PRESERVATION THEOREMS IN ŁUKASIEWICZ MODEL THEORY

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ABSTRACT. We present some model theoretic results for Łukasiewicz predicate logic by using the methods of continuous model theory developed by Chang and Keisler. We prove compactness theorem with respect to the class of all structures taking values in the Łukasiewicz BL-algebra. We also prove some appropriate preservation theorems concerning universal and inductive theories. Finally, Skolemization and Morleyization in this framework are discussed and some natural examples of fuzzy theories are presented.

1. Introduction

In the abstract of [7], Hájek and Cintula wrote: "In the last few decades many formal systems of fuzzy logics have been developed. Since the main differences between fuzzy and classical logics lie at propositional level, the fuzzy predicate logics have developed more slowly (compared to the propositional ones). In this paper we aim to promote interest in fuzzy predicate logics by contributing to the model theory of fuzzy predicate logics." In that paper the authors proved a generalized completeness result for a wide class of fuzzy predicate logics, namely core fuzzy logics. To do this they used a generalized Henkin method for a through survey of the present status of fuzzy predicate logic, see [3].

In the present paper we continue the work on fuzzy predicate logics by developing further model theory of these logics. We choose Łukasiewicz predicate logic. The reason is that the Łukasiewicz BL-algebra is equipped with continuous operations and this is essential in proving the compactness theorem which is a fundamental result in model theory. Here, the compactness theorem says that if each finite subset of a set of sentences has a Łukasiewicz model, then the set itself has a Łukasiewicz model. As consequences of compactness, we prove the most basic model theoretic results including the upward and downward Löwenheim-Skolem theorems. We also prove suitable versions of the most well known preservation theorems, including preservation theorems for universal and inductive theories. At the end, we introduce some fuzzy theories and study their model theoretic properties.

To prove the compactness theorem, we use ultraproduct and its basic properties which we first investigate. We follow the old methods developed by Chang and

Keisler in [2]. We particularize their work by considering $[0,1]_{\text{Ł}}$ (the unit interval equipped with the Łukasiewicz t-norm) as the value space. Note that as the Łukasiewicz t-norm induces continuous connectives, the Chang and Keisler requirement of continuity of connectives is satisfied.

We also use some ideas and techniques from a rather newly developed area of model theory called model theory of metric structures, see [10]. There, besides continuity of connectives one also uses uniform continuity of interpretations of relation and function symbols. We usually use their clear and simple style of writing which is more suitable when one works in model theory.

2. Łukasiewicz BL-algebra

A BL-algebra (see e.g. [6]) is a structure $\mathbb{B} = (B, \vee, \wedge, *, \rightarrow, 0, 1)$ such that $(B, \vee, \wedge, 0, 1)$ is a bounded lattice, $(B, *, 1)$ is a commutative monoid and for any $x, y, z \in B$:

i) $z \leq (x \rightarrow y)$ iff $x * z \leq y$,

ii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,

iii) $x * (x \rightarrow y) = x \wedge y$.

When we work on BL-algebras with domain $[0,1]$, the operation $*$ is called a t-norm. In this case the BL-algebra is called standard or t-norm BL-algebra, see [7] and [8]. In the sequel, we work with a special but important BL-algebra, i.e. Łukasiewicz BL-algebra which is equipped with the Łukasiewicz t-norm: $x * y = \max\{0, x + y - 1\}$. The mathematical importance of this t-norm is that its residuum $\rightarrow$ defined using property (i) in the definition of BL-algebra is continuous. After defining this $\rightarrow$, we do not need $*$ in our model theoretic study anymore.

Definition 2.1. The BL-algebra $\mathbb{B} = ([0,1], \vee, \wedge, *, \rightarrow, 0, 1)$ where $\vee, \wedge, 0, 1$ interpreted as $\max, \min, 0, 1$, and $*$ interpreted as the Łukasiewicz t-norm and $\rightarrow$ interpreted as its residuum, is the Łukasiewicz BL-algebra.

Note that $(r \rightarrow s) = \min\{1, 1 - r + s\}$ is continuous and setting $\neg r = (r \rightarrow 0)$, we have $\neg r = 1 - r$. Also, modified subtraction is defined by $r \dot{-} s = 1 - (r \rightarrow s)$. On the other hand, the operation $\rightarrow$ in Gödel and Product BL-algebras are not continuous, see [6]. The continuity of all connectives is essential in our work.

