EXISTENCE OF EXTREMAL SOLUTIONS FOR IMPULSIVE DELAY FUZZY INTEGRODIFFERENTIAL EQUATIONS IN n-DIMENSIONAL FUZZY VECTOR SPACE

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Abstract. In this paper, we study the existence of extremal solutions for impulsive delay fuzzy integrodifferential equations in n-dimensional fuzzy vector space, by using monotone method. We show that obtained result is an extension of the result of Rodríguez-López [8] to impulsive delay fuzzy integrodifferential equations in n-dimensional fuzzy vector space.

1. Introduction

Fuzzy theory has developed engineering, economics, agriculture, computers, etc. in various fields by many scholars since 1965. Moreover fuzzy integrodifferential equations are a field of increasing interest, due to their applicability to the analysis of phenomena where imprecision in inherent.

Some authors have studied fuzzy integrodifferential equations. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Kwun et al. [3] studied nonlocal controllability for the semilinear fuzzy integrodifferential equations in n-dimensional fuzzy vector space. Kwun et al. [4] studied controllability for the impulsive semilinear nonlocal fuzzy integrodifferential equations in n-dimensional fuzzy vector space.

Many authors have studied extremal solutions for fuzzy differential equations. In [8], Rodríguez-López studied the existence and approximation of extremal solutions for fuzzy differential equation by using monotone iterative technique in one dimensional fuzzy vector space E^1. Nieto and Rodríguez-López [6] studied existence of extremal solutions for quadratic fuzzy equations. Rodríguez-López [7] proved the existence of solutions for impulsive fuzzy differential equations with periodic boundary value using monotone method in one dimensional fuzzy space. Recently, Kwun et al. [5] proved the existence of extremal solutions for impulsive fuzzy differential equations with periodic boundary value in n-dimensional fuzzy vector space.

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In this paper, we study the existence of extremal solutions for the following impulsive delay fuzzy integrodifferential equations in fuzzy vector space.

$$\begin{cases} \frac{dx_i(t)}{dt} = f_i\Big(t, (x_i)_t, \int_0^t q_i(t,s,(x_i)_s)ds\Big), \ t \in J, \\ x_i(t) = \phi_i(t), \ t \in [-r, 0], \\ x_i(t_k^+) = I_k(x_i(t_k)), \ \ t \neq t_k, \ k = 1, 2, \cdots, m, \ i = 1, 2, \cdots, n, \end{cases} \tag{1}$$

where $T > 0$, $J = [0, T]$, $0 < t_1 < \cdots < t_m < t_{m+1} = T$, E_N^i is the set of all upper semi-continuously convex fuzzy numbers on R with $E_N^i \neq E_N^j$ $(i \neq j)$, $f_i : J \times E_N^i \times E_N^i \to E_N^i$ and $q_i : J \times J \times E_N^i \to E_N^i$ are regular continuous fuzzy function, $(x_i)_t = x_i(t + \theta)$, $\theta \in [-r, 0]$. $\phi_i \in C([-r, 0], E_N^i)$ is initial function and $I_k \in C(E_N^i, E_N^i)$ are bounded functions.

2. Preliminaries

In this section, we give basic definitions, terminologies, notations and Lemmas which are most relevant to our investigated and are needed in later chapters. All undefined concepts and notions used here are standard.

A fuzzy set of R^n is a function $u : R^n \to [0, 1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$, its α-level set and $[u]^0 = cl\{x \in R^n : u(x) > 0\}$(the closure of $\{x \in R^n \mid u(x) > 0\}$). Let u, v be fuzzy sets of R^n. It is well known that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0, 1]$ implies $u = v$. Let E^n denote the collection of all fuzzy sets of R^n that satisfies the following conditions:

(1) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $0 \leq \lambda \leq 1$;

(3) $u(x)$ is upper semi-continuous, i.e., $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n$ $(k = 0, 1, 2, \cdots)$, $x_k \to x_0$;

(4) $[u]^0$ is compact.

We call $u \in E^n$ a n-dimension fuzzy number.

Wang et al. [11] defined n-dimensional fuzzy vector space and investigated its properties.

For any $u_i \in E$, $i = 1, 2, \cdots, n$, we call the ordered one-dimension fuzzy number class $u_1, u_2, \cdots, u_n$ (i.e., the Cartesian product of one-dimension fuzzy number $u_1, u_2, \cdots, u_n$) a n-dimension fuzzy vector, denote it as $(u_1,\ u_2, \cdots, u_n)$, and call the collection of all n-dimension fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}$) n-dimensional fuzzy vector space, and denote it as $(E)^n$.

Definition 2.1. [12] If $u \in E^n$, and $[u]^\alpha$ is a hyperrectangle, i.e., $[u]^\alpha$ can be represented by $\prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$, i.e., $[u_{1l}^\alpha, u_{1r}^\alpha] \times [u_{2l}^\alpha, u_{2r}^\alpha] \times \cdots \times [u_{nl}^\alpha, u_{nr}^\alpha]$ for every $\alpha \in [0, 1]$, where $u_{il}^\alpha, u_{ir}^\alpha \in R$ with $u_{il}^\alpha \leq u_{ir}^\alpha$ when $\alpha \in (0, 1]$, $i = 1, 2, \cdots, n$, then we call u a fuzzy n-cell number. We denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

Theorem 2.2. [11] *For any* $u \in L(E^n)$ *with* $[u]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$*, there exists a unique* $(u_1, u_2, \cdots, u_n) \in (E)^n$ *such that* $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ *(*$i = 1, 2, \cdots, n$ *and* $\alpha \in [0,1]$*). Conversely, for any* $(u_1, u_2, \cdots, u_n) \in (E)^n$ *with* $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ *(*$i = 1, 2, \cdots,$ n *and* $\alpha \in [0,1]$*), there exists a unique* $u \in L(E^n)$ *such that* $[u]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$.

Remark 2.3. [11] Theorem 2.2 indicates that fuzzy n-cell numbers and n-dimension fuzzy vectors can represent each other, so $L(E^n)$ and $(E)^n$ may be regarded as identity. If $(u_1, u_2, \cdots, u_n) \in (E)^n$ is the unique n-dimension fuzzy vector determined by $u \in L(E^n)$, then we denote $u = (u_1, u_2, \cdots, u_n)$.

Let $(E_N^i)^n = E_N^1 \times E_N^2 \times \cdots \times E_N^n$, E_N^i $(i = 1, 2, \cdots, n)$ is fuzzy subset of R. Then $(E_N^i)^n \subseteq (E)^n$.

Definition 2.4. [12] The complete metric D_L on $(E_N^i)^n (i = 1, 2, \cdots, n)$ is defined by

$$\begin{aligned} D_L(u,v) &= \sup_{0<\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0<\alpha\leq 1} \max_{1\leq i\leq n} \{|u_{il}^\alpha - v_{il}^\alpha|, |u_{ir}^\alpha - v_{ir}^\alpha|\} \end{aligned}$$

for any $u, v \in (E_N^i)^n$, which satisfies $d_L(u + w, v + w) = d_L(u, v)$.

Definition 2.5. Let $u, v \in C([-r, T], (E_N^i)^n)$

$$H_1(u,v) = \sup_{-r<t\leq T} D_L(u(t), v(t)).$$

Definition 2.6. The derivative $x'(t)$ of a fuzzy process $x \in (E_N^i)^n$ is defined by

$$[x'(t)]^\alpha = \prod_{i=1}^{n} [(x_{il}^\alpha)'(t), (x_{ir}^\alpha)'(t)]$$

provided that equation defines a fuzzy $x'(t) \in (E_N^i)^n$.

Definition 2.7. The fuzzy integral $\int_b^a x(t)dt$, $a, b \in [0, T]$ is defined by

$$\Big[\int_b^a x(t)dt\Big]^\alpha = \prod_{i=1}^{n} \Big[\int_b^a x_{il}^\alpha(t)dt, \int_b^a x_{ir}^\alpha(t)dt\Big]$$

provided that the Lebesgue integrals on the right hand side exist.

Definition 2.8. [7] Let $x, y \in E^1$. We say that $x \leq y$ if and only if $x_l^\alpha \leq y_l^\alpha$ and $x_r^\alpha \leq y_r^\alpha$ for every $\alpha \in [0,1]$.

