A}$-algebra A. An $(\mathbf{A}$-$)$*attachment* is a triple $F = (\Omega\, F, \Sigma\, F, \Vdash)$, where $\Omega\, F$ and $\Sigma\, F$ are $\mathbf{A}$-algebras, and $\Omega\, F \xrightarrow{\Vdash} \mathbf{A}(\Omega\, F, \Sigma\, F)$ is a map. An *attachment morphism* $F_1 \xrightarrow{f} F_2$ is a pair of $\mathbf{A}$-homomorphisms

$(\Omega\, F_1, \Sigma F_1) \xrightarrow{(\Omega\, f, \Sigma\, f)} (\Omega\, F_2, \Sigma\, F_2)$ such that for every $a_1 \in \Omega\, F_1$ and every $a_2 \in \Omega\, F_2$, $(\Vdash_2(a_2))(\Omega\, f(a_1)) = (\Sigma\, f \circ \Vdash_1((\Omega\, f)^{*op}(a_2)))(a_1)$. **AttA** is the category of attachments and their homomorphisms, concrete over the product category $\mathbf{A} \times \mathbf{A}$.

Comparing Definitions 4.7 and 4.8, one arrives at the following functor, which provides the main relationship between attachment systems and attachments.

Proposition 4.9. *There exists a non-full embedding* $\mathbf{AttA} \xhookrightarrow{\mathrm{E}} \mathbf{AttSA}^{op}$ *given by* $\mathrm{E}(F_1 \xrightarrow{f} F_2) = (|\,\Omega\, F_1|, \Sigma\, F_1, \Omega\, F_1, \Vdash_1) \xrightarrow{((\Omega\, f)^*, \Sigma\, f, \Omega\, f)} (|\,\Omega\, F_2|, \Sigma\, F_2, \Omega\, F_2, \Vdash_2)$.

It appears that the concept of algebraically-topological system (even its variety-based topological system instance) provides a proper extension of the notion of attachment, making the latter categorically redundant in mathematics.

Having successfully incorporated the notion of variety-based attachment, we turn to its dual counterpart of [19]. Notice that the idea of dualization stemmed from the wish of C. Guido to define his original attachment concept, based in filters, in terms of ideals. It was the main purpose of [19] to provide a rigid variety-based background for the attachment duality procedure.

Definition 4.10. Given a variety **B**, **ATTSB** is the full subcategory of the category $(\mathbf{Set}, \mathbf{LoB})$-$\mathbf{AlgTopSys}^{\Vvdash}$, the objects of which (*dual attachment systems*) are precisely those systems D for which $\Delta\, D = |\,\Omega\, D|$.

For convenience of the reader, we recall the respective definition from [19].

Definition 4.11. Let **B** be a variety and let $\mathbf{B} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ be a functor such that $B^* = |B|$ for every **B**-algebra B. A *dual* (**B**-)*attachment* is a triple $G = (\Omega\, G, \Sigma\, G, \Vvdash)$, where $\Omega\, G$ and $\Sigma\, G$ are **B**-algebras, and $\Omega\, G \xrightarrow{\Vvdash} \Sigma\, G^{|\,\Omega\, G|}$ is a **B**-homomorphism. A *dual attachment morphism* $G_1 \xrightarrow{f} G_2$ is a pair of **B**-homomorphisms $(\Omega\, G_1, \Sigma\, G_1) \xrightarrow{(\Omega\, f, \Sigma\, f)} (\Omega\, G_2, \Sigma\, G_2)$ such that for every $b_1 \in \Omega\, G_1$ and every $b_2 \in \Omega\, G_2$, $(\Vvdash_2(\Omega\, f(b_1)))(b_2) = (\Sigma\, f \circ \Vvdash_1(b_1))((\Omega\, f)^{*op}(b_2))$. **ATTB** is the category of dual attachments and their morphisms, which is concrete over the product category $\mathbf{B} \times \mathbf{B}$.

Comparing Definitions 4.10 and 4.11, one arrives at another functor, which provides the main relationship between dual attachment systems and dual attachments.

Proposition 4.12. *There exists a non-full embedding* $\mathbf{ATTB} \xhookrightarrow{\mathrm{E}} \mathbf{ATTSB}^{op}$, $\mathrm{E}(G_1 \xrightarrow{f} G_2) = (|\,\Omega\, G_1|, \Sigma\, G_1, \Omega\, G_1, \Vvdash_1) \xrightarrow{((\Omega\, f)^*, \Sigma\, f, \Omega\, f)} (|\,\Omega\, G_2|, \Sigma\, G_2, \Omega\, G_2, \Vvdash_2)$.

Proposition 4.14 essentially says that the concept of dual attachment is categorically redundant in mathematics. It is one of the intended goals of this paper to motivate the interested researchers to use the notion of (dual) attachment system instead of (dual) attachment.

4.3. **Topological Theory Morphisms Induced by (Dual) Attachment Systems.** One of the main justifications of the concept of attachment of C. Guido [23] (which was considered in the framework of complete lattices, but mostly employing frames) was its generated functor from the category L-**Top** of L-topological spaces [28] to the category **Top** of topological spaces, which appeared to be an embedding in the case of *spatial* attachment. In [61], S. Solovyov showed that the functor stems from the realm of systems and is much related to the *hypergraph functor* of the fuzzy community [27]. Following [19], the functor of C. Guido and its extension of [61] take their origin from a particular morphism of topological theories, induced by an attachment. On the other hand, in Subsection 3.4, we provided topological theory morphisms generated by algebraically-topological systems. Moreover, in the previous subsection, we have finally arrived at the conclusion that the concept of algebraically-topological system extends that of attachment. It is the main purpose of this subsection to provide the analogues of the attachment induced topological theory morphisms in the current setting.