Let X be a topological space, $\mathcal{D}$ an ultrafilter over an index set I and $\{x_i\}_{i \in I}$ a family of points in X. Then one sets $\lim_{\mathcal{D}} x_i = x$ if for each open neighborhood U of x one has that $\{i : x_i \in U\} \in \mathcal{D}$. It is a basic fact that X is compact Hausdorff if and only if every such $\{x_i\}_{i \in I}$ has a unique $\mathcal{D}$-limit. Moreover, if X is compact Hausdorff, every continuous operation on X preserves $\mathcal{D}$-limits. For example, if f is unary, then $f(\lim_{\mathcal{D}} x_i) = \lim_{\mathcal{D}} f(x_i)$. In this paper we use $\mathcal{D}$-limits on the compact unit interval.

3. Łukasiewicz Predicate Logic: Syntax & Semantics

In this section we fix our notations for Syntax and Semantics of Łukasiewicz Predicate Logic. We essentially follow the style of writing used in [10].

Throughout this paper $\mathbb{B}$ is the Łukasiewicz BL-algebra. In this algebra we have $\neg r = 1 - r$. In the sequel we use a new defined operation $\dot{-}$ on $[0,1]$, $x \dot{-} y = \neg(x \rightarrow$

y), a similar operation is used in [9]. The logical connective related to this operator will be shown by the same notation. We denote the logical connectives by the same notations as their truth functions in $\mathbb{B}$.

Let L be a first order language. We always assume that L contains a 2-place predicate symbol e for equality although we usually do not display it in L. The family of L-terms is defined as usual. The family of L formulas is defined as follows:

- every $r \in \mathbb{B}$ is an atomic formula
- if R is a n-ary relation symbol and $t_1, ..., t_n$ are terms, then $R(t_1, ..., t_n)$ is an atomic formula. In particular, $e(t_1, t_2)$ is an atomic formula
- If ϕ, ψ are formulas, then so are $\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \to \psi$
- if ϕ is a formula and x is a variable, then $\exists x \psi$ and $\forall x \phi$ are formulas.

A (Łukasiewicz) L-structure is a nonempty set M equipped with
- for each $c \in L$ an element $c^M \in M$,
- for each n-ary $F \in L$ a function $F^M : M^n \to M$,
- for each n-ary $R \in L$, a function $R^M : M^n \to \mathbb{B}$.

It is always assumed that e is interpreted by the characteristic function of equality. So, e^M is the real (or crisp, i.e. always true or false) equality on M.

$\neg\phi$ and $\phi \mathbin{\dot-} \psi$ are used as abbreviations for $\phi \to 0$ and $\neg(\phi \to \psi)$ respectively.

The notion of free variable is defined as usual. For an L-structure M, term $t(\bar{x})$ and $\bar{a} \in M$, $t^M(\bar{a})$ is defined by induction on the complexity of $t(\bar{x})$ and is then an element of M. Similarly, for $\phi(\bar{x})$ and $\bar{a} \in M$, the value of $\phi(\bar{a})$ in M, denoted by $\phi^M(\bar{a})$, is defined in the obvious way. In particular,

$$(\exists x \phi(x))^M = \sup\{\phi^M(a) : a \in M\},$$
$$(\forall x \phi(x))^M = \inf\{\phi^M(a) : a \in M\}.$$

We write $M \vDash \phi(\bar{a})$ if $\phi^M(\bar{a}) = 1$. A sentence is a formula without free variables. Any set of sentences is called a theory. For a theory T, $M \vDash T$ means that $M \vDash \sigma$ for every $\sigma \in T$. The theory of an L-structure M, denoted $Th(M)$, consists of the set of all sentences satisfied in M. Two L-structures M and N are elementarily equivalent, denoted $M \equiv N$, if they have the same theory.

In practice we may use expressions of the form $\phi \leqslant \psi$ and $\phi = \psi$ called *statements*. Note that we have $\phi^M \leqslant \psi^M$ if and only if $M \vDash \phi \to \psi$ and $\phi^M = \psi^M$ if and only if $M \vDash (\phi \to \psi) \wedge (\psi \to \phi)$. So, such statements can be again regarded as formulas.

Two formulas $\phi(\bar{x}), \psi(\bar{x})$ are equivalent, denoted $\phi \equiv \psi$, if $\phi^M(\bar{a}) = \psi^M(\bar{a})$ for every M and $\bar{a} \in M$. For example, we have $\exists x \exists y \phi \equiv \exists y \exists x \phi$ and $\forall x \phi \equiv \neg \exists x \neg \phi$.

