REPRESENTATION THEOREMS OF $L$−SUBSETS AND $L$−FAMILIES ON COMPLETE RESIDUATED LATTICE

H. HAN AND J. FANG

Abstract. In this paper, our purpose is twofold. Firstly, the tensor and residuum operations on $L$−nested systems are introduced under the condition of complete residuated lattice. Then we show that $L$−nested systems form a complete residuated lattice, which is precisely the classical isomorphic object of complete residuated power set lattice. Thus the new representation theorem of $L$−subsets on complete residuated lattice is obtained. Secondly, we introduce the concepts of $L$−family and the system of $L$−subsets, then with the tool of the system of $L$−subsets, we obtain the representation theorem of intersection-preserving $L$−families on complete residuated lattice.

1. Introduction

Since Zadeh proposed $L$−subset theory in 1965, many scholars worked on the connection between $L$−subsets and classical sets. Representation theorem is a main form to establish this connection, whose essence is to search the classical isomorphic object of $L$−power set lattice. Luo [5] first proposed the concept of nested systems, and established a representation theorem of $L$−subsets with them (in his case, $L = [0, 1]$). Then Zhang [13], Shi [7], R. Bělohlávek [1] further studied representation theorems of $L$−subsets based on different forms of nested systems. It is of value to note Xiong [12], Fang and Han [2] studied the representation theorems of $L$−subsets with different tools on the condition that $L$ is only a complete lattice. Recently, a number of related work are constantly in progress, among which literatures [3, 8-10] are newer results.

Recently, scholars usually use complete residuated lattice as the membership degree value lattice of $L$−subsets, e.g. R. Bělohlávek [1]. As a matter of fact, we can prove that $L^X$ is indeed a complete residuated lattice w.r.t the tensor and residuum operations induced from $L$, hence we call it complete residuated power set lattice. One aim of this paper is to give the classical isomorphic object of complete residuated power set lattice, thus establishing a new representation theorem of $L$−subsets on complete residuated lattice.

In scholars’ investigations, there are some special $L$−subsets which are maps from $L^X$ to $L$, such as many-valued filters, lattice-valued convergence structures and so on. In this paper, we call these special $L$−subsets $L$−families. It should be pointed out that $L$−families also have some kind of “levels”, and they can

Received: October 2011; Revised: April 2012; Accepted: July 2012

Key words and phrases: Complete residuated lattices, $L$−subsets, $L$−nested systems, $L$−families, Level $L$−subsets, Representation theorems.
be described with certain “level structures” as well, e.g. G.Jäger [4] discussed the case of lattice-valued uniform convergence spaces and lattice-valued uniform spaces. Hence, it makes sense to find the relations between \( L \)-family and its “levels”. For this purpose, we introduce the concept of the system of \( L \)-subsets. Moreover, we prove that there is a one-to-one correspondence between intersection-preserving \( L \)-families and the systems of \( L \)-subsets. That is the representation theorem of intersection-preserving \( L \)-families on complete residuated lattice.

2. Preliminaries

In this paper, we consider \( L \) a complete lattice, 0 and 1 the smallest and the greatest elements of \( L \) respectively, and \( X \) a nonempty set. An \( L \)-subset in \( X \) is a map \( A : X \to L \). The set of all \( L \)-subsets in \( X \) will be denoted by \( L^X \). Let \( 0_X \) and \( 1_X \) denote the smallest and the greatest elements of \( L^X \). Denote the set of all subsets of \( X \) by \( \mathcal{P}(X) \). In this paper, we do not distinguish between subsets of \( X \) and their characteristic functions. For each \( a \in L \) and \( A \in L^X \), we denote the cut set of \( A \) by \( A_a = \{ x \in X \mid A(x) \geq a \} \). \( L \)-subsets \( (a \land A) : X \to L \) and \( (a \lor A) : X \to L \) mean \( (a \land A)(x) = a \land A(x) \) and \( (a \lor A)(x) = a \lor A(x) \) for each \( x \in X \).

Generally, when \( L \) is a complete lattice, \( L^X \) is also a complete lattice. In the following, some basic facts needed in the sequel are presented.

**Theorem 2.1.** [13, 7] Let \( X \) be a nonempty set and \( L \) be a complete lattice. Then for each \( A \in L^X \), we have \( A = \bigvee_{a \in L}(a \land A_a) \).

**Definition 2.2.** [1] A map \( H : L \to \mathcal{P}(X) \) subjects to the conditions

\begin{itemize}
  \item [\text{LH1}] For \( a, b \in L \), \( a \leq b \) implies \( H(b) \subseteq H(a) \),
  \item [\text{LH2}] For each \( x \in X \), the subset \( \{ a \mid x \in H(a) \} \) of \( L \) is nonempty and has a greatest element,
\end{itemize}

is called an \( L \)-nested system. The family of all \( L \)-nested systems on \( X \) will be denoted by \( H_L(X) \).