Definition 2.9. Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in (E_N^i)^n$. We say that $x \leq_n y$ if for $x_i, y_i \in E_N^i$, $i = 1, 2, \cdots, n$,

$$x_i \leq y_i.$$

Lemma 2.10. [8] *If* $\{f_n\} \subseteq C([c,d],E^1)$, $g \in C([c,d],E^1)$ *are such that*

$$f_n \leq g, \quad \forall n \in N,$$

and $f_n(t)$ *converges to* $f(t)$ *in* E^1, *for all* $t \in [c,d]$, *then* $f \leq g$.

Definition 2.11. [12] Let $x, y \in E^n$. If there exists $z \in E^n$ such that $x = y+z$ then we call $x,\ y$ having Hukuhara difference and y is called the Hukuhara difference of x and y, denoted $x - y$.

Definition 2.12. [2] A fuzzy set $u \in E^n$ is called a Lipschitzian fuzzy set if it is a Lipschitz function of its membership grade in the sense that

$$d_H([u]^\alpha, [u]^\beta) \leq K|\alpha - \beta|$$

for all $\alpha, \beta \in [0,1]$ and some fixed, finite constant K.

Lemma 2.13. [8] *Let* I *be a closed interval in* R, *and* $B \subseteq C(I, E^1)$ *such that, for all* $x \in B$ *and* $t \in I$, $x(t)$ *is a continuous fuzzy number. Consider*

$$\overline{B}_L = \{\overline{x}_L : x \in B\} \subseteq C([0,1] \times I, R),$$
$$\overline{B}_R = \{\overline{x}_R : x \in B\} \subseteq C([0,1] \times I, R),$$

where

$$\overline{x}_L : [0,1] \times I \to R,$$
$$(a,t) \quad \to \overline{x}_L(a,t) = (x(t))_L(a) = (x(t))_{al},$$

and

$$\overline{x}_R : [0,1] \times I \to R,$$
$$(a,t) \quad \to \overline{x}_R(a,t) = (x(t))_R(a) = (x(t))_{ar}.$$

if $\overline{B}_L$ *and* $\overline{B}_R$ *are relatively compact sets in* $(C([0,1] \times I, R), \|\cdot\|_\infty)$, *then* B *is a relatively compact set in* $C(I, E^1)$.

3. **Existence of Extremal Solutions**

In order to prove the existence of extremal solutions for equations in n-dimensional fuzzy vector space, we define

$$f = (f_1, f_2, \cdots, f_n),$$
$$q = (q_1, q_2, \cdots, q_n),$$
$$x_t = ((x_1)_t, (x_2)_t, \cdots, (x_n)_t),$$
$$x(t) = (x_1(t), x_2(t), \cdots, x_n(t)),$$
$$\phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t)),$$

then $f,\ q,\ x_t,\ x(t),\ \phi(t) \in (E_N^i)^n,\ i = 1, 2, \cdots, n$.

We consider the following impulsive delay fuzzy integrodifferential equations in fuzzy vector space $(E_N^i)^n$:

$$\begin{cases} \frac{dx(t)}{dt} = f\Big(t, (x)_t, \int_0^t q(t,s,(x)_s)ds\Big),\ t \in J, \\ x(t) = \phi(t),\ t \in [-r, 0], \\ x(t_k^+) = I_k(x(t_k)), \quad t \neq t_k,\ k = 1, 2, \cdots, m, \end{cases} \tag{2}$$

where $T > 0$, $J = [0,T]$, $0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : J \times (E_N^i)^n \times (E_N^i)^n \to (E_N^i)^n$ and $q : J \times J \times (E_N^i)^n \to (E_N^i)^n$ are regular continuous fuzzy function, $x_t = x(t+\theta)$, $\theta \in [-r,0]$. $\phi \in C([-r,0],(E_N^i)^n)$ is initial function and $I_k \in C((E_N^i)^n,(E_N^i)^n)$ are bounded functions.

To define solutions for the impulsive fuzzy integrodifferential equations, we consider the following space:

$\Omega_i = \Big\{x_i : J \to E_N^i : (x_i)_k \in C(J_k, E_N^i), J_k = (t_k, t_{k+1}]$, and there exist $x_i(0^+),\ x_i(T^-),\ x_i(t_k^+)$, with $x_i(t_k^-) = x_i(t_k), (k = 1,2,\cdots,m,\ i = 1,2,\cdots,n)\Big\}$,

Let $\Omega_i' = \Omega_i \bigcap C([-r,T],E_N^i),\ i = 1,2,\cdots,n,$ and $\Omega' = \prod_{i=1}^n \Omega_i'$.

Definition 3.1. For the partial ordering $\leq_n$, a function $a \in \Omega'$ is a $\leq_n$-lower solution for (2) if

$$\begin{cases} a_t' \leq_n f\Big(t, a_t, \int_0^t q(t,s,a_s)ds\Big),\ t \in J, \\ a(t) \leq_n \phi(t),\ t \in [-r,0], \\ a(t_k^+) \leq_n I_k(a(t_k)),\quad k = 1,2,\cdots,m, \end{cases}$$

we define a $\leq_n$-upper solution $b \in \Omega'$ as a function satisfying the reverse inequalities.

To find extremal solutions for equations (2) by using monotone method, for $M > 0$, let

$$\frac{dx(t)}{dt} = Mx(t) + f\Big(t,(x)_t,\int_0^t q(t,s,(x)_s)ds\Big) - Mx(t).$$

And if $F\Big(t,x_t,\int_0^t q(t,s,x_s)ds\Big) = f\Big(t,x_t,\int_0^t q(t,s,x_s)ds\Big) - Mx(t)$, then we consider the following equations:

$$\begin{cases} \frac{dx(t)}{dt} = Mx(t) + F\Big(t,x_t,\int_0^t q(t,s,x_s)ds\Big),\ t \in J, \\ x(t) = \phi(t),\ t \in [-r,0], \\ x(t_k^+) = I_k(x(t_k)),\quad t \neq t_k,\ k = 1,2,\cdots,m. \end{cases} \tag{3}$$

We define $\widehat{x}_t(\phi) : [-r,T] \to (E_N^i)^n$ for $x_t(\phi) \in C(J,(E_N^i)^n)$, $\theta \in [-r,0]$,

$$\widehat{x}_t(\phi) = \begin{cases} \phi(t),\ t+\theta \in [-r,0], \\ x_t(\phi),\quad t+\theta \in J. \end{cases}$$

Assume the following:

(F1) For $\widehat{x}_t,\ \widehat{y}_t,\ \widehat{\eta}_t,\ \widehat{\xi}_t,\ \widehat{u}_t,\ \widehat{v}_t \in C([-r,T],(E_N^i)^n)$, there exist positive numbers $h,\ k,\ p$ so that

$$\begin{aligned} &d_L\Big(\Big[F\Big(t,\widehat{x}_t,\widehat{u}_t\Big)\Big]^\alpha, \Big[F\Big(t,\widehat{y}_t,\widehat{v}_t\Big)\Big]^\alpha\Big) \\ &\qquad\qquad \leq h d_L\Big([x_t]^\alpha,[y_t]^\alpha\Big) + k d_L\Big([u_t]^\alpha,[v_t]^\alpha\Big), \\ &d_L\Big([q(t,s,\widehat{\eta}_s)]^\alpha,[q(t,s,\widehat{\xi}_s)]^\alpha\Big) \leq p d_L([\eta_s]^\alpha,[\xi_s]^\alpha) \end{aligned}$$

and $F(t,\mathcal{X}_{\{0\}}(0),\mathcal{X}_{\{0\}}(0)) = \mathcal{X}_{\{0\}}(0)$.

(F2) For $x,\ y \in C(J : (E_N^i)^n)$, there exists a $d > 0,\ k = 1, 2, \cdots, m$, so that

$$d_L\Big([I_k(x(t_k))]^\alpha, [I_k(y(t_k))]^\alpha\Big) \le d d_L\Big([x(t)]^\alpha, [y(t)]^\alpha\Big),$$

and $I_k(\mathcal{X}_{\{0\}}(0)) = \mathcal{X}_{\{0\}}(0)$.

(F3) $c(d + hT + kpT^2) < 1$.