Definition 3.1. Assume $\epsilon \geqslant 0$. An injective map $f : M \to N$ is called an ϵ-*embedding* if for each atomic formula $\phi(\bar{x})$

$$|\phi^M(\bar{a}) - \phi^N(f(\bar{a}))| \leqslant \epsilon \quad \forall \bar{a} \in M.$$

If this holds for every $\phi(\bar{x})$, it is called an *elementary ϵ-embedding*. A surjective ϵ-embedding is called an ϵ-*isomorphism*. If $\epsilon = 0$, these are respectively called

embedding, *elementary embedding* and *isomorphism*. M and N are *approximately isomorphic* if they are ϵ-isomorphic for every $\epsilon > 0$.

For example, $f : x \mapsto 0.9x$ on the structure $([0.1], d)$ where d is the usual metric, is a 0.1-embedding. Note that approximately isomorphic models are elementarily equivalent. The notions of substructure $M \subseteq N$, elementary substructure $M \preceq N$, diagram $diag(M)$, and elementary diagram $ediag(M)$ are defined in the obvious way. It is moreover verified that $M \subseteq N$ if and only if $N \vDash diag(M)$ and $M \preceq N$ if and only if $N \vDash ediag(M)$.

4. Basic Facts in Łukasiewicz Model Theory

Our first task in this section is to prove the compactness theorem which is the cornerstone of model theory. Our proof is based on the ultraproduct construction and we do not use Henkin construction or completeness theorem. In fact, Łukasiewicz predicate logic is not complete with respect to Łukasiewicz models (i.e. models with truth values in $\mathbb{B}$, see [6]). Actually, we use suitable versions of the more general results in [2] on continuous model theory and rewrite them in the particular case of Łukasiewicz model theory.

Let I be an index set and $\mathcal{D}$ be an ultrafilter on I. Let M_i, $i \in I$, be a family of L-structures and $M = \prod_{\mathcal{D}} M_i$ be their usual set theoretic ultraproducts consisting of classes of the form

$$[x_i] = \{(y_i) : \{i : x_i = y_i\} \in \mathcal{D}\}.$$

We turn M into an L-structure as follows:
- $c^M = [c^{M_i}]$
- $F^M([x_i], ...) = [F^{M_i}(x_i, ...)]$
- $R^M([x_i], ...) = \lim_{\mathcal{D}} R^{M_i}(x_i, ...)$

These are well-defined. Moreover, an easy induction shows that for every term t, $t^M([a_i^1], ..., [a_i^n]) = [t^{M_i}(a_i^1, ..., a_i^n)]$. Continuity of connectives is used in the following theorem.

Theorem 4.1. (Łoś Theorem) *For every formula $\phi(\bar{x})$ and $[a_i^1], ..., [a_i^n]$*

$$\phi^M([a_i^1], ..., [a_i^n]) = \lim_{\mathcal{D}} \phi^{M_i}(a_i^1, ..., a_i^n).$$

Proof. The atomic cases are obvious. The connective cases are by commutativity of $\mathcal{D}$-limit with continuous functions. Let us consider the case $\exists x \phi(x, \bar{y})$. To make notations simpler, we assume $\bar{y} = \emptyset$. Then for each $[a_i]$ we have

$$\phi^M([a_i]) = \lim_{\mathcal{D}} \phi^{M_i}(a_i) \leqslant \lim_{\mathcal{D}} (\exists x \phi(x))^{M_i}.$$

So,

$$(\exists x \phi(x))^M \leqslant \lim_{\mathcal{D}} (\exists x \phi(x))^{M_i} \quad (*).$$

For the converse let $r = (\exists x \phi(x))^M$ and suppose the inequality $(*)$ is strict. Take an r' such that $r < r' < \lim_{\mathcal{D}} (\exists x \phi(x))^{M_i}$. Then for D-almost all i we have $r' < (\exists x \phi(x))^{M_i}$. So, for $\mathcal{D}$-almost all i there exists a_i such that $r' < \phi^{M_i}(a_i)$. This means that $r < \phi^M([a_i])$ which is a contradiction. We deduce that the equality holds in $(*)$. $\qquad \square$

Using the standard arguments we have

Corollary 4.2. (Compactness) *Every finitely satisfiable set of sentences is satisfiable.*

Definition 4.3. An *approximation* of a statement $\phi \geq r$ (resp $\phi \leq r$) is a statement $\phi \geq s$ (resp $\phi \leq s$) where $r > s$ (resp $r < s$) (we allow s lie outside $[0, 1]$).

It is clear that if $\phi \geq r - \frac{1}{n}$ holds in M_n and $\mathcal{D}$ is non-principal ultrafilter on $\mathbb{N}$ then $\phi \geq r$ holds in $\prod_{\mathcal{D}} M_n$. Therefore, a statement is satisfiable if and only if all its approximations are satisfiable. More generally, we have the following.