Let \( H, G \in H_L(X) \). We define a partial order “\( \leq \)” on \( H_L(X) \) as follows:

\[ H \leq G \iff \forall a \in L, \ H(a) \subseteq G(a). \]

Then \( H_L(X) \) has a smallest element \( H^0 : L \to \mathcal{P}(X) \) defined as follows:

\[ H^0(a) = \begin{cases} X, & a = 0, \\ \emptyset, & a \neq 0. \end{cases} \]

By the above definition, we can prove that the partially ordered set \( (H_L(X), \leq) \) is a complete lattice. That is the following proposition.

**Proposition 2.3.** Let \( X \) be a nonempty set and \( L \) be a complete lattice, then \( (H_L(X), \leq) \) is a complete lattice.

Let \( \{ H_t \mid t \in T \} \subseteq H_L(X) \). By Proposition 2.3, the infimum and supremum of \( H_L(X) \) is defined as:

\[ \left( \bigwedge_{t \in T} H_t \right)(a) = \bigcap_{t \in T} H_t(a), \]

\[ \left( \bigvee_{t \in T} H_t \right) = \bigwedge \{ H \in H_L(X) \mid \forall t \in T, \ H \geq H_t \}. \]
Definition 2.4. [1] Let \((L, \leq)\) be a complete lattice. If there are binary operations \(\otimes\) and \(\rightarrow\) on \(L\) that satisfy:

(R1) \((L, \otimes, 1)\) is a commutative monoid, i.e., \(\otimes\) is commutative, associative, and the identity \(x \otimes 1 = x\) holds for each \(x \in X\),

(R2) adjointness property, i.e. \(x \otimes y \leq z \iff x \leq y \rightarrow z\) holds for all \(x, y, z \in L\),

then \(L\) is called a complete residuated lattice with respect to \(\otimes\) and \(\rightarrow\). Operations \(\otimes\) and \(\rightarrow\) are called tensor and residuum on \(L\) respectively.

In the following proposition, we give a list of some properties of operations \(\otimes\) and \(\rightarrow\).

Proposition 2.5. [1] Let \(L\) be a complete residuated lattice. Then the following holds for all \(x, y, z \in L\) and \(\{y_i\}_{i \in I} \subseteq L\):

(a) \(x \rightarrow x = 1\),
(b) \((x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z)\),
(c) \(x \leq (x \rightarrow y) \rightarrow y\),
(d) If \(y \leq z\), then \(x \rightarrow y \leq x \rightarrow z\),
(e) If \(x \leq y\), then \(y \rightarrow z \leq x \rightarrow z\),
(f) \(x \rightarrow \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \rightarrow y_i)\),
(g) \(\bigvee_{i \in I} y_i \rightarrow y = \bigwedge_{i \in I} (y_i \rightarrow y)\).

3. Representation Theorem of \(L\)-subsets on Complete Residuated Lattice

In this section, \(L\) is always assumed to be a complete residuated lattice, we shall define tensor and residuum operations on \(H_L(X)\) and show that \(H_L(X)\) forms a complete residuated lattice with respect to these two operations. Then the classical isomorphic object of complete residuated power set lattice will be given, and the new representation theorem of \(L\)-subsets is obtained.

Definition 3.1. Let \(H \in H_L(X)\). An \(L\)-subset \(\theta_H : X \rightarrow L\) defined by

\[ \forall x \in X, \theta_H(x) = \bigvee \{a \in L \mid x \in H(a)\} \]

is called an \(L\)-subset induced by \(H\).

By the above definition, the tensor and residuum operations on \(H_L(X)\) can be defined in a natural way. For this purpose, we need two lemmas to show rationality of the definition.

Lemma 3.2. Let \(H, G \in H_L(X)\). Define a map \(H \otimes G : L \rightarrow \mathcal{P}(X)\) as follows:

\[ \forall a \in L, (H \otimes G)(a) = \{x \mid \theta_H(x) \otimes \theta_G(x) \geq a\} \]

then \(H \otimes G \in H_L(X)\), i.e., \(H \otimes G\) is an \(L\)-nested system.

Proof. By Definition 2.2, we check that \(H \otimes G\) satisfies (LH1) and (LH2) as follows.

(LH1) For \(a, b \in L\), if \(a \leq b\), then

\[ (H \otimes G)(b) = \{x \mid \theta_H(x) \otimes \theta_G(x) \geq b\} \subset \{x \mid \theta_H(x) \otimes \theta_G(x) \geq a\} = (H \otimes G)(a). \]
(LH2) For each $x \in X$, since \[ \{ a \mid x \in (H \otimes G)(a) \} = \{ a \mid \theta_H(x) \otimes \theta_G(x) \geq a \}, \]
\[ \theta_H(x) \otimes \theta_G(x) \] is the greatest element of the subset \[ \{ a \mid x \in (H \otimes G)(a) \} \] of $L$. □

**Lemma 3.3.** Let $H, G \in H_L(X)$. Define a map $H \rightarrow G : L \rightarrow \mathcal{P}(X)$ as follows:
\[ \forall a \in L, \ (H \rightarrow G)(a) = \{ x \mid \theta_H(x) \rightarrow \theta_G(x) \geq a \}, \]
then $H \rightarrow G \in H_L(X)$, i.e., $H \rightarrow G$ is an $L$-nested system.