To follow the sequence of appearance of the concepts, we begin with the category **AttSA**. There exist the following three functors (cf. the respective functors of Subsection 3.4):

(1) $\mathbf{Set}\times\mathbf{AttSA} \xrightarrow{T_\Omega^{AttS}} \mathbf{LoA}$, which is defined by $T_\Omega^{AttS}((X_1, D_1) \xrightarrow{(f,g)} (X_2, D_2)) = (\Omega\, D_1)^{X_1} \xrightarrow{((f,(\Omega\, g)^{op})^{\leftarrow})^{op}} (\Omega\, D_2)^{X_2}$;

(2) $\mathbf{Set} \times \mathbf{AttSA} \xrightarrow{K_{\Omega\,\times}^{AttS}} \mathbf{Set} \times \mathbf{LoA}$, $K_{\Omega\,\times}^{AttS}((X_1, D_1) \xrightarrow{(f,g)} (X_2, D_2)) = (|\,\Omega\, D_1| \times X_1, \Sigma\, D_1) \xrightarrow{(\Delta\, g\times f,(\Sigma\, g)^{op})} (|\,\Omega\, D_2| \times X_2, \Sigma\, D_2)$;

(3) $\mathbf{Set} \times \mathbf{LoA} \xrightarrow{T_\Sigma^{AttS}} \mathbf{LoA} = \mathbf{Set} \times \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$.

From Lemma 3.19 and the comments afterwards as well as Proposition 4.3, one obtains the following topological theory morphism:

$$\begin{array}{ccc} \mathbf{Set} \times \mathbf{AttSA} & \xrightarrow{K_{\Omega\,\times}^{AttS}} & \mathbf{Set} \times \mathbf{LoA} \\ {\scriptstyle T_\Omega^{AttS}}\downarrow & \swArrow{\eta^{AttS}} & \downarrow{\scriptstyle T_\Sigma^{AttS}} \\ \mathbf{LoA} & \xrightarrow[\Phi^{op}=1_{\mathbf{LoA}}]{} & \mathbf{LoA}, \end{array}$$

where $T_\Sigma^{AttS} K_{\Omega\,\times}^{AttS} \xrightarrow{\eta^{AttS}} T_\Omega^{AttS}$ is given by the formula $T_\Omega^{AttS}(X, D) \xrightarrow{(\eta_{(X,D)}^{AttS})^{op}} T_\Sigma^{AttS} K_{\Omega\,\times}^{AttS}(X, D) = (\Omega\, D)^X \xrightarrow{(-)^{\Vdash}_{(X,D)}} (\Sigma\, D)^{|\,\Omega\, D| \times X}$, $\alpha^{\Vdash}_{(X,D)}(a, x) = \alpha_X^{\Vvdash_\Omega}(a, x) = (\models(a, \alpha(x)) =)(\Vdash(a))(\alpha(x))$ (for a better understanding, we used the satisfaction relation $\models$). The just defined morphism of topological theories provides an extension of the respective morphism of [61] (at that time never defined explicitly, due to the lack of its underlying machinery), whose fixed-basis restriction (the reader is advised to recall Subsection 3.3) gives rise to the above-mentioned functor of C. Guido and the hypergraph functor.

The case of the category **ATTSB** is similar (use Proposition 4.1), the respective natural transformation $T_{\Sigma}^{ATTS} K_{\Omega\times}^{ATTS} \xrightarrow{\eta^{ATTS}} T_{\Omega}^{ATTS}$ defined by the formula $T_{\Omega}^{ATTS}(X,D) \xrightarrow{(\eta^{ATTS}_{(X,D)})^{op}} T_{\Sigma}^{ATTS} K_{\Omega\times}^{ATTS}(X,D) = (\Omega\, D)^X \xrightarrow{(-)^{\Vdash\!\vdash}_{(X,D)}} (\Sigma\, D)^{|\,\Omega\, D|\times X}$, $\alpha^{\Vdash\!\vdash}_{(X,D)}(a,x) = \alpha_X^{\Vdash\!\vdash\Omega}(a,x) = (\models(a,\alpha(x)) =)(\Vdash\!\vdash(\alpha(x)))(a)$. The obtained topological theory morphism is an analogue of that of [19] for dual attachments.

The reader will easily recall the topological theory morphism given just after Lemma 3.21. It appears that the morphism has not yet been considered in the attachment framework. It will be the topic of our subsequent papers to study the morphism and its generated functor between the categories of topological structures.

4.4. **Dual Attachment Pairs.** We have already mentioned in the paper the concept of duality in the setting of attachments (see [19] for a throughout discussion of the topic). As follows from the obtained results, every (dual) attachment (system) gives rise to a functor between the categories of catalg topological spaces. The concept of attachment duality provides a machinery to determine, when the fixed-basis functors generated by an attachment and a dual attachment coincide, the respective pair of attachments for such an occurrence being called a *dual attachment pair*. It appears that algebraically-topological systems provide a convenient framework to develop further the theory of attachment duality started in [19]. Moreover, it is precisely this ability to incorporate attachment duality, which shows the advantage of the latter structures over topological systems. As the reader will see, the notion of attachment duality requires an extended version of systems, which do have an algebra instead of a set as their first component (cf. Example 3.4).

Let D be a system from the category $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys** such that there exists an $\mathbf{A}$-algebra A with $\|A\|_{\mathbf{E}} = \Delta\, D$, $\|A\|_{\mathbf{B}} = \Omega\, D$ and, moreover, $\models(a_1,a_2) = \models(a_2,a_1)$ for every $a_1,a_2 \in A$.

From the first item of Proposition 4.1, one obtains a $\mathbf{B}$-homomorphism $\Omega\, D \xrightarrow{\Vdash\!\vdash} (\|\,\Sigma\, D\|_{\mathbf{B}})^{(|\,\Delta\, D| = |\,\Omega\, D|)}$, which is given by $\Vdash\!\vdash(b) = \models(-,b)$ and, therefore, $G = (|\,\Omega\, D|, \|\,\Sigma\, D\|_{\mathbf{B}}, \Omega\, D, \Vdash\!\vdash)$ is a dual attachment system, or, more precisely, an object of the category **ATTSB**. On the other hand, by the first item of Proposition 4.3, there exists a map $(|\,\Omega\, D| = |\,\Delta\, D|) \xrightarrow{\Vdash} \mathbf{E}(\Delta\, D, \|\,\Sigma\, D\|_{\mathbf{E}})$ defined by $\Vdash(e) = \models(-,e)$ and, therefore, $F = (|\,\Delta\, D|, \|\,\Sigma\, D\|_{\mathbf{E}}, \Delta\, D, \Vdash)$ is an attachment system, or, more precisely, an object of the category **AttSE**. Moreover, for every $a_1,a_2 \in A$, $(\Vdash(a_1))(a_2) = \models(a_2,a_1) = \models(a_1,a_2) = (\Vdash\!\vdash(a_2))(a_1)$. Following the terminology of [19], (F,G) is called a dual attachment pair.