Lemma 3.2. *If $x \in \Omega'$ is an integral solution of (3), then $x \in \Omega'$ is given by*

$$\begin{aligned} x(t) &= x_t(\phi)(0) \\ &= S(t)\phi(0) + \int_0^t S(t-s)F\Big(s, \widehat{x}_s(\phi), \int_0^s q(s,\tau,\widehat{x}_\tau(\phi))d\tau\Big)ds \\ &\qquad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)),\ t \in J, \\ x(t) &= \phi(t),\ t \in [-r, 0], \end{aligned}$$

where $S(t) = \exp\{\int_0^t M dt\}$ is continuous with $|S(t)| \le c,\ c > 0,$ for all $t \in J$.

Theorem 3.3. *If hypotheses (F1)-(F3) are hold. Then the equations (3) have a unique solution $x \in \Omega'$.*

Proof. Let $\delta > 0$ satisfy

$$d_L\Big([S(t)\phi(0)]^\alpha, [\phi(0)]^\alpha\Big) \le \frac{\delta}{3},$$

for $t \in [-r, T]$ and

$$c(d + hT + kpT^2) \sup_{\theta\in[-r,0]} D_L\Big(x_t(\phi), \mathcal{X}_{\{0\}}(0)\Big) \le \frac{\delta}{3}.$$

Let we define K, the nonempty closed bounded subset of $\in \Omega'$, by

$$K = \Big\{\eta \in \Omega' \mid \eta(0) = \phi(0),\ H_1(\eta, \phi) \le \delta\Big\}.$$

Define a mapping G on K by

$$\begin{aligned} Gx_t(\phi)(0) = S(t)\phi(0) &+ \int_0^t S(t-s)F\Big(s, \widehat{x}_s(\phi), \int_0^s q(s,\tau,\widehat{x}_\tau(\phi))d\tau\Big)ds \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)). \end{aligned}$$

For $G\widehat{x}_t(\phi)$, if $-r \le t + \theta \le 0$, then $G\widehat{x}_t(\phi)(\theta) = \phi(t + \theta)$ and hence

$$H_1(G\widehat{x}_t, \phi) \le \frac{\delta}{3}.$$

If $0 < t + \theta \le T$, then

$$
\begin{aligned}
& d_L\Big([G\widehat{x}_t(\phi)(\theta)]^\alpha, [\phi(\theta)]^\alpha\Big) \\
&\quad = d_L\bigg(\Big[S(t+\theta)\phi(0) + \int_0^{t+\theta} S(t+\theta-s)F\Big(s, \widehat{x}_s(\phi), \int_0^s q(s,\tau,\widehat{x}_\tau(\phi))d\tau\Big)ds \\
&\qquad\qquad + \sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k(x(t_k))\Big]^\alpha, \Big[\phi(\theta)\Big]^\alpha\bigg) \\
&\quad \le d_L\Big([S(t+\theta)\phi(0)]^\alpha, [\phi(0)]^\alpha\Big) + d_L\Big([\phi(0)]^\alpha, [\phi(\theta)]^\alpha\Big) \\
&\qquad + \int_0^{t+\theta} d_L\bigg(\Big[S(t+\theta-s)F\Big(s, \widehat{x}_s(\phi), \int_0^s q(s,\tau,\widehat{x}_\tau(\phi))d\tau\Big)\Big]^\alpha, \\
&\qquad\qquad [S(t)F(s, \mathcal{X}_{\{0\}}(0), \mathcal{X}_{\{0\}}(0))]^\alpha\bigg)ds \\
&\qquad + d_L\bigg(\Big[\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k(x(t_k))\Big]^\alpha, \Big[\sum_{0<t_k<t} S(t-t_k)I_k(\mathcal{X}_{\{0\}}(0))\Big]^\alpha\bigg) \\
&\quad \le \frac{\delta}{3} + \frac{\delta}{3} + c\int_0^{t+\theta} \Big(hd_L\Big([x_s(\phi)]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \\
&\qquad\qquad + kp\int_0^s d_L\Big([x_\tau(\phi)]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big)d\tau\Big)ds \\
&\qquad + cdd_L\Big([x(t)]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big).
\end{aligned}
$$

$$
\begin{aligned}
& D_L\Big(G\widehat{x}_t(\phi)(\theta), \phi(\theta)\Big) \\
&\quad = \sup_{0<\alpha\le 1} d_L\Big([G\widehat{x}_t(\phi)(\theta)]^\alpha, [\phi(\theta)]^\alpha\Big) \\
&\quad \le \frac{\delta}{3} + \frac{\delta}{3} + c\int_0^{t+\theta} \sup_{0<\alpha\le 1} \Big(hd_L\Big([x_s(\phi)]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \\
&\qquad\qquad + kp\int_0^s d_L\Big([x_\tau(\phi)]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big)d\tau\Big)ds \\
&\qquad + cd\sup_{0<\alpha\le 1} d_L\Big([x(t)]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \\
&\quad \le \frac{\delta}{3} + \frac{\delta}{3} + ch\int_0^{t+\theta} D_L\Big(x_s(\phi), \mathcal{X}_{\{0\}}(0)\Big)ds \\
&\qquad + ckp\int_0^{t+\theta}\int_0^s D_L\Big(x_\tau(\phi), \mathcal{X}_{\{0\}}(0)\Big)d\tau ds + cdD_L\Big(x(t), \mathcal{X}_{\{0\}}(0)\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&H_1\Big(G\widehat{x}_t,\phi\Big)\\
&\quad=\sup_{\theta\in[-r,0]} D_L\Big(G\widehat{x}_t(\phi),\phi(\theta)\Big)\\
&\quad\le\frac{\delta}{3}+\frac{\delta}{3}+ch\sup_{\theta\in[-r,0]}\int_0^{t+\theta} D_L\Big(x_s(\phi),\mathcal{X}_{\{0\}}(0)\Big)ds\\
&\qquad+ckp\sup_{\theta\in[-r,0]}\int_0^{t+\theta}\int_0^{s} D_L\Big(x_\tau(\phi),\mathcal{X}_{\{0\}}(0)\Big)d\tau ds\\
&\qquad+cd\sup_{\theta\in[-r,0]} D_L\Big(x(t),\mathcal{X}_{\{0\}}(0)\Big)\\
&\quad\le\frac{\delta}{3}+\frac{\delta}{3}+chT\sup_{\theta\in[-r,0]} D_L\Big(x_s(\phi),\mathcal{X}_{\{0\}}(0)\Big)\\
&\qquad+ckpT^2\sup_{\theta\in[-r,0]} D_L\Big(x_s(\phi),\mathcal{X}_{\{0\}}(0)\Big)+cd\sup_{\theta\in[-r,0]} D_L\Big(x(t),\mathcal{X}_{\{0\}}(0)\Big)\\
&\quad\le\frac{\delta}{3}+\frac{\delta}{3}+c(d+hT+kpT^2)\sup_{\theta\in[-r,0]} D_L\Big(x_t(\phi),\mathcal{X}_{\{0\}}(0)\Big)\\
&\quad\le\delta.
\end{aligned}
$$

Therefore we have

$$G\widehat{x}_t(\phi)\in K,\quad \text{i.e.}\quad G:K\to K.$$

Furthermore, if $x_t(\phi), y_t(\phi)\in K$, then we have

$$
\begin{aligned}
&d_L\Big([G\widehat{x}_t(\phi)]^\alpha,[G\widehat{y}_t(\phi)]^\alpha\Big)\\
&\quad=d_L\Big(\Big[S(t+\theta)\phi(0)+\int_0^{t+\theta}S(t+\theta-s)F\Big(s,\widehat{x}_s(\phi),\int_0^s q(s,\tau,\widehat{x}_\tau(\phi))d\tau\Big)ds\\
&\qquad\qquad+\sum_{0<t_k<t+\theta}S(t+\theta-t_k)I_k(x(t_k))\Big]^\alpha,\\
&\qquad\Big[S(t+\theta)\phi(0)+\int_0^{t+\theta}S(t+\theta-s)F\Big(s,\widehat{y}_s(\phi),\int_0^s q(s,\tau,\widehat{y}_\tau(\phi))d\tau\Big)ds\\
&\qquad\qquad+\sum_{0<t_k<t+\theta}S(t+\theta-t_k)I_k(y(t_k))\Big]^\alpha\Big)\\
&\quad\le\int_0^{t+\theta}d_L\Big(\Big[S(t+\theta-s)F\Big(s,\widehat{x}_s(\phi),\int_0^s q(s,\tau,\widehat{x}_\tau(\phi))d\tau\Big)\Big]^\alpha,\\
&\qquad\qquad\Big[S(t+\theta-s)F\Big(s,\widehat{y}_s(\phi),\int_0^s q(s,\tau,\widehat{y}_\tau(\phi))d\tau\Big)\Big]^\alpha\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+d_L\Big(\Big[\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k(x(t_k))\Big]^\alpha,\\
&\qquad\qquad\Big[\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k(y(t_k))\Big]^\alpha\Big)\\
&\le c\int_0^{t+\theta}\Big(hd_L\Big([x_s(\phi)]^\alpha,[y_s(\phi)]^\alpha\Big)+kp\int_0^s d_L\Big([x_\tau(\phi)]^\alpha,[y_\tau(\phi)]^\alpha\Big)d\tau\Big)ds\\
&\quad +cdd_L\Big([x(t)]^\alpha,[y(t)]^\alpha\Big).
\end{aligned}
$$