By Lemmas 3.2 and 3.3, for $H, G \in H_L(X)$, $H \otimes G$ and $H \rightarrow G$ are both $L$-nested systems. The defined operations $\otimes$ and $\rightarrow$ are called tensor and residuum on $H_L(X)$ respectively.

Based on the operations on $H_L(X)$ defined above, we obtain the following important theorem.

**Theorem 3.4.** The complete lattice $H_L(X)$ forms a complete residuated lattice with respect to $(\otimes, \rightarrow)$.

*Proof.* (1) By Proposition 2.3, we know that $(H_L(X), \leq)$ is a complete lattice, and its greatest element $H^1$ is defined by: \( \forall a \in L, \ H^1(a) = X. \)

(2) We prove that $(H_L(X), \otimes, H^1)$ is a commutative monoid, i.e. (R1) holds.

Firstly, we need to prove that $\otimes$ satisfies commutative law. For each $H, G \in H_L(X)$ and each $a \in L$, we have
\[ (H \otimes G)(a) = \{ x \mid \theta_H(x) \otimes \theta_G(x) \geq a \} = \{ x \mid \theta_G(x) \otimes \theta_H(x) \geq a \} = (G \otimes H)(a). \]

By the arbitrariness of $a$, we obtain $H \otimes G = G \otimes H$.

Secondly, we prove that $\otimes$ has a unit element. For each $H \in H_L(X)$ and each $a \in L$, the following holds:
\[ (H \otimes H^1)(a) = \{ x \mid \theta_H(x) \otimes \theta_{H^1}(x) \geq a \} = \{ x \mid \theta_H(x) \geq a \} = H(a). \]

By the arbitrariness of $a$, we know that $H \otimes H^1 = H$, which means $H^1$ is the unit element.

Thirdly, we prove $\otimes$ satisfies the associative law. Let $H, G, M \in H_L(X)$ and $a \in L$. For all $x \in X$, since
\[ \theta_{H \otimes G}(x) = \bigvee \{ a \in L \mid x \in (H \otimes G)(a) \} = \bigvee \{ a \in L \mid \theta_H(x) \otimes \theta_G(x) \geq a \} = \theta_H(x) \otimes \theta_G(x), \]
it is observed that
\[ ((H \otimes G) \otimes M)(a) = \{ x \mid \theta_{H \otimes G}(x) \otimes \theta_M(x) \geq a \} = \{ x \mid \theta_H(x) \otimes \theta_G(x) \otimes \theta_M(x) \geq a \} = \{ x \mid \theta_H(x) \otimes \theta_G(x) \otimes \theta_M(x) \geq a \} = (H \otimes (G \otimes M))(a). \]
By the arbitrariness of $a$, $(H \otimes G) \otimes M = H \otimes (G \otimes M)$ holds.

(3) We need to prove that $(\otimes, \to)$ satisfies (R2). For each $H, G, M \in H_L(X)$, it remains to prove $H \otimes G \leq M \iff H \leq G \to M$. In fact, this can be proved by the following equations:

\[
(H \otimes G) \leq M \iff \forall x \in X, \theta_{H \otimes G}(x) \leq \theta_M(x)
\]
\[
\forall x \in X, \theta_H(x) \otimes \theta_G(x) \leq \theta_M(x) 
\]
\[
\forall x \in X, \theta_H(x) \leq \theta_G(x) \to \theta_M(x)
\]
\[
\forall x \in X, \theta_H(x) \leq \theta_{G \to M}(x)
\]
\[
H \leq G \to M.
\]

Finally, it follows from the above (1)-(3) and Definition 2.4 that $(H_L(X), \leq)$ is a complete residuated lattice w.r.t the operations $\otimes$ and $\to$ defined in Lemmas 3.2 and 3.3.

In order to obtain the new representation theorem of $L$–subsets on complete residuated lattice, we need the following lemma for preparation.

**Lemma 3.5.** Let $M, L$ be complete residuated lattices. If $f : M \to L$ is an isomorphism between complete lattices, then $f$ preserves the residuum operation.

The classical isomorphic object of complete residuated power set lattice is obtained from the following theorem.

**Theorem 3.6.** Let $X$ be a nonempty set and $L$ be a complete residuated lattice. Then $(H_L(X), \lor, \land, \otimes, \to) \cong (L^X, \lor, \land, \otimes, \to)$.

**Proof.** Define a map $f : H_L(X) \to L^X$ by $f(H) = \bigvee_{a \in L} \{a \land H(a)\}$ for each $H \in H_L(X)$. As it is known that $f$ is a bijection and preserves intersection and union operations, we only need to prove $f$ preserves tensor and residuum operations.