Proposition 4.4. *Let T be a theory. Then a sentence ϕ is satisfied in a model of T if and only if every approximation of ϕ is satisfied in a model of T.*

Note that it is not true that if $T \vDash \sigma$ then some finite part of T does so. This holds for approximations of σ. In particular, if $T \vDash \sigma$, then for each $r < 1$ there exists a finite $T_0 \subseteq T$ such that $T_0 \vDash \sigma \geqslant r$.

Proposition 4.5. *If $M \equiv N$ then there exists a model K such that $M \preceq K$ and $N \preceq K$. If $f : M \to N_1$ and $g : M \to N_2$ are elementary, then there are K and elementary $f' : N_1 \to K$, $g' : N_2 \to K$ such that $f'f = g'g$.*

Proof. We prove the first part, the second part can be proved similarly. Let $\Sigma = ediag(M) \cup ediag(N)$. Any finite part of $ediag(M)$ is equivalent to one sentence say $\phi(\bar{a}) \in ediag(M)$. Then since $M \equiv N$, $\exists \bar{x}\phi(\bar{x})$ is approximately satisfiable in N. This shows that Σ is finitely satisfiable. Now, any model $K \vDash \Sigma$ elementarily extends both M and N. $\qquad \square$

A similar result as 4.5 is proved in [4]. The following results of classical model theory can be proved similar to their classical counterparts. We just give a short proof for the upward theorem.

Proposition 4.6. (Tarski-Vaught) *Assume $M \subseteq N$. Then $M \preceq N$ if and only if for every $\phi(x)$ with parameters in M, one has that $(\exists x\phi(x))^M = (\exists x\phi(x))^N$.*

Proposition 4.7. (Downward Löwenheim-Skolem) *Assume $X \subseteq M$ and $|X| \leqslant \kappa \leqslant |M|$ where κ is infinite. Then there exists $N \preceq M$ such that $X \subseteq N$ and $|N| = \kappa$.*

Proposition 4.8. (Upward Löwenheim-Skolem) *If T is satisfiable, then it has models of any cardinality $\kappa \geqslant |L| + \aleph_0$.*

Proof. Let $\{c_i : i \in \kappa\}$ be a set of new constant symbols. Add the sentences $c_i \neq c_j$, where $i \neq j$, to T and use the compactness theorem. $\qquad \square$

In particular, every infinite model has arbitrarily large elementary extensions. We recall that Downward and Upward Löwenheim-Skolem Theorems for general fuzzy structured are proved in [5].

Proposition 4.9. (Union of Chain) *Let $M_0 \preceq M_1 \preceq \ldots$ be an elementary chain of length κ of L-structures. Then there exists an L-structure $M = \bigcup_{i < \kappa} M_i$ such that $M_i \preceq M$ for each i.*

Proposition 4.10. (Axiomatizability) *A class $\mathcal{K}$ of L-structures is elementary if and only if it is closed under elementary equivalence and ultraproduct.*

Proof. We prove the 'if' part. Let Γ be the set of all sentences satisfied in every member of $\mathcal{K}$. Obviously, $\mathcal{K} \vDash \Gamma$. Conversely, assume $M \models \Gamma$. Let Λ be the set of all finite sets of sentences satisfied in M. For $\tau \in \Lambda$ set $\hat{\tau} = \{\tau' \in \Lambda : \tau \subseteq \tau'\}$. The family of sets of the form $\hat{\tau}$ is contained in a non-principal ultrafilter $\mathcal{D}$ over Λ. Now assume $\tau \in \Lambda$. Note that there is no $r < 1$ such that $\mathcal{K} \models \wedge_{\sigma \in \tau} \sigma \leqslant r$. Since, otherwise, $M \models \sigma \leqslant r$ for some $\sigma \in \tau$ which is impossible. Thus, $\wedge_{\sigma \in \tau} \sigma \geqslant 1$ is approximately (and hence exactly) satisfied in a member of $\mathcal{K}$. Let $M_\tau \in \mathcal{K}$ be such that $M_\tau \models \tau$. Then we have $M \equiv \prod_{\mathcal{D}} M_\tau \in \mathcal{K}$. □