$$
\begin{aligned}
&D_L\Big(G\widehat{x}_t(\phi),G\widehat{y}_t(\phi)\Big)\\
&\quad=\sup_{0<\alpha\le 1} d_L\Big([G\widehat{x}_t(\phi)]^\alpha,[G\widehat{y}_t(\phi)]^\alpha\Big)\\
&\quad\le ch\int_0^{t+\theta}\sup_{0<\alpha\le 1} d_L\Big([x_s(\phi)]^\alpha,[y_s(\phi)]^\alpha\Big)ds\\
&\qquad +ckp\int_0^{t+\theta}\int_0^s\sup_{0<\alpha\le 1} d_L\Big([x_\tau(\phi)]^\alpha,[y_\tau(\phi)]^\alpha\Big)d\tau ds\\
&\qquad +cd\sup_{0<\alpha\le 1} d_L\Big([x(t)]^\alpha,[y(t)]^\alpha\Big)\\
&\quad\le ch\int_0^{t+\theta} D_L\Big(x_s(\phi),y_s(\phi)\Big)ds+ckp\int_0^{t+\theta}\int_0^s D_L\Big(x_\tau(\phi),y_\tau(\phi)\Big)d\tau ds\\
&\qquad +cdD_L\Big(x(t),y(t)\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&H_1\Big(G\widehat{x}_t,G\widehat{y}_t\Big)\\
&\quad=\sup_{\theta\in[-r,0]} D_L\Big(G\widehat{x}_t(\phi),G\widehat{y}_t(\phi)\Big)\\
&\quad\le ch\sup_{\theta\in[-r,0]}\int_0^{t+\theta} D_L\Big(x_s(\phi),y_s(\phi)\Big)ds\\
&\qquad +ckp\sup_{\theta\in[-r,0]}\int_0^{t+\theta}\int_0^s D_L\Big(x_\tau(\phi),y_\tau(\phi)\Big)d\tau ds+cd\sup_{\theta\in[-r,0]} D_L\Big(x(t),y(t)\Big)\\
&\quad\le chT\sup_{\theta\in[-r,0]} D_L\Big(x_t(\phi),y_t(\phi)\Big)+ckpT^2\sup_{\theta\in[-r,0]} D_L\Big(x_t(\phi),y_t(\phi)\Big)\\
&\qquad +cd\sup_{\theta\in[-r,0]} D_L\Big(x(t),y(t)\Big)\\
&\quad\le c(d+hT+kpT^2)H_1(x_t,y_t).
\end{aligned}
$$

By hypothesis (F3), G is a contraction mapping. Using the Banach fixed point theorem, the equations (3) have a unique fixed point $x_t(\phi) \in K$. This complete the proof of theorem. □

Now, for showing the existence of extremal solutions of equations (2), the following Lemmas 3.4 and 3.5 are, respectively, extensions of Lemmas 2.10 and 2.13 (see [8]) to n-dimensional fuzzy vector space.

Lemma 3.4. *Let* $\{P_n\} = \{(P_n)_1 \times (P_n)_2 \times \cdots \times (P_n)_n\} \subset C([c,d]_n, (E_N^i)^n)$, $P = (P_1 \times P_2 \times \cdots \times P_n)$, $Q = (Q_1 \times Q_2 \times \cdots \times Q_n) \in C([c,d]_n, (E_N^i)^n)$ *are such that*

$$P_n \leq_n Q$$

and $P_n(t)$ *converges to* $P(t)$ *in* $(E_N^i)^n$*, for all* $t \in [c,d]_n$*, then* $P \leq_n Q$*.*

Proof. By Lemma 2.10, for each $i = 1, 2, \cdots, n$, $(P_n)_i \leq Q_i$ and $(P_n)_i(t)$ converges to $P_i(t)$ in E^1, then $P_i \leq Q_i$. Therefore $P_n \leq_n Q$ and $P_n(t)$ converges to $P(t)$ in $(E_N^i)^n$, for all $t \in [c,d]_n$, then $P \leq_n Q$. □

Lemma 3.5. *Let* $H = H_1 \times H_2 \times \cdots \times H_n \subset \Omega'$. *We consider*

$$\overline{H}_l = \{\overline{x}_l : x \in H\} \subseteq C([0,1] \times [-r,T], R^n),$$

$$\overline{H}_r = \{\overline{x}_r : x \in H\} \subseteq C([0,1] \times [-r,T], R^n),$$

where $\overline{x}_l(\alpha,t) = (x(t))_l^\alpha$, $\overline{x}_r(\alpha,t) = (x(t))_r^\alpha$. *If* $\overline{H}_l$ *and* $\overline{H}_r$ *are relatively compact sets in* $(C([0,1] \times [-r,T], R^n), ||\cdot||_\infty)$*, then* H *is a relatively compact set in* $C([-r,T], (E_N^i)^n)$*.*

Proof. By Lemma 2.13, for each $i = 1, 2, \cdots, n$, $\overline{H}_{il}$ and $\overline{H}_{ir}$ are relatively compact sets in $C([0,1] \times [-r,T], R)$. Then

$$\overline{H}_l = \overline{H}_{1l} \times \overline{H}_{2l} \times \cdots \times \overline{H}_{nl}$$

and

$$\overline{H}_r = \overline{H}_{1r} \times \overline{H}_{2r} \times \cdots \times \overline{H}_{nr}$$

are relatively compact sets. In consequence, H is a relatively compact set. □

Assume the following:

(F4) For all $t \in J$, $a_t, b_t, x_t, y_t \in \Omega'$, $a_t \leq_n x_t \leq_n y_t \leq_n b_t$,

$$F\Big(t, x_t, \int_0^t q(t,s,x_s)ds\Big) \leq_n F\Big(t, y_t, \int_0^t q(t,s,y_s)ds\Big).$$

(F5) For $\phi \in C([-r,T], (E_N^i)^n)$, α, $\beta \in [0,1]$ and $\forall \varepsilon > 0$, there exists $\delta_2 > 0$ such that $|\alpha - \beta| < \delta_2$,

$$d_L\Big([\phi(0)]^\alpha, [\phi(0)]^\beta\Big) \leq \frac{\varepsilon}{3c}.$$

(F6) For $\widehat{x}_t$, $\widehat{y}_t$, $\widehat{u}_t \in C([-r,T], (E_N^i)^n)$, α, $\beta \in [0,1]$ and $\forall \varepsilon > 0$, there exists $\delta_3 > 0$ such that $|\alpha - \beta| < \delta_3$, $d_L([x_t]^\alpha, [x_t]^\beta) < \varepsilon$,

$$d_L\Big([F(t, \widehat{x}_t, \widehat{y}_t)]^\alpha, [F(t, \widehat{x}_t, \widehat{y}_t)]^\beta\Big) \leq \frac{\varepsilon}{3cT}.$$

$$d_L\Big([q(s,\tau,\widehat{u}_t)]^{\alpha},[q(s,\tau,\widehat{u}_t)]^{\beta}\Big)\le\frac{\varepsilon}{6csT}.$$

(F7) For $x\in(E_N^i)^n$, $\alpha,\beta\in[0,1]$ and $\forall\varepsilon>0$, there exists $\delta_4>0$ such that $|\alpha-\beta|<\delta_4$,

$$d_L\Big(\Big[\sum_{0<t_k<t}I_k(x(t_k))\Big]^{\alpha},\Big[\sum_{0<t_k<t}I_k(x(t_k))\Big]^{\beta}\Big)\le\frac{\varepsilon}{3c}.$$

We can now employ Lemma 3.5 with

$$H=\{(a_n)_t\in\Omega':a_t\le(a_n)_t\le b_t\}.$$

Then the set H is closed bounded convex.