For each $x \in X$ and $H, G \in H_L(X)$, the following equations hold:

\[
f(H \otimes G)(x) = \bigvee_{a \in L} \{a \land (H \otimes G)(a)(x)\}
\]
\[
= \bigvee_{\{a \in L \mid x \in (H \otimes G)(a)\}} \theta_H(x) \otimes \theta_G(x)
\]
\[
= \bigvee_{\{a \in L \mid x \in H(a)\}} \bigvee_{\{a \in L \mid x \in G(a)\}} \theta_H(x) \otimes \theta_G(x)
\]
\[
= \bigvee_{a \in L} \{a \land (H(a))(x)\} \otimes \bigvee_{a \in L} \{a \land G(a)(x)\}
\]
\[
f(H)(x) \otimes f(G)(x)
\]
\[
= (f(H) \otimes f(G))(x).
\]

By the arbitrariness of $x$ we have $f(H \otimes G) = f(H) \otimes f(G)$, i.e., $f$ preserves tensor operation. By Lemma 3.5 we obtain that $f$ also preserves residuum operation. Therefore, $f$ is an isomorphism between $(H_L(X), \lor, \land, \otimes, \to)$ and $(L^X, \lor, \land, \otimes, \to)$, i.e., $(H_L(X), \lor, \land, \otimes, \to) \cong (L^X, \lor, \land, \otimes, \to)$, as desired.
Consequently, Theorem 3.6 approves that $H_L(X)$ is the classical isomorphic object of $L$–power set lattice $L^X$ under the condition that $L$ is a complete residuated lattice. That is the new representation theorem of $L$–subsets on complete residuated lattice.

4. The Level $L$–subsets of $L$–families and Their Representation

In this section, we introduce the concept of $L$–families and define the system of $L$–subsets on the condition that $L$ is a complete residuated lattice. It is proved that there is a one-to-one correspondence between all intersection-preserving $L$–families and all systems of $L$–subsets, that is the representation theorem of intersection-preserving $L$–families on complete residuated lattice.

In the following, we introduce the concept of $L$–families, then discuss the union and intersection operations of $L$–families w.r.t $L$–partial order $S(-,-)$.

**Definition 4.1.** Let $L$ be a complete residuated lattice, $X$ be a nonempty set. An $L$–family in $X$ is a map $A : L^X \to L$. For each $B \in L^X$, $A(B)$ is called the membership degree of $B$ in $A$.

**Example 4.2.** Let $L$ be a complete residuated lattice, $X$ be a nonempty set. If $F : L^X \to L$ satisfies:

(F1) $F(1_X) = 1$, $F(0_X) = 0$,
(F2) $F(A) \land F(B) \leq F(A \land B)$,
(F3) $A \leq B \Rightarrow F(A) \leq F(B)$,

for all $A, B \in L^X$. Then $F$ is an $L$–family in $X$.

For each $A, B \in L^X$, define $S : L^X \times L^X \to L$ as follows (which can be seen in [1]):

\[ S(A, B) = \bigwedge_{x \in X} (A(x) \Rightarrow B(x)), \]

Then, $S$ is a binary $L$–relation on $L^X$. $S(A, B)$ is called subsethood degree of $A$ in $B$.

It is easily checked that $S(-,-)$ is an $L$–partial order on $L^X$, which means that $S(-,-)$ fulfills:

(P1) $S(A, A) = 1$ ;
(P2) If $S(A, B) = 1$ and $S(B, A) = 1$, then $A = B$ ;
(P3) $S(A, B) \circ S(B, C) \leq S(A, C)$ .

The pair $(L^X, S(-,-))$ forms an $L$–partially ordered set.

Definition 4.3 gives the union and intersection of $L$–family $A$ w.r.t $L$–partial order $S(-,-)$.

**Definition 4.3.** [1] Let $A : L^X \to L$ be an $L$–family in $X$. Define $L$–subsets $\sup A$, $\inf A \in L^X$ such that for all $x \in X$,

\[ \sup A(x) = \bigvee_{B \in L^X} (A(B) \circ B(x)), \]
\[ \inf A(x) = \bigwedge_{C \in L^X} (A(C) \Rightarrow C(x)). \]
Then \(\sup \mathcal{A}\) is called the union of \(\mathcal{A}\) w.r.t \(S(\cdot, \cdot)\) and \(\inf \mathcal{A}\) is called the intersection of \(\mathcal{A}\) w.r.t \(S(\cdot, \cdot)\).

Based on the union and intersection of an \(L\)-family, we have the following result.

**Proposition 4.4.** Let \(\mathcal{A} : L^X \to L\) be an \(L\)-family in \(X\). Then for all \(B \in L^X\), the following equations hold:

\[
S(\sup \mathcal{A}, B) = \bigwedge_{C \in L^X} (\mathcal{A}(C) \rightarrow S(C, B)),
\]

\[
S(B, \inf \mathcal{A}) = \bigwedge_{C \in L^X} (\mathcal{A}(C) \rightarrow S(B, C)).
\]

Next, we give the concept of level \(L\)-subsets of an \(L\)-family.

**Definition 4.5.** Let \(\mathcal{A} : L^X \to L\) be an \(L\)-family. An \(L\)-subset \(A_\alpha\) defined by

\[
A_\alpha = \bigwedge \{ C \mid \mathcal{A}(C) \geq \alpha \}
\]

is called a level \(L\)-subset of \(A\), where \(\alpha \in L\). \(\{A_\alpha\}_{\alpha \in L}\) is called the system of level \(L\)-subsets of \(A\).