5. Preservation

In this section we prove appropriate versions of some preservation theorems in Łukasiewicz predicate logic. The proofs are usually modified versions of their classical counterparts, see [1]. As stated before, $\phi \to \psi$ is satisfied in a model M if and only if $\phi^M \leqslant \psi^M$. For this reason, to simplify the arguments, we sometimes use $\leqslant$ instead of $\to$. The reader can also check the following equivalences (and similar ones) where x is not free in ϕ

$$(\exists x \psi \to \phi) \equiv \forall x(\psi \to \phi), \qquad (\phi \to \forall x \psi) \equiv \forall x(\phi \to \psi).$$

Below by T we mean a theory over Łukasiewicz predicate logic. A formula of the form $\forall \bar{x} \phi$ where ϕ is quantifier-free is called universal. A theory is called universal if it is equivalent to a set of universal sentences. Following [10], we say an L-theory T has *quantifier-elimination* if every formula can be approximated by quantifier-free formulas, i.e. for each $\phi(\bar{x})$ and $\epsilon > 0$ there exists a quantifier-free $\theta(\bar{x})$ such that for each $\bar{a} \in M \vDash T$, one has that $|\phi^M(\bar{a}) - \theta^M(\bar{a})| \leqslant \epsilon$. A relation $P : M^n \to [0,1]$ is *definable* if there exists a sequence $\phi_k(\bar{x})$ of L-formulas such that $\lim_k \phi_k = P$ uniformly on M^n. Uniform convergence makes this notion intrinsic so that the sequence ϕ_k convergence uniformly to a definable relation in any $N \equiv M$. If T has quantifier-elimination, then definable relations are approximated by quantifier-free formulas. Fuzzy nature of Łukasiewicz logic does not avoid us to define the notion of definable set as a subset of M^n whose characteristic function is definable. The set of definable subsets of M^n is closed under Boolean operations.

Proposition 5.1. *A theory T is universal if and only if it is closed under substructure.*

Proof. We prove the if part. Assume T is closed under substructure. Let $T_\forall$ be the set of all universal consequences of T. Every model of T is a model of $T_\forall$. Conversely assume $M \vDash T_\forall$. We claim that $T \cup diag(M)$ is finitely satisfiable. Assume not. Then for some $\phi(\bar{a}) \in diag(M)$, the set $T \cup \{\phi(\bar{a})\}$ is not satisfiable. Thus, by Proposition 4.4, for some $r > 0$ we must have that $T \vDash \phi(\bar{a}) \leqslant 1 - r$. So, since $\bar{a}$ is not used in T, we must have that $T \vDash \exists \bar{x} \phi(\bar{x}) \leqslant 1 - r$ or in other words, $T \vDash r \leqslant \forall \bar{x}(1 - \phi(\bar{x}))$. So, using the rules, we have that $\forall \bar{x}(r \to \neg \phi(\bar{x})) \in T_\forall$. However, this sentence is not satisfied in M, a contradiction. Finally assume $N \vDash T \cup ediag(M)$. Then we have that $M \subseteq N \vDash T$ and hence $M \vDash T$. □

T is said to be *model-complete* if for every models M, N of T, $M \subseteq N$ implies that $M \preceq N$.

Proposition 5.2. *Assume T is model-complete. Then every formula is T-equivalent to a countable set of universal formulas.*

Proof. Let $\phi(\bar{x})$ be a formula and

$$\Gamma = \{\theta(\bar{x}) : \ \theta(\bar{x}) \text{ is universal and } T \vDash \forall \bar{x}(\phi(\bar{x}) \rightarrow \theta(\bar{x}))\}.$$

Γ is closed under conjunction. We claim that $T \cup \Gamma \vDash \phi$. Assume not. Then there exists a model $M \vDash T \cup \Gamma(\bar{d})$ such that $M \nvDash \phi(\bar{d})$ where $\bar{d}$ is a sequence of new constant symbols of length $\bar{x}$. Note that $T \cup \mathrm{diag}(M) \cup \{\phi(\bar{d})\}$ is satisfiable since otherwise $T, \phi(\bar{d}) \vDash \psi(\bar{a}, \bar{d}) \rightarrow r$ for some $\psi(\bar{a}, \bar{d}) \in diag(M)$ and $r < 1$. Then $\forall \bar{y}\,(\psi(\bar{y}, \bar{x}) \rightarrow r) \in \Gamma$ and so $\psi^M(\bar{a}, \bar{d}) \leqslant r$ which is impossible. Now, by compactness, there exists a model $N \vDash T, \phi(\bar{d})$ such that $M \subseteq N$. So, by model-completeness, $\phi(\bar{d})$ is satisfied in M. This is a contradiction. Therefore, we have that $T \cup \Gamma \vDash \phi$ and so for each n there exists $\theta_n \in \Gamma$ such that $T, \theta_n \vDash 1 - \frac{1}{n} \leqslant \phi$. Thus, ϕ is equivalent to $\{\theta_n : n \geqslant 1\}$ over T. $\qquad\square$