The construction can also be done in a different manner. From the second item of Proposition 4.3, we get a map $(|\,\Delta\, D| = |\,\Omega\, D|) \xrightarrow{\Vdash} \mathbf{B}(\Omega\, D, \|\,\Sigma\, D\|_{\mathbf{B}})$, $\Vdash(b) = \models(b,-)$ and then $\bar{F} = (|\,\Omega\, D|, \|\,\Sigma\, D\|_{\mathbf{B}}, \Omega\, D, \Vdash)$ is an object of **AttSB**. On the other hand, from the second item of Proposition 4.1, we get an $\mathbf{E}$-homomorphism $\Delta\, D \xrightarrow{\Vdash} (\|\,\Sigma\, D\|_{\mathbf{E}})^{(|\,\Omega\, D| = |\,\Delta\, D|)}$, $\Vdash\!\vdash(e) = \models(e,-)$ and then $\bar{G} = (|\,\Delta\, D|, \|\,\Sigma\, D\|_{\mathbf{E}}, \Delta\, D, \Vdash\!\vdash)$ is an object of **ATTSE**. For every $a_1,a_2 \in A$, $(\Vdash(a_1))(a_2) = \models(a_1,a_2) = \models(a_2,a_1) = (\Vdash\!\vdash(a_2))(a_1)$. We obtain a dual attachment pair $(\bar{F},\bar{G})$, different from (F,G).

To have a usable term, we call the systems D with the above conditions *dual attachment system pairs.*

Definition 4.13. The category $(\mathbf{E},\mathbf{C},\mathbf{B})$-**DAttSPr** is the full subcategory of $(\mathbf{E},\mathbf{C},\mathbf{B})$-**AlgTopSys**, comprising precisely the dual attachment system pairs.

Categories of the form $(\mathbf{E},\mathbf{C},\mathbf{B})$-**DAttSPr** provide a convenient framework to study the concept of duality for attachment systems, providing not only objects but also the respective morphisms, the case of which was not touched on in [19]. It will be the topic of our further research to study the categories $(\mathbf{E},\mathbf{C},\mathbf{B})$-**DAttSPr** and their induced concepts in full detail.

4.5. **Attachment Topology.** This subsection shows probably one of the most interesting concepts related to attachment system theory, which has never been considered before, namely, the topology induced by attachment systems. It is important to notice that in [19] we doubted even the possibility of its existence, because we were concentrating on attachment generated topological theory morphisms. This paper clarifies the question at last.

Since the category **AttSA** of Definition 4.6 is a subcategory of a particular category of algebraically-topological systems, which has a spatialization procedure described in Proposition 3.6, one easily obtains the respective attachment system machinery (recall the notations of Example 2.8).

Proposition 4.14. *There exists a functor* $\mathbf{AttSA} \xrightarrow{\mathrm{AttSpat}} \mathbf{LoA}\text{-}\mathbf{Top}$ *defined by* $\mathrm{AttSpat}(D_1 \xrightarrow{f} D_2) = (|\,\Omega\, D_1|, \Sigma\, D_1, \tau_1) \xrightarrow{(\Delta\, f, (\Sigma\, f)^{op})} (|\,\Omega\, D_2|, \Sigma\, D_2, \tau_2)$*, where* $\tau_i = \{(\Vdash_i(-))(b)\,|\, b \in |\,\Omega\, D_i|\}$.

The following example is bound to provide more intuition for the concept.

Example 4.15. Let L be a frame and let $D = (|L|, \mathbf{2}, L, \Vdash)$ be an object of **AttSFrm**. Then $\mathrm{AttSpat}\, D$ can be considered as a crisp topological space $(|L|, \tau)$, where τ consists of the sets $Ext(b) = \{a \in L\,|\, b \in \Vdash(a)\}$ (the notation "*Ext*" comes from the realm of topological systems of S. Vickers) for every $b \in L$, where "$b \in \Vdash(a)$" is a shorthand for "$(\Vdash(a))(b) = \top$". To get even more clarity, suppose that L is a finite chain and define the map $L \xrightarrow{\Vdash} \mathbf{Frm}(L, \mathbf{2})$ by

$$L \xrightarrow{\Vdash(a)} \mathbf{2} : b \mapsto \begin{cases} \top, & a < b \\ \bot, & \text{otherwise} \end{cases}$$

for $a \neq \top$ and let $\Vdash(\top) = \Vdash(\bot)$. It follows that $Ext(b) = \{a \in L\,|\, a < b\} \bigcup \{\top\}$ for $b \neq \bot$ and $Ext(\bot) = \varnothing$. If L has more than two elements, then the obtained topology is neither discrete no indiscrete.

To get better acquainted with the new concept, below we show a simple property of attachment system topology. To begin with, we recall two properties of attachments, introduced by C. Guido [23].

Definition 4.16. Let $D = (|L|, \mathbf{2}, L, \Vdash)$ be an object of **AttSFrm**. D is called *spatial* provided that for every distinct $a, b \in L$, there exists $c \in L$ such that

$a \in \Vdash(c)$ and $b \notin \Vdash(c)$. D is called *symmetrical* provided that for every $a, b \in L$, if $b \in \Vdash(a)$, then $a \in \Vdash(b)$.

Proposition 4.17. *If a frame attachment system* $D = (|L|, \mathbf{2}, L, \Vdash)$ *is both spatial and symmetrical, then* $\mathrm{AttSpat}\, D$ *is a* T_0 *space.*

Proof. Given distinct $a, b \in L$, either $a \nleqslant b$ or $b \nleqslant a$. In the former case, by spatiality of D, there exists $c \in L$ such that $a \in \Vdash(c)$ and $b \notin \Vdash(c)$. Since D is symmetrical, $c \in \Vdash(a)$ and $c \notin \Vdash(b)$ and, therefore, $a \in Ext(c)$ and $b \notin Ext(c)$. □

It will be the topic of our future research to consider set-theoretic properties of attachment system topology, whereas here we will continue exploring its categorical features.