In the following, we introduce the concept of the system of \(L\)-subsets, with which we establish the representation theorem of intersection-preserving \(L\)-families on complete residuated lattice.

**Definition 4.6.** Let \(\{H_\alpha\}_{\alpha \in L} \subseteq L^X\) satisfy:

(C1) If \(M \subseteq L\), then \(H_\lor M = \bigvee_{\alpha \in M} H_\alpha\),

(C2) \(H_0 = 0_X\),

then \(\{H_\alpha\}_{\alpha \in L}\) is called the system of \(L\)-subsets.

The following example shows that the system of level \(L\)-subsets of an intersection-preserving \(L\)-family is the system of \(L\)-subsets, that is to say the system of \(L\)-subsets defined above exists.

**Example 4.7.** Let \(\mathcal{A} : L^X \to L\) be an \(L\)-family which preserves arbitrary intersections, \(\{A_\alpha\}_{\alpha \in L}\) be the system of level \(L\)-subsets of \(\mathcal{A}\). Then \(\{A_\alpha\}_{\alpha \in L}\) is the system of \(L\)-subsets.

**Proof.** That \(\{A_\alpha\}_{\alpha \in L}\) satisfies (C2) in Definition 4.6 is obvious. We verify (C1) as follows:

1. For all \(\alpha \in M\), we have \(A_\alpha = \bigwedge_{\mathcal{A}(C) \geq \alpha} C \leq \bigwedge_{\mathcal{A}(C) \geq \lor M} C = A_\lor M\), thus \(\bigvee_{\alpha \in M} A_\alpha \leq A_\lor M\).

2. Under the condition that \(\mathcal{A}\) is an intersection-preserving map, we have

\[
\mathcal{A}\left(\bigvee_{\alpha \in M} A_\alpha\right) \geq \mathcal{A}(A_\alpha) = \mathcal{A}\left(\bigwedge_{\mathcal{A}(C) \geq \alpha} C\right) = \bigwedge_{\mathcal{A}(C) \geq \alpha} \mathcal{A}(C) \geq \alpha
\]

for all \(\alpha \in M\). Hence, \(\mathcal{A}\left(\bigvee_{\alpha \in M} A_\alpha\right) \geq \lor M\), which implies \(A_\lor M \leq \bigvee_{\alpha \in M} A_\alpha\).

To sum up, we have \(A_\lor M = \bigvee_{\alpha \in M} A_\alpha\), as desired.

The following Lemma 4.8 shows the sufficient and necessary condition that an \(L\)-family can be represented by its system of level \(L\)-subsets.
Before this, we introduce a term first. Let $\mathcal{A} : L^X \to L$ be an $L$–family, \{\mathcal{B}_\alpha\}_{\alpha \in L}$ be a family of $L$–subsets. Then we say that \mathcal{A} can be represented by \{\mathcal{B}_\alpha\}_{\alpha \in L} if it satisfies $\mathcal{A}(C) = \bigvee \{ \alpha \mid C \supseteq \mathcal{B}_\alpha \}$ for all $C \in L^X$.

A map $\mathcal{A} : L^X \to L$ is called intersection-preserving iff $\mathcal{A} \left( \bigwedge_{j \in J} B_j \right) = \bigwedge_{j \in J} \mathcal{A}(B_j)$ holds for all $\{B_j\}_{j \in J} \subseteq L^X$.

**Lemma 4.8.** Let $\mathcal{A} : L^X \to L$ be an $L$–family, \{\mathcal{A}_\alpha\}_{\alpha \in L}$ be the system of level $L$–subsets of $\mathcal{A}$. Then $\mathcal{A}$ can be represented by \{\mathcal{A}_\alpha\}_{\alpha \in L} iff $\mathcal{A}$ is an intersection-preserving map.

**Proof.** Necessity. For $\{B_j\}_{j \in J} \subseteq L^X$, put $\alpha = \bigwedge_{j \in J} \mathcal{A}(B_j)$. Then $B_j \supseteq \mathcal{A}_\alpha$ holds for all $j \in J$. This shows $\bigwedge_{j \in J} B_j \supseteq \mathcal{A}_\alpha$. Since $\mathcal{A} : L^X \to L$ can be represented by \{\mathcal{A}_\alpha\}_{\alpha \in L}$, it follows that

$$\mathcal{A} \left( \bigwedge_{j \in J} B_j \right) = \bigvee \{ \beta \mid \bigwedge_{j \in J} B_j \supseteq \mathcal{A}_\beta \} \geq \alpha = \bigwedge_{j \in J} \mathcal{A}(B_j).$$

On the other hand, for each $C, D \in L^X$ with $C \subseteq D$, we have $\mathcal{A}(C) \subseteq \mathcal{A}(D)$. This implies $\mathcal{A} \left( \bigwedge_{j \in J} B_j \right) \subseteq \bigwedge_{j \in J} \mathcal{A}(B_j)$. Therefore, $\mathcal{A} \left( \bigwedge_{j \in J} B_j \right) = \bigwedge_{j \in J} \mathcal{A}(B_j)$.