A L-theory T is said to have built-in Skolem functions if for every formula $\phi(\bar{x}, y)$ and $k \geqslant 1$ there exists a function symbol $f_k \in L$ such that

$$T \vDash \forall \bar{x}\big[(\exists y \phi(\bar{x}, y) \dot{-} \frac{1}{k}) \rightarrow \phi(\bar{x}, f_k(\bar{x}))\big].$$

In other words, in every model of T the following statement is satisfied:

$$\forall \bar{x}\big[\exists y \phi(\bar{x}, y) \dot{-} \frac{1}{k} \leqslant \phi(\bar{x}, f_k(\bar{x}))\big].$$

As in classical case, if T has built-in Skolem functions then T is model-complete.

An extension $T^* \supseteq T$ in a language $L^* \supseteq L$ is *conservative* if every model of T has an expansion to a model of T^*. A *Skolemization* of an L-theory T is a conservative extension $T^* \supseteq T$ in a language $L^* \supseteq L$ such that T^* has built-in Skolem functions.

Proposition 5.3. *Every L-theory T has a Skolemization in a language L^* with $|L^*| = |L| + \aleph_0$.*

Proof. Set $L_0 = L$ and $T_0 = T$. For each L_0-formula $\phi(\bar{x}, y)$ with $|\bar{x}| = n$ consider a sequence $f_\phi^1, f_\phi^2, ...$ of new n-ary function symbols and let $\phi_k(\bar{x})$ be the formula

$$\exists y \phi(\bar{x}, y) \dot{-} \frac{1}{k} \leqslant \phi(\bar{x}, f_\phi^k(\bar{x})).$$

Let L_1 be the language containing L_0 and all these new function symbols. Let T_1 be the L_1-theory consisting of axioms of T_0 and universal closure of ϕ_k for every ϕ and k. Repeating the argument we obtain chains

$$L_0 \subseteq L_1 \subseteq \cdots$$
$$T_0 \subseteq T_1 \subseteq \cdots$$

Let $L^* = \cup L_i$ and $T^* = \cup T_i$. It is easy to verify that T^* is a conservative extension of T having built-in Skolem function. $\qquad\square$

A similar construction is *Morleyization* of a theory where one adds sufficient predicate symbols (called *Skolem relations*) to the language in order to obtain a conservative extension which is inductive and admits quantifier-elimination. An advantage of this construction is that it requires a definitional expansion of the language.

Let T be an L-theory. A theory $T' \supseteq T$ is said to be a *model-completion* of T if (i) every model of T can be extended to a model of T'. (ii) if a model A of T is embedded in two models B, C of T' then $(B, a)_{a \in A} \equiv (C, a)_{a \in A}$. Using elementary chains, Abraham Robinson has shown that a first order theory can have at most one model-completion. It is easy to verify that the same proof works in Łukasiewicz model theory.

Proposition 5.4. *Every theory has at most one model-completion.*

A theory T is said to be *inductive* if the union of any chain of models of T is a model of T.

Proposition 5.5. (Chang-Łoś-Suszko) *A theory T is inductive if and only if it can be axiomatized by a set of $\forall\exists$-sentences.*

Proof. For the nontrivial direction suppose T is preserved under union of chains. Let $T_{\forall\exists}$ be the set of $\forall\exists$-consequences of T. Assume $M_0 \vDash T_{\forall\exists}$. It follows that $T \cup diag_\forall(M_0)$ is approximately satisfiable. Since otherwise for some $\forall\bar{x}\phi(\bar{x}, \bar{a}) \in diag_\forall(M_0)$ and $r < 1$ we must have that $T \vDash \forall\bar{x}\phi(\bar{x}, \bar{a}) \to r$. But this implies that $T \vDash \forall\bar{y}\exists\bar{x}(\phi(\bar{x}, \bar{y}) \to r)$ and hence $M_0 \vDash \forall\bar{x}\phi(\bar{x}, \bar{a}) \to r$ which is a contradiction. Now suppose $N_0 \vDash T \cup diag_\forall(M_0)$. Then $M_0 \subseteq N_0 \vDash T$. Moreover, every existential sentence of L_{M_0} true in N_0 must hold in M_0, since, if $\exists\bar{x}\phi(\bar{x}, \bar{a})$ holds in N_0 but not in M_0, then for some $r < 1$, $\forall\bar{x}(\phi(\bar{x}, \bar{a}) \to r)$ must hold in M_0 and hence in N_0 which is impossible. Now, this later fact means that $diag(N_0) \cup ediag(M_0)$ is satisfiable. So, by compactness, N_0 has an extension M_1 which is an elementary extension of M_0. Repeating the construction (as $M_1 \vDash T_{\forall\exists}$), we get a chain

$$M_0 \subseteq N_0 \subseteq M_1 \subseteq N_1 \ldots$$

where each N_k is a model of T and M_k's form an elementary chain. The union of this chain is both a model of T and an elementary extension of M_0. Thus M_0 is a model of T, and T is equivalent to $T_{\forall\exists}$. $\square$