Lemma 3.6. *Consider a function $\mathcal{A}$ defined by*

$$\begin{aligned}\mathcal{A}:[a,b]_n&\quad\to\Omega'\\ \eta&\quad\mapsto\mathcal{A}\eta=x_\eta,\end{aligned}$$

which satisfies:
(i) $\mathcal{A}([a,b]_n)\subseteq[a,b]_n$,
(ii) $\mathcal{A}$ *is* $\le_n$*-nondecreasing.*

Proof. For a fixed $\eta\in[a,b]_n$, we consider the problem

$$\begin{cases}x'(t)=M\eta(t)+F\Big(t,\eta(t),\int_0^t q(t,s,\eta(s))ds\Big),\ t\in[0,T],\\ x(t)=\phi(t),\quad t\in[-r,0],\\ x(t_k^+)=I_k(\eta(t_k)),\quad t\ne t_k,\ k=1,2,\cdots,m,\end{cases}\tag{4}$$

which has, by Theorem 3.3, a unique solution $x_\eta\in\Omega'$.

We prove that $\mathcal{A}$ is $\le_n$-nondecreasing. $\mathcal{A}a\ge_n a$, and $\mathcal{A}b\le_n b$. Indeed, let $\eta,\zeta\in[a,b]$ be such that $\eta\le_n\zeta$, then $\mathcal{A}\eta$ and $\mathcal{A}\zeta$ are functions in Ω' and

$$\begin{aligned}(\mathcal{A}\eta)'(t)&=M\eta(t)+F\Big(t,\eta(t),\int_0^t q(t,s,\eta(s))ds\Big)\\ &\le_n M\zeta(t)+F\Big(t,\zeta(t),\int_0^t q(t,s,\zeta(s))ds\Big)=(\mathcal{A}\zeta)'(t),\quad t\in[0,T],\\ (\mathcal{A}\eta)(t)&=(\mathcal{A}\phi)(t)\le_n(\mathcal{A}\varphi)(t)=(\mathcal{A}\zeta)(t),\ t\in[-r,0],\\ (\mathcal{A}\eta)(t_k^+)&=I_k(\eta(t_k))\le_n I_k(\zeta(t_k))=(\mathcal{A}\zeta)(t_k^+),\ k=1,2,\cdots,m,\end{aligned}$$

so we obtain that $\mathcal{A}\eta\le_n\mathcal{A}\zeta$ on $[-r,T]$. Moreover, let $\mathcal{A}a\in\Omega'$, which satisfies, the properties of the $\le_n$-lower solution and the partial ordering,

$$\begin{aligned}&a'(t)\le_n Ma(t)+F\Big(t,a(t),\int_0^t q(t,s,a(s))ds\Big)=(\mathcal{A}a)(t),\quad t\in[0,T],\\ &a(t)\le_n(\mathcal{A}\phi)(t)=(\mathcal{A}a)(t),\ t\in[-r,0],\\ &a(t_k^+)\le_n I_k(a(t_k))=(\mathcal{A}a)(t_k^+),\ k=1,2,\cdots,m,\end{aligned}$$

then $a\le_n\mathcal{A}a$ on $[-r,T]$. Similarly, $b\ge_n\mathcal{A}b$ on $[-r,T]$.

This prove that $\mathcal{A} : [a,b]_n \to [a,b]_n$ and $\mathcal{A}$ is nondecreasing. Define the sequences $\{a_n\}, \{b_n\}$ such that $a_0 = a,\ b_0 = b,\ a_{n+1} = \mathcal{A}a_n$ and $b_{n+1} = \mathcal{A}b_n$. It can be proved that $\{a_n\}$ is nondecreasing, $\{b_n\}$ is nonincreasing, and

$$a = a_0 \le_n a_1 \le_n \cdots \le_n a_n \le_n b_n \le_n \cdots \le_n b_1 \le_n b_0 = b.$$

Note that a_n is the solution to

$$\begin{cases} x'(t) = Ma_{n-1}(t) + F\Big(t, a_{n-1}(t), \int_0^t q(t,s,a_{n-1}(s))ds\Big),\ t \in [0,T], \\ x(t) = \phi(t), \quad t \in [-r,0], \\ x(t_k^+) = I_k(a_{n-1}(t_k)), \quad k = 1,2,\cdots,m, \end{cases}$$

and b_n is the solution to

$$\begin{cases} x'(t) = Mb_{n-1}(t) + F\Big(t, b_{n-1}(t), \int_0^t q(t,s,b_{n-1}(s))ds\Big),\ t \in [0,T], \\ x(t) = \phi(t), \quad t \in [-r,0], \\ x(t_k^+) = I_k(b_{n-1}(t_k)), \quad k = 1,2,\cdots,m. \end{cases}$$

□

Theorem 3.7. *Let $a, b \in \Omega'$ be, respectively, $\le_n$-lower and $\le_n$-upper solutions for equations (2) with $a \le_n b$ on $[-r,T]$. By condition of Lemma 3.6 and hypotheses (F1)-(F7), there exist monotone sequences $\{a_n\} \uparrow \rho,\ \{b_n\} \downarrow \gamma$ in Ω', where $a_0 = a,\ b_0 = b$, and ρ, γ are the extremal solutions to equations (2) in the fuzzy functional interval*

$$[a,b]_n := \{x \in \Omega' : a \le_n x \le_n b \ on\ [-r,T]\}.$$

Proof. We prove that $\{a_n\}$ and $\{b_n\}$ are uniformly equicontinuous in $C([-r,T],(E_N^i)^n)$. In consequence, there exist convergent subsequences $\{a_{n_j}\} \to \rho,\ \{b_{n_j}\} \to \gamma$ in $C([-r,T],(E_N^i)^n)$, hence, by monotonicity $\{a_n\} \to \rho,\ \{b_n\} \to \gamma$ in $C([-r,T],(E_N^i)^n)$, that is, $\{a_n\} \to \rho,\ \{b_n\} \to \gamma$ in Ω'. In order to apply given in Lemma 3.5, we have to prove that $\{a_n\}, \{b_n\}$ are equicontinuous in $C([-r,T],(E_N^i)^n)$. Indeed, for each $n \in \mathrm{N}$, using the integral representation of a_n in $[-r,T]$, we obtain

$$\begin{aligned} a_n(t) &= S(t)a_n(0) + \int_0^t S(t-s)F\Big(s, (\widehat{a}_{n-1})_s(s), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\tau))d\tau\Big)ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(a_{n-1}(t_k)),\ t \in [0,T], \\ a_n(t) &= \phi(t),\ t \in [-r,0]. \end{aligned}$$