Sufficiency. On one hand, for all $\alpha \in L$, we have

$$\mathcal{A}(\mathcal{A}_\alpha) = \mathcal{A} \left( \bigwedge \mathcal{A}(C) \right) = \bigwedge_{\mathcal{A}(C) \geq \alpha} \mathcal{A}(C) \geq \alpha.$$

By this, for each $D \in L^X$, with $D \supseteq \mathcal{A}_\alpha$, $\mathcal{A}(D) \geq \alpha$ always hold. So $\mathcal{A}(D) \geq \bigvee \{ \alpha \mid D \supseteq \mathcal{A}_\alpha \}$ holds for each $D \in L^X$.

On the other hand, for each $D \in L^X$, put $\alpha_D = \mathcal{A}(D)$, then we have $D \supseteq \mathcal{A}_{\alpha_D}$. Thus $\mathcal{A}(D) = \alpha_D \leq \bigvee \{ \alpha \mid D \supseteq \mathcal{A}_\alpha \}$. Hence, $\mathcal{A}(D) = \bigvee \{ \alpha \mid D \supseteq \mathcal{A}_\alpha \}$ holds for every $D \in L^X$, which means $\mathcal{A}$ can be represented by \{\mathcal{A}_\alpha\}_{\alpha \in L}$.

We collect our main results in the following two theorems.

The following theorem shows that when $L$–family $\mathcal{A}$ preserves arbitrary intersections, any $L$–family which can be represented by the system of level $L$–subsets of $\mathcal{A}$ is equal to $\mathcal{A}$.

**Theorem 4.9.** Let $\mathcal{A} : L^X \to L$ be an $L$–family which preserves arbitrary intersections, \{\mathcal{A}_\alpha\}_{\alpha \in L} be the system of level $L$–subsets of $\mathcal{A}$. If $\mathcal{B} : L^X \to L$ is a map that can be represented by \{\mathcal{A}_\alpha\}_{\alpha \in L}$, then

1. $\mathcal{B}$ is an $L$–family that preserves arbitrary intersections,
2. \( \mathcal{B} = \mathcal{A} \).

**Proof.** (1) In order to show $\mathcal{B}$ is an intersection-preserving map, we need to check $\mathcal{B} \left( \bigwedge_{j \in J} D_j \right) = \bigwedge_{j \in J} \mathcal{B}(D_j)$ for all $\{D_j\}_{j \in J} \subseteq L^X$. First of all, since $\mathcal{B}$ can be represented by \{\mathcal{A}_\alpha\}_{\alpha \in L}$, we have $\mathcal{B}(C) \subseteq \mathcal{B}(D)$ whenever $C, D \in L^X$ with $C \subseteq D$.

On one hand, put $\alpha = \bigwedge_{j \in J} \mathcal{B}(D_j)$, then $\alpha \leq \mathcal{B}(D_j)$ holds for all $j \in J$. Put $\mathcal{B}(D_j) = \bigvee \{ \beta \mid D_j \supseteq \mathcal{A}_\beta \} = \gamma$, as \{\mathcal{A}_\alpha\}_{\alpha \in L} is the system of $L$–subsets it satisfies (C1) of Definition 4.6, we have $D_j \supseteq \mathcal{A}_\gamma$. Actually, that is because: If we put $M = \{ \beta \mid D_j \supseteq \mathcal{A}_\beta \}$, then $\gamma = \bigvee M$. We have $\mathcal{A}_\gamma = \mathcal{A}_{\bigvee M} = \bigvee_{\beta \in M} \mathcal{A}_\beta \leq \mathcal{B}(D_j).$
Theorem 4.12.

\[ \bigvee_{B \in M} D_j = D_j. \] Thus \( D_j \geq A_\alpha \) for all \( j \in J \). Furthermore, \( \bigwedge_{j \in J} D_j \geq A_\alpha \), which implies \( B(B) = \bigvee \{ \beta \mid \bigwedge_{j \in J} D_j \geq A_\beta \} \geq \alpha = \bigwedge_{j \in J} B(D_j). \)

On the other hand, since \( \bigwedge_{j \in J} D_j \leq D_j \) holds for all \( j \in J \), \( B(B) \leq B(D_j) \) holds for each \( j \in J \). We have with this \( B(B) \leq B(D_j) \).

By the above proof, \( B(B) = \bigvee \{ \bigwedge_{j \in J} D_j \} \).

Proof. (1) It is similar to the proof of Theorem 4.9 (1).

(2) As \( \mathcal{A} \) is an \( L \)-family which preserves arbitrary intersections together with Lemma 4.8, we obtain \( B(C) = \bigvee \{ \alpha \mid C \geq A_\alpha \} = \mathcal{A}(C) \) for every \( C \in L^X \). Therefore, \( B = \mathcal{A} \).

Let \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \) be the system of \( L \)-subsets. If \( \mathcal{A} \) is an \( L \)-family that can be represented by \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \), then there is a question whether the system of level \( L \)-subsets of \( \mathcal{A} \) is precisely \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \) or not. The following Theorem 4.10 gives the answer.