We now give some simple examples of Łukasiewicz theories. A theory T is called κ-*categorical* if every two models of T of cardinality κ are approximately isomorphic. In classical model theory there are several methods for constructing $\aleph_0$-categorical theories. Usually, with some modifications, these methods work in the Łukasiewicz context too. Here we use the simplest ways and give some examples of complete categorical theories. Note that the easy method used in these examples can not be similarly applied for continuous model theory (i.e. the framework of [10]) since one must additionally take care of the continuity of interpretations of relation and function symbols.

Example 5.6. In this example we adopt the classical theory of random graphs for the present framework. Let $L = \{R\}$ and T_G be the theory of Łukasiewicz graphs axiomatized by

(i) $R(x, y) = R(y, x)$

(ii) $R(x, x) = 0$.

Let T_{RG} be T_G extended by the following set of axioms

$$\forall \bar{x}^1 \ldots \forall \bar{x}^n \Big[\big(\bigwedge_{i \neq i'; \, j, j'} \neg e(x_j^i, x_{j'}^{i'}) \big) \to \exists y \big(\bigwedge_{i, j} R(y, x_j^i) \leftrightarrow r_i \big) \Big]$$

where $n \geqslant 1$, $r_1, ..., r_n \in [0, 1]$ are distinct and $\bar{x}^1, ..., \bar{x}^n$ are disjoint tuples of variables with lengths $k_1, ..., k_n$. These axioms state that if the tuples $\bar{x}^1, ..., \bar{x}^n$ are interpreted by pairwise disjoint tuples of elements in a model, then there exists y such that for each i, j, $R(y, x_j^i)$ is sufficiently close to r_i. Note that T_{RG} can be axiomatized by a smaller set of axioms, namely T_G augmented by those which only use rational r_i's. Our aim is to show that T_{RG} is complete. First we show that T_{RG} has a model. Let call an L-structure (M, R) rational if R^M takes values in $[0, 1]_{\mathbb{Q}}$. Let G_0 be any countable rational Łukasiewicz graph. It is not hard to construct a countable rational Łukasiewicz graph $G_1 \supseteq G_0$ such that for any pairwise disjoint finite subsets $X_1, ..., X_k \subseteq G_0$ and distinct rational numbers $0 \leqslant r_1, ..., r_k \leqslant 1$ there exists a $g \in G_1$ such that for any i and $x \in X_i$, $R(x, g) = r_i$. Repeating the argument, one obtains a chain $G_0 \subseteq G_1 \subseteq G_2 \ldots$ of countable Łukasiewicz graphs union of which satisfies the above axioms.

Construction of the above graph can be similarly done by replacing the rational numbers with any countable dense subset of $[0, 1]$. The models constructed using different such subsets are clearly not isomorphic. But, they are ϵ-isomorphic for any $\epsilon > 0$. Indeed, an easy back and forth argument shows that any two countable models of T_{RG} are ϵ-isomorphic for each ϵ. In particular, using a Vaught's type test for completeness we have that

Proposition 5.7. *T_{RG} is $\aleph_0$-categorical and hence complete.*

One can similarly show that T_{RG} has quantifier-elimination. Moreover, any countable model M of T_{RG} is countably universal in the sense that every finite or countable model of T_G can be ϵ-embedded in M for any $\epsilon > 0$.

Example 5.8. In this example, we simulate the classical theory of a unary predicate P where both P and its complement are interpreted by infinite sets. Let $L = \{P\}$ where P is a unary relation symbol. For each $r < s$ in the unit interval there exists a set of sentences stating that $P^{-1}[r, s]$ is infinite. Let T be the union of all these sets for r, s. Then T states that the inverse image of every nontrivial interval is infinite. In particular, if $M \vDash T$, the range of P^M is dense in $[0, 1]$. However, it may be the rationals or any other dense subset of $[0, 1]$. This means that T has many non-isomorphic countable models. All these models are approximately isomorphic. Hence T is $\aleph_0$-categorical according to our definition. It is also complete and has quantifier-elimination. If the language contains two unary relation symbols P, R, we may impose further conditions stating that every intersection

$P^{-1}[r,s] \cap R^{-1}[r',s']$ is infinite. The resulting theory is again $\aleph_0$-categorical and hence complete.