Proposition 3.9 provides a way from spaces to systems, which in general is not an embedding. Unfortunately, the functor can not be directly used in the framework of **AttSA**, since the latter category imposes a restriction on its objects. On the other hand, a particular subcategory of the category **LoA-Top** can still be mapped to the category of attachment systems, thereby providing a full embedding.

Definition 4.18. LoA-AttSTop is the full subcategory of **LoA-Top**, comprising precisely those spaces (X, A, τ) for which there exists a bijection $|\tau| \xrightarrow{\hbar} X$ (*attachment system spaces*).

The new category in hand, we are able to obtain the following important result.

Proposition 4.19. *There exists a full embedding* $\mathbf{LoA\text{-}AttSTop} \xhookrightarrow{\mathrm{AttE}} \mathbf{AttSA}$, *which explicitly is given by the formula* $\mathrm{AttE}((X_1, A_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \tau_2)) = (|\tau_1|, A_1, \tau_1, \Vdash_1) \xrightarrow{(\hbar_2^{-1} \circ f \circ \hbar_1, \varphi, (f,\varphi)^{\leftarrow})} (|\tau_2|, A_2, \tau_2, \Vdash_2)$, *where* $\tau_i \xrightarrow{\Vdash_i} \mathbf{A}(\tau_i, A_i)$ *is defined by* $(\Vdash_i(\alpha))(\beta) = \beta(\hbar_i(\alpha))$.

Proof. Straightforward application of Proposition 3.9 provides the correctness of the definition and its functorial properties, e.g., preservation of continuity follows from the fact that for every $\alpha \in |\tau_1|$ and every $\beta \in \tau_2$, $(\Vdash_1(\alpha))((f,\varphi)^{\leftarrow}(\beta)) = \varphi^{op} \circ \beta \circ f(\hbar_1(\alpha)) = \varphi^{op} \circ \beta(\hbar_2 \circ \hbar_2^{-1} \circ f \circ h_1(\alpha)) = (\varphi^{op} \circ \Vdash_2(\hbar_2^{-1} \circ f \circ \hbar_1(\alpha)))(\beta)$.

For the embedding property, notice that $\mathrm{AttE}(X_1, A_1, \tau_1) = \mathrm{AttE}(X_2, A_2, \tau_2)$ implies $\tau_1 = \tau_2$ and, therefore, $X_1 = X_2$.

To show fullness, we notice that given an arbitrary attachment system morphism $(|\tau_1|, A_1, \tau_1, \Vdash_1) \xrightarrow{f} (|\tau_2|, A_2, \tau_2, \Vdash_2)$, for every $\beta \in \tau_2$ and every $x \in X_1$, it follows that $(\Omega\, f(\beta))(x) = (\Omega\, f(\beta))(\hbar_1 \circ \hbar_1^{-1}(x)) = (\Vdash_1(\hbar_1^{-1}(x)))(\Omega\, f(\beta)) = (\Sigma\, f \circ \Vdash_2(\Delta\, f(\hbar_1^{-1}(x))))(\beta) = \Sigma\, f \circ \beta(\hbar_2(\Delta\, f(\hbar_1^{-1}(x)))) = (\Sigma\, f \circ \beta \circ (\hbar_2 \circ \Delta\, f \circ \hbar_1^{-1}))(x) = ((\hbar_2 \circ \Delta\, f \circ \hbar_1^{-1}, (\Sigma\, f)^{op})^{\leftarrow}(\beta))(x)$, i.e., $\tau_1 \ni \Omega\, f(\beta) = ((\hbar_2 \circ \Delta\, f \circ \hbar_1^{-1}), (\Sigma\, f)^{op})^{\leftarrow}(\beta)$. As a consequence, we obtain that $|(X_1, A_1, \tau_1)| \xrightarrow{(\hbar_2 \circ \Delta\, f \circ \hbar_1^{-1}, (\Sigma\, f)^{op})} |(X_2, A_2, \tau_2)|$ is continuous and $\mathrm{AttE}(\hbar_2 \circ \Delta\, f \circ \hbar_1^{-1}, (\Sigma\, f)^{op}) = f$. □

After looking at Propositions 4.14 and 4.19, the question arises as to whether an analogue of Proposition 3.10 holds in the attachment system setting. Similar to the problem with the definition of the embedding AttE, it appears that the image of the category **AttSA** under AttSpat is (in general) not necessarily contained in the category **LoA-AttSTop**. As a possible solution, we can restrict the domain of AttSpat. The following provides the subcategory in question, which is preceded by an additional definition required for the occasion (the new concept has already appeared in [61], motivated by the notion of attachment spatiality of C. Guido recalled in Definition 4.16, which in its turn was related to the well-known notion of *spatial locale* [33]).

Definition 4.20. An attachment system D is called Σ-*spatial* provided that for every distinct $a_1, a_2 \in \Omega\, D$, there exists $a \in \Omega\, D$ such that $(\Vdash(a))(a_1) \neq (\Vdash(a))(a_2)$. **SpAttSA** is the full subcategory of **AttSA** of Σ-spatial attachment systems.

The next lemma justifies our introduction of a new category.

Lemma 4.21.