**Theorem 4.10.** Let \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \) be the system of \( L \)-subsets. If \( \mathcal{A} : L^X \rightarrow L \) is an \( L \)-family that can be represented by \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \), \( \{ A_\alpha \}_{\alpha \in \mathcal{A}} \) is the system of level \( L \)-subsets of \( \mathcal{A} \), then we have

1. \( \mathcal{A} \) is an \( L \)-family which preserves arbitrary intersections.
2. \( \mathcal{H}_\alpha = A_\alpha \) holds for all \( \alpha \in L \).

Proof. (1) It is similar to the proof of Theorem 4.9 (1).

(2) As \( \mathcal{A} \) can be represented by \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \), for every \( \alpha \in L \), \( \mathcal{A}(C) = \bigvee \{ \beta \mid \mathcal{H}_\alpha \geq \mathcal{H}_\beta \} \) holds. Since \( \mathcal{H}_\alpha \geq \mathcal{H}_\alpha \), we have \( \mathcal{A}(\mathcal{H}_\alpha) \geq \alpha \). So \( A_\alpha \leq \mathcal{H}_\alpha \) holds since \( \{ A_\alpha \}_{\alpha \in \mathcal{A}} \) is the system of level \( L \)-subsets of \( \mathcal{A} \). In the following we prove \( \mathcal{H}_\alpha \leq A_\alpha \) for all \( \alpha \in L \).

It is sufficient to verify for a given \( \alpha \in L \) and each \( D \in L^X \) that \( D \geq A_\alpha \) \( \Rightarrow \) \( D \geq \mathcal{H}_\alpha \). For \( D \geq A_\alpha \), \( \mathcal{A}(D) \geq \mathcal{A}(A_\alpha) \) holds since \( \mathcal{A} \) is an order-preserving map. It follows from the property of \( \mathcal{A} \) being an intersection-preserving map that \( \mathcal{A}(A_\alpha) = \mathcal{A}(D) \geq \mathcal{A}(\mathcal{H}_\alpha) \geq \alpha \), which means \( \mathcal{A}(D) \geq \alpha \). Letting \( \xi = \mathcal{A}(D) \), it follows from \( \{ \mathcal{H}_\alpha \}_{\alpha \in L} \) satisfying (C1) that \( D \geq \mathcal{H}_\xi \).

In fact, if we denote \( M = \{ \beta \mid D \geq \mathcal{H}_\beta \} \), then \( \xi = \bigvee M \). Therefore, \( \mathcal{H}_\xi \leq \bigvee_{\beta \in M} D = D \). Since \( \xi \geq \alpha \), we have \( D \geq \mathcal{H}_\xi \geq \mathcal{H}_\alpha \). Hence, \( \mathcal{H}_\alpha \leq A_\alpha \) for all \( \alpha \in L \).

The above Theorems 4.9 and 4.10 establish the one-to-one correspondence between all intersection-preserving \( L \)-families and all systems of \( L \)-subsets. That is called representation theorem of intersection-preserving \( L \)-families.

Next, we present a kind of \( L \)-families which satisfy some additional conditions and we call them principal \( L \)-families.

**Definition 4.11.** Let \( \mathcal{A} : L^X \rightarrow L \) be an \( L \)-family in \( X \). If there exists some \( B \in L^X \) such that \( \mathcal{A}(C) = S(B, C) \) for all \( C \in L^X \), then \( \mathcal{A} \) is called a principal \( L \)-family.

The following theorem describes the property of a principal \( L \)-family.

**Theorem 4.12.** A map \( \mathcal{A} : L^X \rightarrow L \) is a principal \( L \)-family iff \( \mathcal{A}(C) = S(\inf \mathcal{A}, C) \) holds for all \( C \in L^X \).
Proof. The sufficiency is easily proved by Definition 4.11, we prove the necessity as follows.

Suppose there exists some \( B \in L^X \) such that \( \mathcal{A}(C) = S(B, C) \) for each \( C \in L^X \), it remains to prove \( B = \inf \mathcal{A} \).

First, we have
\[
S(B, \inf \mathcal{A}) = \bigwedge_{x \in X} (\inf \mathcal{A}(x) \to B(x))
\]
\[
= \bigwedge_{x \in X} \left( \bigwedge_{C \in L^X} (\mathcal{A}(C) \to C(x)) \to B(x) \right)
\]
\[
\geq \bigwedge_{x \in X} (\mathcal{A}(B) \to B(x))
\]
\[
= \mathcal{A}(B)
\]
\[
= 1.
\]

Then, by \( \mathcal{A}(B) = S(B, B) = 1 \) the following holds:
\[
S(\inf \mathcal{A}, B) = \bigwedge_{x \in X} (\inf \mathcal{A}(x) \to \bigwedge_{j \in J} B_j(x))
\]
\[
= \bigwedge_{x \in X} \left( \bigwedge_{j \in J} (\mathcal{A}(B_j) \to B_j(x)) \right)
\]
\[
\geq \bigwedge_{j \in J} \mathcal{A}(B_j)
\]
\[
= 1.
\]

To sum up, it follows from \( S(\inf \mathcal{A}, B) = 1 \) and \( S(B, \inf \mathcal{A}) = 1 \) that \( B = \inf \mathcal{A} \). \( \square \)