Example 5.9. An interesting source of examples is the theories of equivalence relations. An equivalence relation is a Łukasiewicz structure (M,E), where E is a binary relation on M, such that

- $E(x,x) = 1$
- $E(x,y) = E(y,x)$
- $E(x,y) \wedge E(y,z) \leqslant E(x,z)$

In two valued model theory, there are several interesting $\aleph_0$-categorical theories of equivalence relations. The Łukasiewicz variant of these theories can be obtained with some more effort. We leave this task to another paper. We also recall that in the literature, an order is defined as a structure (M,O,E) where E is an equivalence relation and O satisfies the first and the third above axioms (with E replaced by O) and $O(x,y) \wedge O(y,x) \leqslant E(x,y)$.

Example 5.10. Let us see what does it mean "fuzzy group" in the context of Łukasiewicz model theory. By a *Lukasiewicz group* we mean a Łukasiewicz structure of the form $(M,\cdot,\cdot^{-1},\mu\,1)$, where μ is a unary relation on M, such that $(M,\cdot,\cdot^{-1},1)$ is a group and

- $\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y)$
- $\mu(x^{-1}) = \mu(x)$
- $\mu(1) = 1$

Thus, a group is a Łukasiewicz group with $\mu = 1$. In fuzzy algebra, one defines a fuzzy subgroup of a group G (usually called a fuzzy group) as a function $\mu : G \to [0,1]$ such that the above conditions hold for every $x,y \in G$. For a language L and L-structures M,N, define *weak substructure* $M \subseteq_w N$ similar to substructure $M \subseteq N$ except that for every relation symbol R (other than equality) and $\bar{a} \in M$, replace $R^M(\bar{a}) = R^N(\bar{a})$ with $R^M(\bar{a}) \leqslant R^N(\bar{a})$. Then, a fuzzy subgroup of a group G is just a Łukasiewicz group H (defined again on the ground set G) which is also a weak substructure of G. So, fuzzy group theorists study Łukasiewicz groups which are weak substructure of a fixed group.

6. Conclusion

Classical model theory is a logical framework mainly used for studying classical algebraic structures. Similarly, Łukasiewicz model theory can be considered as a logical framework for studying structures studied in Łukasiewicz fuzzy algebra. Compactness and its consequences, in particular the usual preservation theorems, are essential for this purpose. The few examples given above show that parts of properties considered in fuzzy algebra can be expressed and studied in this framework. However, further examples must be given to clarify the situation. The authors hope to continue this line of research in the future.

References

[1] J. Barwise (editor), *Handbook of mathematical logic*, North-Holland, 1977.

[2] Chang and Keisler, *Continuous Model Theory*, Annals of Mathematical Studies, Princeton University Press, **58** (1966).

[3] P. Cintula and P. Hájek, *Triangular norm based predicate fuzzy logics*, Fuzzy sets and systems, **161** (2010), 311-346.

[4] P. Dellunde and F. Esteva, *On elementary extensions for fuzzy predicate logics*, In Proceedings of IPMU, (2010), 747-756.

[5] G. Gerla, *The category of the fuzzy models and Lowenheim-Skolem theorem*, Mathematics of Fuzzy Systems, (1986), 121-141.

[6] P. Hájek, *Metamathematics of Fuzzy Logic, Trends in Logic*, Kluwer Academic Publishers, Dordercht, **4** (1998).

[7] P. Hájek and P. Cintula, *On theories and models in fuzzy predicate logics*, Journal of Symbolic Logic, **71** (2006), 863-880.

[8] F. Montagna, *On the predicate logics of continuous t-norm BL-algebras*, Archive for Mathematical Logic, **44** (2005), 97-114.

[9] I. B. Yaacov, A. P. Pedersen, *A Proof of completeness for continnuous first-order logic*, Journal of Symbolic Logic, **75** (2010), 168-190.

[10] I. B. Yaacov, A. Berenstein, C. W. Henson and A. Usvyatsov, *Model theory for metric structures*, In Model Theory with Applications to Algebra and Analysis, Vol. II, eds. Z. Chatzidakis, D. Macpherson, A. Pillay, and A.Wilkie, Lecture Notes series of the London Mathematical Society, No. 350, Cambridge University Press, (2008), 315-427.

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