(1) *Given a* Σ*-spatial attachment system* D *let* $\mathrm{AttSpat}\, D = (|\,\Omega\, D|, \Sigma\, D, \tau)$. *There exists an* $\mathbf{A}$*-isomorphism* $\Omega\, D \xrightarrow{Ext} \tau$ *defined by* $Ext(a) = (\Vdash(-))(a)$. *In particular,* $\mathrm{AttSpat}\, D$ *is an attachment system space.*

(2) *Given an attachment system space* (X, A, τ)*,* $\mathrm{AttE}(X, A, \tau)$ *is* Σ*-spatial.*

Proof. To show the first item, notice that surjectivity of the map follows from Proposition 4.14, whereas injectivity is the consequence of Σ-spatiality. Moreover, for every $\lambda \in \Lambda_{\mathbf{A}}$, every $\langle a_i \rangle_{n_\lambda} \in (\Omega\, D)^{n_\lambda}$ and every $a \in \Omega\, D$, one obtains that $(Ext(\omega_\lambda^{\Omega\, D}(\langle a_i \rangle_{n_\lambda})))(a) = (\Vdash(a))(\omega_\lambda^{\Omega\, D}(\langle a_i \rangle_{n_\lambda})) = \omega_\lambda^{\Sigma\, D}(\langle (\Vdash(a))(a_i) \rangle_{n_\lambda}) = \omega_\lambda^{\Sigma\, D}(\langle (Ext(a_i))(a) \rangle_{n_\lambda}) = (\omega_\lambda^{\tau}(\langle Ext(a_i) \rangle_{n_\lambda}))(a)$.

For the second item, notice that given distinct $\alpha_1, \alpha_2 \in \tau$, there exists $x \in X$ such that $\alpha_1(x) \neq \alpha_2(x)$ and, therefore, $(\Vdash(\hbar^{-1}(x)))(\alpha_1) = \alpha_1(\hbar \circ \hbar^{-1}(x)) = \alpha_1(x) \neq \alpha_2(x) = (\Vdash(\hbar^{-1}(x)))(\alpha_2)$. ☐

With the help of the above-mentioned lemma, we immediately obtain the restriction $\mathbf{SpAttSA} \underset{\overline{\mathrm{AttE}}}{\overset{\overline{\mathrm{AttSpat}}}{\rightleftarrows}} \mathbf{LoA\text{-}AttSTop}$, which (as might have been expected) appears to be an equivalence.

Proposition 4.22. *The above two functors* $\mathbf{LoA\text{-}AttSTop} \xhookrightarrow{\overline{\mathrm{AttE}}} \mathbf{SpAttSA}$ *and* $\mathbf{SpAttSA} \xrightarrow{\overline{\mathrm{AttSpat}}} \mathbf{LoA\text{-}AttSTop}$ *provide an equivalence between the categories* **LoA-AttSTop** *and* **SpAttSA**.

Proof. Straightforward computations similar to the proof of Proposition 3.10 show that given a Σ-spatial attachment system D, it follows that $(\overline{\mathrm{AttE}}\, \overline{\mathrm{AttSpat}} D = (|\tau|, \Sigma\, D, \tau, \tilde{\Vdash})) \xrightarrow{\varepsilon_D = (|Ext^{-1}|, 1_{\Sigma\, D}, Ext^{op})} D$ is an $\overline{\mathrm{AttE}}$-co-universal arrow for D. Moreover, it is easy to see that ε_D is an isomorphism.

Given an attachment system space (X, A, τ), there exists a morphism of the category $\mathbf{Set} \times \mathbf{LoA}$ defined by $|(X, A, \tau)| \xrightarrow{\eta_{(X,A,\tau)} = (\hbar^{-1}, 1_A)} |(\overline{\mathrm{AttSpat}}\, \overline{\mathrm{AttE}}(X, A, \tau) = \overline{\mathrm{AttSpat}}(|\tau|, A, \tau, \Vdash) = (|\tau|, A, \tilde{\tau}))|$. To prove its continuity, notice that given $\alpha \in \tau$

and $x \in X$, $((\hbar^{-1}, 1_A)^{\leftarrow}(Ext(\alpha)))(x) = (Ext(\alpha))(\hbar^{-1}(x)) = (\Vdash(\hbar^{-1}(x)))(\alpha) = \alpha(\hbar \circ \hbar^{-1}(x)) = \alpha(x)$. We show that $\eta_{(X,A,\tau)}$ is an $\overline{\text{AttSpat}}$-universal arrow for (X, A, τ), i.e., every space morphism $(X, A, \tau) \xrightarrow{(f,\varphi)} (\overline{\text{AttSpat}}D = (|\,\Omega\, D|, \Sigma\, D, \sigma))$ has a unique system morphism $(\overline{\text{AttE}}(X, A, \tau) = (|\tau|, A, \tau, \tilde{\Vdash})) \xrightarrow{g} D$, making the following triangle commute

$$\begin{array}{ccc}
(X, A, \tau) & \xrightarrow{\eta_{(X,A,\tau)}} & \overline{\text{AttSpat}}\,\overline{\text{AttE}}(X, A, \tau) \\
& \searrow^{(f,\varphi)} & \Big\downarrow{\scriptstyle \overline{\text{AttSpat}}g} \\
& & \overline{\text{AttSpat}}D.
\end{array}$$

The morphism in question can be defined by $|\tau| \xrightarrow{\Delta\, g} |\,\Omega\, D| = |\tau| \xrightarrow{\hbar} X \xrightarrow{f} |\,\Omega\, D|$, $\Sigma\, D \xrightarrow{\Sigma\, g} A = \Sigma\, D \xrightarrow{\varphi^{op}} A$ and $\Omega\, D \xrightarrow{\Omega\, g} \tau = \Omega\, D \xrightarrow{Ext} \sigma \xrightarrow{(f,\varphi)^{\leftarrow}|_{\sigma}^{\tau}} \tau$. To verify its continuity, notice that given $\alpha \in \tau$ and $a \in \Omega\, D$, $(\tilde{\Vdash}(\alpha))(\Omega\, g(a)) = (\Omega\, g(a))(\hbar(\alpha)) = ((f, \varphi)^{\leftarrow} \circ Ext(a))(\hbar(a)) = (\varphi^{op} \circ Ext(a) \circ f \circ \hbar)(\alpha) = \Sigma\, g \circ \Vdash(f \circ \hbar(\alpha))(a) = (\Sigma\, g \circ \Vdash(\Delta\, g(\alpha)))(a)$. Commutativity of the above diagram is clear. Uniqueness of g follows from reversing the just presented equation sequence.