Corollary 4.13. If \( \mathcal{A} : L^X \to L \) is a principal \( L \)-family, then \( \mathcal{A} \) is an intersection-preserving map, that is for \( \{B_j \mid j \in J\} \subseteq L^X \),
\[
\mathcal{A}\left( \bigwedge_{j \in J} B_j \right) = \bigwedge_{j \in J} \mathcal{A}(B_j).
\]

Proof. Let \( \{B_j \}_{j \in J} \subseteq L^X \). By Theorem 4.12, we have
\[
\mathcal{A}\left( \bigwedge_{j \in J} B_j \right) = S\left( \inf \mathcal{A}, \bigwedge_{j \in J} B_j \right)
\]
\[
= \bigwedge_{x \in X} (\inf \mathcal{A}(x) \to \bigwedge_{j \in J} B_j(x))
\]
\[
= \bigwedge_{x \in X} \left( \bigwedge_{j \in J} (\mathcal{A}(B_j) \to B_j(x)) \right)
\]
\[
= \bigwedge_{j \in J} \mathcal{A}(B_j)
\]
\[
= \mathcal{A}(B_j),
\]
which means that \( \mathcal{A} \) is an intersection-preserving map. \( \square \)

The next proposition shows when \( \mathcal{A} \) is a principal \( L \)-family, \( \inf \mathcal{A} \) can be represented by the system of level \( L \)-subsets of \( \mathcal{A} \) as the following form.

Proposition 4.14. Let \( \mathcal{A} : L^X \to L \) be a principal \( L \)-family, \( \{A_\alpha \}_{\alpha \in L} \) be the system of level \( L \)-subsets of \( \mathcal{A} \). Then \( \inf \mathcal{A} = \bigwedge_{\alpha \in L} (\alpha \to A_\alpha) \) holds.
Proof. (1) Recall that \( \inf \mathcal{A} = \bigwedge_{C \in L^X} (\mathcal{A}(C) \rightarrow C) \), and let \( \alpha_c = \mathcal{A}(C) \) for each \( C \in L^X \). Then \( C \geq \mathcal{A}_\alpha \). Thus for each \( x \in X \),
\[
\inf \mathcal{A}(x) = \bigwedge_{C \in L^X} (\mathcal{A}(C) \rightarrow C(x)) \\
\geq \bigwedge_{C \in L^X} (\alpha_c \rightarrow \mathcal{A}_\alpha(x)) \\
\geq \bigwedge_{\alpha \in L} (\alpha \rightarrow \mathcal{A}_\alpha(x)).
\]

(2) Moreover, for all \( \alpha \in L \), the following equations hold:
\[
\mathcal{A}(\mathcal{A}_\alpha) = \mathcal{S}(\inf \mathcal{A}, \mathcal{A}_\alpha) \\
= \bigwedge_{x \in X} (\inf \mathcal{A}(x) \rightarrow \mathcal{A}_\alpha(x)) \\
= \bigwedge_{x \in X} \left( \inf \mathcal{A}(x) \rightarrow \bigwedge_{\mathcal{A}(C) \geq \alpha} C(x) \right) \\
= \bigwedge_{x \in X} \bigwedge_{\mathcal{A}(C) \geq \alpha} (\inf \mathcal{A}(x) \rightarrow C(x)) \\
= \bigwedge_{\mathcal{A}(C) \geq \alpha} \mathcal{S}(\inf \mathcal{A}, C) \\
= \bigwedge_{\mathcal{A}(C) \geq \alpha} \mathcal{A}(C) \\
\geq \alpha,
\]
i.e. \( \mathcal{A}(\mathcal{A}_\alpha) \geq \alpha \) for all \( \alpha \in L \). From this, we obtain for each \( x \in X \),
\[
\inf \mathcal{A}(x) = \bigwedge_{C \in L^X} (\mathcal{A}(C) \rightarrow C(x)) \\
\leq \bigwedge_{\alpha \in L} (\mathcal{A}(\mathcal{A}_\alpha) \rightarrow \mathcal{A}_\alpha(x)) \\
\leq \bigwedge_{\alpha \in L} (\alpha \rightarrow \mathcal{A}_\alpha(x)).
\]

From the above proof, it follows by the arbitrariness of \( x \) in \( X \) that \( \inf \mathcal{A} = \bigwedge_{\alpha \in L} (\alpha \rightarrow \mathcal{A}_\alpha) \).

\[\square\]

5. Conclusion

In this paper, we establish representation theorems of \( L \)-subsets and \( L \)-families on complete residuated lattice. The new representation theorem of \( L \)-subsets shows that not only \( L \)-power set lattice but also its classical isomorphic object namely \( L \)-nested systems form complete residuated lattices, and they are still isomorphic.

References


Hui Han*, Department of Mathematics, Ocean University of China, 266100 Qingdao, P.R. China

E-mail address: hanhui200801@163.com

Jinming Fang, Department of Mathematics, Ocean University of China, 266100 Qingdao, P.R. China

E-mail address: jinming-fang2163.com

*Corresponding author