The set $\{\{a_n\} : n \in \mathrm{N}\}$ is uniformly equicontinuous in the variable $t,\ \alpha$. For $t+\theta \in [-r,0]$, then for $\alpha, \beta \in [0,1]$, given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|\alpha - \beta| < \delta_1$,

$$d_L\Big([(a_n)_t(\phi)]^\alpha, [(a_n)_t(\phi)]^\beta\Big) < \varepsilon.$$

By hypotheses (F5)-(F7), for $t+\theta \in [0,T]$, then for $\alpha, \beta \in [0,1]$, given $\varepsilon > 0$ there exist $\delta_2, \delta_3, \delta_4 > 0$ such that $|\alpha - \beta| < \min\{\delta_2, \delta_3, \delta_4\}$,

$$
\begin{aligned}
& d_L\Big([(a_n)_t(\phi)]^\alpha, [(a_n)_t(\phi)]^\beta\Big) \\
&\quad = d_L\Big(\Big[S(t+\theta)\phi(0) \\
&\qquad\qquad + \int_0^{t+\theta} S(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)ds \\
&\qquad\qquad\qquad + \sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\alpha, \\
&\qquad\quad \Big[S(t+\theta)\phi(0) \\
&\qquad\qquad + \int_0^{t+\theta} S(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)ds \\
&\qquad\qquad\qquad + \sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\beta\Big) \\
&\quad \le d_L\Big([S(t+\theta)\phi(0)]^\alpha, [S(t+\theta)\phi(0)]^\beta\Big) \\
&\qquad + \int_0^{t+\theta} d_L\Big(\Big[S(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)\Big]^\alpha, \\
&\qquad\qquad \Big[S\Big(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)\Big]^\beta\Big)ds \\
&\qquad + d_L\Big(\Big[\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\alpha, \\
&\qquad\qquad\qquad \Big[\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\beta\Big) \\
&\quad \le c d_L\Big([\phi(0)]^\alpha, [\phi(0)]^\beta\Big) \\
&\qquad + c\int_0^{t+\theta} d_L\Big(\Big[F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)\Big]^\alpha, \\
&\qquad\qquad\qquad \Big[F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)\Big]^\beta\Big)ds \\
&\qquad + c d_L\Big(\Big[\sum_{0<t_k<t+\theta} I_k((a_{n-1})(t_k))\Big]^\alpha, \Big[\sum_{0<t_k<t+\theta} I_k((a_{n-1})(t_k))\Big]^\beta\Big) \\
&\quad \le c\frac{\varepsilon}{3c} + c\int_0^{t+\theta}\Big(\frac{\varepsilon}{6cT} + \int_0^s \frac{\varepsilon}{6csT}d\tau\Big)ds + c\frac{\varepsilon}{3c}.
\end{aligned}
$$

$$
\begin{aligned}
& H_1\Big((a_n)_t, (a_n)_t\Big) \\
&\quad = \sup_{\theta\in[-r,0]} D_L\Big((a_n)_t, (a_n)_t\Big)
\end{aligned}
$$

$$= \sup_{\theta\in[-r,0]} \sup_{0<\alpha,\beta\le 1} d_L\Big([(a_n)_t(\phi)]^\alpha, [(a_n)_t(\phi)]^\beta\Big)$$
$$\le \sup_{\theta\in[-r,0]} \Big(c\frac{\varepsilon}{3c} + c\int_0^{t+\theta}\Big(\frac{\varepsilon}{6cT} + \int_0^s \frac{\varepsilon}{6csT}d\tau\Big)ds + c\frac{\varepsilon}{3c}\Big)$$
$$\le c\frac{\varepsilon}{3c} + c\int_0^T\Big(\frac{\varepsilon}{6cT} + \int_0^s \frac{\varepsilon}{6csT}d\tau\Big)ds + c\frac{\varepsilon}{3c}$$
$$= \varepsilon.$$

Thus the set $\{\{a_n\} : n \in \mathrm{N}\}$ is uniformly equicontinuous in the variable $\alpha \in [0,1]$. And for $k = 1, 2, \cdots, m$,

$$d_L\Big([(a_n)_t(\phi)]^\alpha, [(a_n)_{t-t'}(\phi)]^\alpha\Big)$$
$$= d_L\Big(\Big[S(t+\theta)\phi(0)$$
$$+\int_0^{t+\theta} S(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)ds$$
$$+\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\alpha,$$
$$\Big[S(t-t'+\theta)\phi(0)$$
$$+\int_0^{t-t'+\theta} S(t-t'+\theta-s)F\Big(s, (\widehat{a}_{n-1})_{s-s'}(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_{\tau-\tau'}(\phi))d\tau\Big)ds$$
$$+\sum_{0<(t-t')_k<t-t'+\theta} S(t-t'+\theta-t_k)I_k((a_{n-1})((t-t')_k))\Big]^\alpha\Big)$$
$$\le d_L\Big(\int_0^{t+\theta}\Big[S(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)\Big]^\alpha ds,$$
$$\int_0^{t-t'+\theta}\Big[S(t-t'+\theta-s)F\Big(s, (\widehat{a}_{n-1})_{s-s'}(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_{\tau-\tau'}(\phi))d\tau\Big)\Big]^\alpha ds\Big)$$
$$+d_L\Big(\Big[\sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\alpha,$$
$$\Big[\sum_{0<(t-t')_k<t-t'+\theta} S(t-t'+\theta-t_k)I_k((a_{n-1})((t-t')_k))\Big]^\alpha\Big)$$
$$\le chd_L\Big(\int_0^{t+\theta}[(a_{n-1})_s(\phi)]^\alpha ds, \int_0^{t-t'+\theta}[(a_{n-1})_{s-s'}(\phi)]^\alpha ds\Big)$$
$$+ckpd_L\Big(\int_0^{t+\theta}\int_0^s[(a_{n-1})_\tau(\phi)]^\alpha d\tau ds, \int_0^{t-t'+\theta}\int_0^s[(a_{n-1})_{\tau-\tau'}(\phi)]^\alpha d\tau ds\Big)$$
$$+cdd_L\Big([(a_{n-1})(t_k)]^\alpha, [(a_{n-1})((t-t')_k]^\alpha\Big).$$

Then by hypotheses (F1)-(F3), provided we take $t > t' > 0$, then we have

$$\begin{aligned}
&H_1\Big((a_n)_t,(a_n)_{t-t'}\Big)\\
&\quad= \sup_{\theta\in[-r,0]} D_L\Big((a_n)_t(\phi),(a_n)_{t-t'}(\phi)\Big)\\
&\quad= \sup_{\theta\in[-r,0]}\sup_{0<\alpha\le 1} d_L\Big([(a_n)_t(\phi)]^\alpha,[(a_n)_{t-t'}(\phi)]^\alpha\Big)\\
&\quad\le ch\sup_{\theta\in[-r,0]}\sup_{0<\alpha\le 1} d_L\Big(\int_0^{t+\theta}[(a_{n-1})_s(\phi)]^\alpha ds,\int_0^{t-t'+\theta}[(a_{n-1})_{s-s'}(\phi)]^\alpha ds\Big)\\
&\qquad +ckp\sup_{\theta\in[-r,0]}\sup_{0<\alpha\le 1} d_L\Big(\int_0^{t+\theta}\int_0^s[(a_{n-1})_\tau(\phi)]^\alpha d\tau ds,\\
&\qquad\qquad \int_0^{t-t'+\theta}\int_0^s[(a_{n-1})_{\tau-\tau'}(\phi)]^\alpha d\tau ds\Big)\\
&\qquad\qquad +cd\sup_{\theta\in[-r,0]}\sup_{0<\alpha\le 1} d_L\Big([(a_{n-1})(t_k)]^\alpha,[(a_{n-1})((t-t')_k]^\alpha\Big)\\
&\quad\le ch\sup_{\theta\in[-r,0]} D_L\Big(\int_0^{t+\theta}(a_{n-1})_s(\phi)ds,\int_0^{t-t'+\theta}(a_{n-1})_{s-s'}(\phi)ds\Big)\\
&\qquad +ckp\sup_{\theta\in[-r,0]} D_L\Big(\int_0^{t+\theta}\int_0^s(a_{n-1})_\tau(\phi)d\tau ds,\int_0^{t-t'+\theta}\int_0^s(a_{n-1})_{\tau-\tau'}(\phi)d\tau ds\Big)\\
&\qquad\qquad +cd\sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})(t_k),(a_{n-1})((t-t')_k\Big)\\
&\quad\le chT\sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})_t(\phi),(a_{n-1})_{t-t'}(\phi)\Big)\\
&\qquad +ckpT^2\sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})_t(\phi),(a_{n-1})_{t-t'}(\phi)\Big)\\
&\qquad\qquad +cd\sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})(t_k),(a_{n-1})((t-t')_k\Big)\\
&\quad\le c(d+hT+kpT^2)H_1\Big((a_{n-1})_t(\phi),(a_{n-1})_{t-t'}(\phi)\Big).
\end{aligned}$$

Hence $H_1((a_n)_t(\phi),(a_n)_{t-t'}(\phi)) \to 0$ as $t' \to 0$ the set $\{\{a_n\} : n \in \mathrm{N}\}$ is uniformly equicontinuous in the variable $t \in [-r,T]$. The case of equicontinuity from the right is similar.