The last step shows that $\eta_{(X,A,\tau)}$ is an isomorphism, i.e., $|(|\tau|, A, \tilde{\tau})| \xrightarrow{(\hbar, 1_A)} |(X, A, \tau)|$ is continuous. For every $\alpha \in \tau$ and every $\beta \in |\tau|$, $((\hbar, 1_A)^{\leftarrow}(\alpha))(\beta) = \alpha \circ \hbar(\beta) = (\Vdash(\beta))(\alpha) = (Ext(\alpha))(\beta)$. As a result, $(\hbar, 1_A)^{\leftarrow}(\alpha) = Ext(a) \in \tilde{\tau}$. $\square$

An experienced reader will immediately recall the equivalence between the categories of sober topological spaces and spatial locales from the theory of pointless topology [33]. A natural question arises as to whether Proposition 4.22 will result in the just mentioned equivalence in the case of frames. Brief inspection shows that attachment system spaces are not sober spaces. Indeed, by [33, p. 43] every sober space (X, τ) has a bijection $X \to \{X \backslash S \mid S \text{ is an irreducible closed subset of } X\} \subseteq \tau$ that is something different from just having a bijection $X \to \tau$. It follows that attachment system topology is not a particular subcase of classical pointless topology and, therefore, deserves to be studied on its own. It will be the topic of our further research to fully develop the respective theory.

5. **Conclusion**

This paper introduced the concept of *algebraically-topological system* as a generalization of topological systems of S. Vickers [67] and their variety-based version of [62]. Apart from extending the classical results of the theory of (variety-based) topological systems to the new framework, the concept incorporated the notion of attachment of C. Guido [23], showing its categorical redundancy in mathematics. The proposed substitution for the concept, called *attachment system*, appears to have many good properties. For example, attachment systems give rise to several morphisms of topological theories, providing a functor, which incorporates the *hypergraph functor* of lattice-valued mathematics [27]. Moreover, attachment systems have an internal topology, the existence of which has never been mentioned in the attachment theory before (the theory itself, however, is quite young). As the main result in this direction, we proved the equivalence between the categories of spatial attachment systems and attachment system spaces, which provides an attachment system analogue of the well-known sobriety-spatiality equivalence of, e.g., [33], but not coinciding with it. In view of the obtained results, we call attachment system topology *variety-based pointless topology.* Its main advantage is the possibility of doing pointless topology in an arbitrary variety of algebras and that provides an opening for developing lattice-valued analogue of pointless topology, whose theory has already been started long ago by, e.g., B. Hutton [30]. As often happens with new settings, several open problems arise, some of which we list below.

In [66], we showed that the categories of catalg topological spaces and systems are respectively topological and (under reasonable assumptions on the corresponding topological theory) essentially algebraic over their ground categories. With the notions of topological algebra and algebraically-topological system in hand, the first problem springs into mind at once.

Problem 5.1. Under what conditions are the categories $(\mathbf{E}, \mathbf{C}, \mathbf{B})$-**TopAlg** and $(\mathbf{E}, \mathbf{C}, \mathbf{B})$-**AlgTopSys** respectively topological and essentially algebraic over their ground categories?

The problem initiates a rather far-reaching study on incorporation the theory of topological algebras into the framework of catalg topology, presented at the beginning of the paper. As has already been said, topological algebras and algebraically-topological systems provide an enrichment (in the category-theoretic sense) of variety-based topology. The next problem is then immediate.

Problem 5.2. In what way should one try to enrich catalg topology in order to incorporate the theory of topological algebras and their related concepts?

The answer to the problem could extend the catalg influence to the realm of classical topological algebra, which up to now has been untouched in the theory (despite the term "algebraic" in its name).

When looking at the morphisms between topological theories considered in this paper and keeping in mind their generated functors (including the above hypergraph functor), one can pose the following problem.

Problem 5.3. Is it possible to obtain the most essential properties of functors between categories of topological structures through the properties of their generated topological theory morphisms?

The problem is a part of our current research on developing a fruitful topological theory (in a general sense) based on the setting of topological theory morphisms (in the sense of this paper).

The framework of attachment system topology introduced in this paper, apart from being a challenging topic of study by itself, raises an important question.

Problem 5.4. To which extent (if any) is it possible to incorporate the classical sobriety-spatiality equivalence of, e.g., [33] into the setting of Proposition 4.22?

The problem induces the study on differences and similarities between the topologies induced by locales and those of attachment systems based on frames. In particular, by analogy with [61, Lemma 5], one can obtain the following simple result.

Proposition 5.5. *Let* $\mathbf{A}$ *be a variety of algebras and let* $\mathbf{A} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ *be a functor, which takes every* $\mathbf{A}$*-algebra* A *to its underlying set. There exists an embedding* $\mathbf{A} \overset{\mathrm{E}}{\hookrightarrow} \mathbf{SpAttSA}^{op}$ *given by* $\mathrm{E}(A_1 \xrightarrow{\varphi} A_2) = (|A_1|, A_1, A_1, \Vdash_1) \xrightarrow{(\varphi^*,\varphi,\varphi)} (|A_2|, A_2, A_2, \Vdash_2)$*, where* $\Vdash_i(a) = 1_{A_i}$ *for every* $a \in A_i$*.*

Example 5.6. The case of **Frm** gives rise to (at least) two possible definitions of the respective functor $\mathbf{Frm} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ (cf. [20, 45, 47, 48] for the constructions and notations):

(1) $(A \xrightarrow{\varphi} B)^* = |A| \xrightarrow{(\varphi^{\dashv})^{op}} |B|$, $\varphi^{\dashv}(b) = \bigwedge\{a \in A \,|\, b \leqslant \varphi(a)\}$;

(2) $(A \xrightarrow{\varphi} B)^* = |A| \xrightarrow{(\varphi^{\vdash})^{op}} |B|$, $\varphi^{\vdash}(b) = \bigvee\{a \in A \,|\, \varphi(a) \leqslant b\}$.

The second item actually gives a functor $\mathbf{Frm} \xrightarrow{(-)^*} \mathbf{LoCSLat}(\bigwedge)$, where $\mathbf{CSLat}(\bigwedge)$ is the variety of $\bigwedge$-*semilattices* (cf. the examples after Definition 2.3).

In view of Proposition 5.5, the functor AttSpat of Proposition 4.14, restricted to the category **LoA**, provides (attachment system) spaces of the form $(|A|, A, \tau)$, where $\tau = \{A \xrightarrow{a} A \,|\, a \in A\}$ (recall from Subsection 2.1 our notations with respect to constant maps). Having the result in hand, we ask a challenging question.