In consequence, $\{\{a_n\} : n \in \mathrm{N}\}$ is uniformly equicontinuous in the variable $\alpha \in [0,1]$, $t \in [-r,T]$. This proves that $\{\{a_n\} : n \in \mathrm{N}\}$ is uniformly equicontinuous in $C([-r,T],(E_N^i)^n))$. And proceeding similarly, and also for $\{b_n\}$. Hence $\{a_n\},\{b_n\} \in B$. Next, we have to prove that ρ,γ are solutions to (2). To check that ρ is a solution to (2), we prove that

$$\rho_t(\phi) = S(t)\phi(0) + \int_0^t S(t-s)F\Big(s, \widehat{\rho}_s(\phi), \int_0^s q(s,\tau,\widehat{\rho}_\tau(\phi))d\tau\Big)ds + \sum_{0<t_k<t} S(t-t_k)I_k(\rho(t_k)),\ t \in [0,T],$$

$$\rho(t) = \phi(t),\ t \in [-r,0],$$

and ρ is a solution to (2). The previous problem have solution since $\rho \in \Omega'$, and $a \leq_n a_n \leq_n b$ on Ω', for every n, hence, $a \leq_n \rho \leq_n b$ on Ω'. For $k = 1, 2, \cdots, m$, we prove that the limit of the following expression is zero as n tends to $+\infty$.

$$\begin{aligned}
&d_L\Big([(a_n)_t(\phi)]^\alpha, [\rho_t(\phi)]^\alpha\Big)\\
&= d_L\Big(\Big[S(t+\theta)\phi(0) + \int_0^{t+\theta} S(t+\theta-s)F\Big(s, (\widehat{a}_{n-1})_s(\phi), \int_0^s q(s,\tau,(\widehat{a}_{n-1})_\tau(\phi))d\tau\Big)ds\\
&\qquad + \sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k((a_{n-1})(t_k))\Big]^\alpha,\\
&\qquad \Big[S(t+\theta)\phi(0) + \int_0^{t+\theta} S(t+\theta-s)F\Big(s, \widehat{\rho}_s(\phi), \int_0^s q(s,\tau,\widehat{\rho}_\tau(\phi))d\tau\Big)ds\\
&\qquad + \sum_{0<t_k<t+\theta} S(t+\theta-t_k)I_k(\rho(t_k))\Big]^\alpha\Big)\\
&\leq ch\int_0^{t+\theta} d_L\Big([(a_{n-1})_s(\phi)]^\alpha[\rho_s(\phi)]^\alpha\Big)ds\\
&\quad + ckp\int_0^{t+\theta}\int_0^s d_L\Big([(a_{n-1})_\tau(\phi)]^\alpha[\rho_\tau(\phi)]^\alpha\Big)d\tau ds\\
&\quad + cdd_L\Big([(a_{n-1})(t_k)]^\alpha, [\rho(t_k)]^\alpha\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
H_1\Big((a_n)_t, \rho_t\Big) &= \sup_{\theta\in[-r,0]} D_L\Big((a_n)_t(\phi), \rho_t(\phi)\Big)\\
&= \sup_{\theta\in[-r,0]}\sup_{0<\alpha\leq 1} d_L\Big([(a_n)_t(\phi)]^\alpha, [\rho_t(\phi)]^\alpha\Big)\\
&\leq ch\sup_{\theta\in[-r,0]}\sup_{0<\alpha\leq 1}\int_0^{t+\theta} d_L\Big([(a_{n-1})_s(\phi)]^\alpha[\rho_s(\phi)]^\alpha\Big)ds
\end{aligned}$$

$$+ckp \sup_{\theta\in[-r,0]} \sup_{0<\alpha\le 1} \int_0^{t+\theta}\int_0^s d_L\Big([(a_{n-1})_\tau(\phi)]^\alpha [\rho_\tau(\phi)]^\alpha\Big) d\tau ds$$

$$+cd \sup_{\theta\in[-r,0]} \sup_{0<\alpha\le 1} d_L\Big([(a_{n-1})(t_k)]^\alpha, [\rho(t_k)]^\alpha\Big)$$

$$\le chT \sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})_t(\phi), \rho_t(\phi)\Big)$$

$$+ckpT^2 \sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})_t(\phi), \rho_t(\phi)\Big)$$

$$+cd \sup_{\theta\in[-r,0]} D_L\Big((a_{n-1})_t(\phi), \rho(t_k)\Big)$$

$$\le c(d + hT + kpT^2) H_1\Big((a_{n-1})_t, \rho_t\Big).$$

Therefore $H_1((a_n)_t, \rho_t) \to 0$ as $n \to +\infty$.

Using that I_k and F are continuous convergence of a_{n-1} towards ρ, we obtain that ρ is a solution to (2). For function γ, we follow a similar procedure. Finally, if x is a solution to (2) such that $a \le_n x \le_n b$, using that $\mathcal{A}$ is nondecreasing, we obtain

$$a_n = \mathcal{A}^n a \le_n \mathcal{A}^n x = x \le_n \mathcal{A}^n b = b_n,$$

then, by Lemma 3.4,

$$\rho \le_n x \le_n \gamma.$$

In conclusion, x exists between ρ and γ. □

4. **Example**

We consider the following two one-dimensional impulsive delay fuzzy integrodifferential equations

$$\begin{cases} \frac{dx_i(t)}{dt} = f_i\Big(t, (x_i)_t, \int_0^t q_i(t,s,(x_i)_s)ds\Big),\ t \in J, \\ x_i(t) = \phi_i(t),\ t \in J, \\ x_i(t_k^+) = I_k(x_i(t_k)),\quad t \ne t_k,\ k = 1,2,\cdots,m,\ i = 1,2, \end{cases} \tag{5}$$

where $T > 0, J = [0,T],\ 0 < t_1 < \cdots < t_m < t_{m+1} = T$, $E_N^i, i = 1,2$, is the set of all upper semi-continuously convex fuzzy numbers on R with $E_N^i \ne E_N^j$ $(i \ne j)$, $f_i : J \times E_N^i \times E_N^i \to E_N^i$ and $q_i : J \times J \times E_N^i \to E_N^i$ are regular continuous fuzzy function, $(x_i)_t = x_i(t+\theta),\ \theta \in [-r,0]$. $\phi_i \in C([-r,0], E_N^i)$ is initial function and $I_k \in C(E_N^i, E_N^i)$ are bounded functions.

Let for $i = 1,2$,

$$f_i\Big(t, (x_i)_t, \int_0^t q_i(t,s,(x_i)_s)ds\Big) = \widetilde{2}t(x_i)_t^2 + \int_0^t (t-s)(x_i)_s ds,\ t \in [0,T].$$

For a positive constant M, let

$$\frac{dx_i(t)}{dt} = M(x_i)_t + \widetilde{2}t(x_i)_t^2 + \int_0^t (t-s)(x_i)_s ds - M(x_i)_t.$$

Put

$$F_i\Big(t,(x_i)_t,\int_0^t q_i(t,s,(x_i)_s)ds\Big)=\widetilde{2}t(x_i)_t^2+\int_0^t(t-s)(x_i)_s ds-M(x_i)_t,$$

$$x_i(t)=\phi_i(t),\quad i=1,2,\quad t\in[-r,0],$$

$$\Delta x_i(t_k)=x_i(t_k^+)-x_i(t_k^-)$$

is impulsive effect at $t=t_k(k=1,2,\cdots,m)$. Let

$$\begin{aligned}
&F\Big(t,x_t,\int_0^t q(t,s,x_s)ds\Big)\\
&\quad=\Big(F_1\Big(t,(x_1)_t,\int_0^t q_1(t,s,(x_1)_s)ds\Big),F_2\Big(t,(x_2)_t,\int_0^t q_2(t,s,(x_2)_s)ds\Big)\Big)\\
&\quad=\Big(\widetilde{2}t(x_1)_t^2+\int_0^t(t-s)(x_1)_s^2ds-M(x_1)_t,\widetilde{2}t(x_2)_t^2+\int_0^t(t-s)(x_2)_s^2ds-M(x_2)_t\Big),\\
&\Delta x(t_k)=(\Delta x_1(t_k),\Delta x_2(t_k)),\quad t\neq t_k,\ k=1,2,\cdots,m,\\
&I_k(x(t_k))=(I_k(x_1(t_k)),I_k(x_2(t_k)))\\
&\qquad\qquad=(x_1(t_k^+)-x_1(t_k^-),x_2(t_k^+)-x_2(t_k^-))\\
&\qquad\qquad=(x_1(t_k^+)-x_1(t_k^-),x_2(t_k^+)-x_2(t_k^-)),\quad t\neq t_k,\ k=1,2,\cdots,m.
\end{aligned}$$