Problem 5.7. Is it possible to incorporate pointless topology into attachment topology?

In case of a positive answer, the pointless topology has an opening for a much wider variety-based development, which should be studied appropriately.

In [56] S. Solovyov has presented a generalization of the catalg framework, called *lattice-valued catalg topology*, which extended the theory of (L, M)-*fuzzy topological spaces* of T. Kubiak and A. Šostak [37]. The concluding problem of this paper, therefore, is simple to state.

Problem 5.8. Provide the extension of the framework of this paper to the theory of lattice-valued topological algebras and algebraically-topological systems.

All of the above-mentioned problems will be addressed to our forthcoming papers. The reader is kindly invited to join the research.

Acknowledgements. The authors are much grateful to the anonymous referee for helping them to improve both the linguistic and the mathematical quality of the manuscript, removing several of its deficiencies.

S. Solovyov is grateful to the Department of Mathematics and Statistics of York University (Toronto, Ontario, Canada) for the opportunity of spending the first six months of the year 2012 at its quarters, during which period the first (and also the second) revised version of this manuscript was prepared.

References

[1] M. Abel and A. Šostak, *Towards the theory of L-bornological spaces*, Iranian Journal of Fuzzy Systems, **8(1)** (2011), 19–28.

[2] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and concrete categories: the joy of cats*, Dover Publications (Mineola, New York), 2009.

[3] D. Aerts, E. Colebunders, A. van der Voorde and B. van Steirteghem, *State property systems and closure spaces: a study of categorical equivalence*, Int. J. Theor. Phys., **38(1)** (1999), 359–385.

[4] D. Aerts, E. Colebunders, A. van der Voorde and B. van Steirteghem, *On the amnestic modification of the category of state property systems*, Appl. Categ. Struct., **10(5)** (2002), 469–480.

[5] F. Bayoumi and S. E. Rodabaugh, *Overview and comparison of localic and fixed-basis topological products*, Fuzzy Sets and Systems, **161(18)** (2010), 2397–2439.

[6] J. Bénabou, *Treillis locaux et paratopologies*, Semin. de Topologie et de Geometrie differentielle Ch. Ehresmann, **1(2)** (1957/58).

[7] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182–190.

[8] D. M. Clark and B. A. Davey, *Natural dualities for the working algebraist*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, **57** (1998).

[9] P. M. Cohn, *Universal algebra*, D. Reidel Publ. Comp., 1981.

[10] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued topological systems*, In: U. Bodenhofer, B. De Baets, E. P. Klement, S. Saminger-Platz, eds., Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009.

[11] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued predicate transformers and interchange systems*, In: P. Cintula, E. P. Klement, L. N. Stout, eds., Abstracts of the 31st Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2010.

[12] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Interweaving algebra and topology: Lattice-valued topological systems*, Fuzzy Sets and Systems, **192** (2012), 58–103.

[13] J. T. Denniston and S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, Quaest. Math., **32(2)** (2009), 139–186.

[14] C. H. Dowker and D. Papert, *On Urysohn's lemma*, General Topology and its Relations to modern Analysis and Algebra 2, Proc. 2nd Prague topol. Sympos. 1966, (1967), 111–114.

[15] C. H. Dowker and D. Papert, *Quotient frames and subspaces*, Proc. Lond. Math. Soc., **III(16)** (1966), 275–296.

[16] C. H. Dowker and D. Papert Strauss, *Separation axioms for frames*, Topics in Topol., Colloqu. Keszthely 1972, Colloquia Math. Soc. Janos Bolyai, **8** (1974), 223–240.

[17] C. Ehresmann, *Gattungen von lokalen Strukturen*, Jahresber. Dtsch. Math.-Ver., **60** (1957), 49–77.

[18] A. Frascella, *Attachment and topological systems in varieties of algebras*, Ph.D. thesis, Department of Mathematics "Ennio De Giorgi", University of Salento, Italy, 2011.

[19] A. Frascella, C. Guido and S. Solovyov, *Dual attachment pairs in categorically-algebraic topology*, Appl. Gen. Topol., **12(2)** (2011), 101–134.

[20] G. Gierz, K. Hofmann and etc., *Continuous lattices and domains*, Cambridge University Press, 2003.

[21] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., **18** (1967), 145–174.

[22] J. A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl., **43** (1973), 734–742.

[23] C. Guido, *Fuzzy points and attachment*, Fuzzy Sets and Systems, **161(16)** (2010), 2150–2165.

[24] C. Guido and V. Scarciglia, *L-topological spaces as spaces of points*, Fuzzy Sets and Systems, **173(1)** (2011), 45–59.

[25] C. Guido and S. Solovyov, *Topological systems versus attachment relation*, submitted.

[26] J. Gutiérrez García and S. E. Rodabaugh, *Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies*, Fuzzy Sets and Systems, **156(3)** (2005), 445–484.

[27] U. Höhle, *A note on the hypergraph functor*, Fuzzy Sets and Systems, **131(3)** (2002), 353–356.

[28] U. Höhle and A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, In: U. Höhle, S. E. Rodabaugh, eds., Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, **3** (1999), 123–272.
[29] B. Hutton, *Uniformities on fuzzy topological spaces*, J. Math. Anal. Appl., **58** (1977), 559–571.
[30] B. Hutton, *Products of fuzzy topological spaces*, Topology Appl., **11** (1980), 59–67.
[31] J. R. Isbell, *Atomless parts of spaces*, Math. Scand., **31** (1972), 5–32.
[32] G. Jäger, *Lattice-valued categories of lattice-valued convergence spaces*, Iranian Journal of Fuzzy Systems, **8(2)** (2011), 67–89.
[33] P. T. Johnstone, *Stone spaces*, Cambridge University Press, 1982.
[34] G. M. Kelly, *Basic concepts of enriched category theory*, Cambridge University Press, 1982.
[35] J. C. Kelly, *Bitopological spaces*, Proc. Lond. Math. Soc., **13(III)** (1963), 71–89.
[36] D. Kruml and J. Paseka, *Algebraic and categorical aspects of quantales*, In: M. Hazewinkel, ed., Handbook of Algebra, Elsevier, **5** (2008), 323–362.
[37] T. Kubiak and A. Šostak, *Foundations of the theory of* (L, M)*-fuzzy topological spaces*, In: U. Bodenhofer, B. De Baets, E. P. Klement, S. Saminger-Platz, eds., Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009.
[38] F. W. Lawvere, *Functorial semantics of algebraic theories*, Ph.D. thesis, Columbia University, 1963.
[39] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl., **56** (1976), 621–633.
[40] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer-Verlag, 1998.
[41] E. G. Manes, *Algebraic theories*, Springer-Verlag, 1976.
[42] L. Pontrjagin, *Topological groups*, Oxford University Press, 1939.
[43] P. M. Pu and Y. M. Liu, *Fuzzy topology I: neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., **76** (1980), 571–599.
[44] S. E. Rodabaugh, *Categorical frameworks for stone representation theories*, In: S. E. Rodabaugh, E. P. Klement, U. Höhle, eds., Applications of Category Theory to Fuzzy Subsets, Theory and Decision Library: Series B: Mathematical and Statistical Methods, Kluwer Academic Publishers, **14** (1992), 177–231.
[45] S. E. Rodabaugh, *Powerset operator based foundation for point-set lattice-theoretic (poslat) fuzzy set theories and topologies*, Quaest. Math., **20(3)** (1997), 463–530.
[46] S. E. Rodabaugh, *Categorical foundations of variable-basis fuzzy topology*, In: U. Höhle, S. E. Rodabaugh, eds., Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series, Dordrecht: Kluwer Academic Publishers, **3** (1999), 273–388.
[47] S. E. Rodabaugh, *Powerset operator foundations for poslat fuzzy set theories and topologies*, In: U. Höhle, S. E. Rodabaugh, eds., Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series, Dordrecht: Kluwer Academic Publishers, **3** (1999), 91–116.
[48] S. E. Rodabaugh, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, Int. J. Math. Math. Sci., **2007** (2007), 1–71.
[49] S. E. Rodabaugh, *Functorial comparisons of bitopology with topology and the case for redundancy of bitopology in lattice-valued mathematics*, Appl. Gen. Topol., **9(1)** (2008), 77–108.
[50] S. E. Rodabaugh, *Necessity of non-stratified and anti-stratified spaces in lattice-valued topology*, Fuzzy Sets and Systems, **161(9)** (2010), 1253–1269.
[51] S. E. Rodabaugh, *Relationship of algebraic theories to powersets over objects in* **Set** *and* **Set**×**C**, Fuzzy Sets and Systems, **161(3)** (2010), 453–470.
[52] K. I. Rosenthal, *Quantales and their applications*, Pitman Research Notes in Mathematics, Addison Wesley Longman, **234** (1990).
[53] J. D. H. Smith, *Modes and modals*, Discuss. Math., Algebra Stoch. Methods, **19(1)** (1999), 9–40.

[54] S. Solovjovs, *Embedding topology into algebra*, In: U. Bodenhofer, B. De Baets, E. P. Klement, S. Saminger-Platz, eds., Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009.

[55] S. Solovjovs, *Categorically-algebraic topology*, In: Abstracts of the International Conference on Algebras and Lattices (Jardafest), Charles University, Prague, 2010.

[56] S. Solovjovs, *Lattice-valued categorically-algebraic topology*, In: Abstracts of the 91st Peripatetic Seminar on Sheaves and Logic (PSSL 91), University of Amsterdam, Amsterdam, 2010.

[57] S. Solovjovs, *Variable-basis categorically-algebraic dualities*, In: D. Dubois, M. Grabisch, R. Mesiar, E. P. Klement, eds., Abstracts of the 32nd Linz Seminar on Fuzzy Set Theory, 2011.

[58] S. Solovyov, *Categorically-algebraic topology and its applications*, submitted.

[59] S. Solovyov, *Sobriety and spatiality in varieties of algebras*, Fuzzy Sets and Systems, **159(19)** (2008), 2567–2585.

[60] S. Solovyov, *Categorically-algebraic dualities*, Acta Univ. M. Belii, Ser. Math., **17** (2010), 57–100.

[61] S. Solovyov, *Hypergraph functor and attachment*, Fuzzy Sets and Systems, **161(22)** (2010), 2945–2961.

[62] S. Solovyov, *Variable-basis topological systems versus variable-basis topological spaces*, Soft Comput., **14(10)** (2010), 1059–1068.

[63] S. Solovyov, *Localification of variable-basis topological systems*, Quaest. Math., **34(1)** (2011), 11–33.

[64] S. Solovyov, *On algebraic and coalgebraic categories of variety-based topological systems*, Iranian Journal of Fuzzy Systems, **8(5)** (2011), 13–30.

[65] S. Solovyov, *Powerset operator foundations for catalg fuzzy set theories*, Iranian Journal of Fuzzy Systems, **8(2)** (2011), 1–46.

[66] S. Solovyov, *Categorical foundations of variety-based topology and topological systems*, Fuzzy Sets and Systems, **192** (2012), 176–200.

[67] S. Vickers, *Topology via logic*, Cambridge University Press, 1989.

[68] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–365.

[69] Q. Y. Zhang, *Algebraic generations of some fuzzy powerset operators*, Iranian Journal of Fuzzy Systems, **8(5)** (2011), 31–58.

Anna Frascella, Department of Mathematics "E. De Giorgi", University of Salento, P. O. Box 193, 73100 Lecce, Italy
E-mail address: frascella_anna@libero.it

Cosimo Guido, Department of Mathematics "E. De Giorgi", University of Salento, P. O. Box 193, 73100 Lecce, Italy
E-mail address: cosimo.guido@unisalento.it

Sergey A. Solovyov*, Department of Mathematics, University of Latvia, Zellu iela 8, LV-1002 Riga, Latvia and Institute of Mathematics and Computer Science, University of Latvia, Raina bulvaris 29, LV-1459 Riga, Latvia
E-mail address: sergejs.solovjovs@lu.lv, sergejs.solovjovs@lumii.lv

*Corresponding author