We consider the following equations

$$\begin{cases}\frac{dx(t)}{dt}=Mx(t)+F\Big(t,x_t,\int_0^t q(t,s,x_s)ds\Big),\ t\in[0,T],\\ x(t)=\phi(t),\ t\in[-r,0],\\ x(t_k^+)=I_k(x(t_k)),\quad t\neq t_k,\ k=1,2,\cdots,m.\end{cases}\tag{6}$$

The α-level set of fuzzy numbers are the following:

$[\widetilde{0}]^\alpha=[\alpha-1,1-\alpha]$, $[\widetilde{2}]^\alpha=[\alpha+1,3-\alpha]$ for all $\alpha\in[0,1]$, $M=1$. Then α-level set of $F\Big(t,\widehat{x}_t,\int_0^t q(t,s,\widehat{x}_s)ds\Big)$ is

$$\begin{aligned}
&\Big[F\Big(t,\widehat{x}_t,\int_0^t q(t,s,\widehat{x}_s)ds\Big)\Big]^\alpha\\
&\quad=\Big[\widetilde{2}t(x_1)_t^2+\int_0^t(t-s)(x_1)_s ds-(x_1)_t\Big]^\alpha\\
&\qquad\qquad\times\Big[\widetilde{2}t(x_2)_t^2+\int_0^t(t-s)(x_2)_s ds-(x_2)_t\Big]^\alpha\\
&\quad=\Big([\widetilde{2}]^\alpha\cdot t[(x_1)_t^2]^\alpha+\int_0^t(t-s)[(x_1)_s]^\alpha ds-[(x_1)_t]^\alpha\Big)\\
&\qquad\qquad\times\Big([\widetilde{2}]^\alpha\cdot t[(x_2)_t^2]^\alpha+\int_0^t(t-s)[(x_2)_s]^\alpha ds-[(x_2)_t]^\alpha\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big([\alpha+1, 3-\alpha] \cdot t[(x_{1l}^{\alpha})_t^2, (x_{1r}^{\alpha})_t^2] + \int_0^t (t-s)[(x_{1l}^{\alpha})_s, (x_{1r}^{\alpha})_s]ds - [(x_{1l}^{\alpha})_t, (x_{1r}^{\alpha})_t]\Big) \\
&\times\Big([\alpha+1, 3-\alpha] \cdot t[(x_{2l}^{\alpha})_t^2, (x_{2r}^{\alpha})_t^2] + \int_0^t (t-s)[(x_{2l}^{\alpha})_s, (x_{2r}^{\alpha})_s]ds - [(x_{2l}^{\alpha})_t, (x_{2r}^{\alpha})_t]\Big) \\
&= \Big[(\alpha+1)t(x_{1l}^{\alpha})_t^2 + \int_0^t (t-s)(x_{1l}^{\alpha})_s ds - (x_{1l}^{\alpha})_t, \\
&\qquad\qquad (3-\alpha)t(x_{1r}^{\alpha})_t^2 + \int_0^t (t-s)(x_{1r}^{\alpha})_s ds - (x_{1r}^{\alpha})_t\Big] \\
&\times\Big[(\alpha+1)t(x_{2l}^{\alpha})_t^2 + \int_0^t (t-s)(x_{2l}^{\alpha})_s ds - (x_{2l}^{\alpha})_t, \\
&\qquad\qquad (3-\alpha)t(x_{2r}^{\alpha})_t^2 + \int_0^t (t-s)(x_{2r}^{\alpha})_s ds - (x_{2r}^{\alpha})_t\Big]
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
&d_L\Big(\Big[F\Big(t, \widehat{x}_t, \int_0^t q(t,s,\widehat{x}_s)ds\Big)\Big]^{\alpha}, \Big[F\Big(t, \widehat{y}_t, \int_0^t q(t,s,\widehat{y}_s)ds\Big)\Big]^{\alpha}\Big) \\
&= d_L\Big(\prod_{i=1}^{2}\Big(\Big[(\alpha+1)t(x_{il}^{\alpha})_t^2 + \int_0^t (t-s)(x_{il}^{\alpha})_s ds - (x_{il}^{\alpha})_t, \\
&\qquad\qquad (3-\alpha)t(x_{ir}^{\alpha})_t^2 + \int_0^t (t-s)(x_{ir}^{\alpha})_s ds - (x_{ir}^{\alpha})_t\Big], \\
&\qquad \Big[(\alpha+1)t(y_{il}^{\alpha})_t^2 + \int_0^t (t-s)(y_{il}^{\alpha})_s ds - (y_{il}^{\alpha})_t, \\
&\qquad\qquad (3-\alpha)t(y_{ir}^{\alpha})_t^2 + \int_0^t (t-s)(y_{ir}^{\alpha})_s ds - (y_{ir}^{\alpha})_t\Big]\Big)\Big) \\
&\le t \max_{1\le i\le 2}\{(\alpha+1)|(x_{il}^{\alpha})_t^2 - (y_{il}^{\alpha})_t^2|, (3-\alpha)|(x_{ir}^{\alpha})_t^2 - (y_{ir}^{\alpha})_t^2|\} \\
&+ \int_0^t (t-s) \max_{1\le i\le 2}\{|(x_{il}^{\alpha})_s - (y_{il}^{\alpha})_s|, |(x_{ir}^{\alpha})_s - (y_{ir}^{\alpha})_s|\}ds \\
&+ \max_{1\le i\le 2}\{|(x_{il}^{\alpha})_t - (y_{il}^{\alpha})_t|, |(x_{ir}^{\alpha})_t - (y_{ir}^{\alpha})_t|\}ds \\
&\le T(3-\alpha) \max_{1\le i\le 2}\{|(x_{il}^{\alpha})_t - (y_{il}^{\alpha})_t||(x_{il}^{\alpha})_t + (y_{il}^{\alpha})_t|, \\
&\qquad\qquad |(x_{ir}^{\alpha})_t - (y_{ir}^{\alpha})_t||(x_{ir}^{\alpha})_t + (y_{ir}^{\alpha})_t|\} \\
&+\Big(\frac{T^2}{2} + 1\Big) \max_{1\le i\le 2}\{|(x_{il}^{\alpha})_t - (y_{il}^{\alpha})_t|, |(x_{ir}^{\alpha})_t - (y_{ir}^{\alpha})_t|\} \\
&\le 3T|(x_{ir}^{\alpha})_t + (y_{ir}^{\alpha})_t| d_L([x_t]^{\alpha}, [y_t]^{\alpha}) + \Big(\frac{T^2}{2} + 1\Big) d_L([x_t]^{\alpha}, [y_t]^{\alpha}) \\
&= h d_L([x_t]^{\alpha}, [y_t]^{\alpha}) + k d_L([x_t]^{\alpha}, [y_t]^{\alpha}),
\end{aligned}
$$

where h and k satisfy the hypothesis (F1). Since I_k is a bounded function, we know that the hypothesis (F2) holds. Choose T such that $T < (1-cd)/ch$. Then all conditions stated in Theorem 3.3 are satisfied, so the equations (6) have a unique fuzzy solution. Since ϕ is a continuous function on $[-r,T]$, hypothesis (F5) satisfied. Next, we show that the hypothesis (F6) satisfies. For $\alpha,\beta\in[0,1]$ and given $\varepsilon=\frac{1}{3cT}\delta_3>0$, there exist

$$\delta_3=\frac{\varepsilon}{\Big(3T|(x^{\alpha}_{ir})_t+(x^{\beta}_{ir})_t|+\frac{T^2}{2}+1\Big)}>0$$

such that

$$|\alpha-\beta|<\delta_3,\quad d_L([x_t]^{\alpha},[x_t]^{\beta})<\varepsilon,$$

then

$$\begin{aligned}
&d_L\Big(\Big[F\Big(t,\widehat{x}_t,\int_0^t q(t,s,\widehat{x}_s)ds\Big)\Big]^{\alpha},\Big[F\Big(t,\widehat{x}_t,\int_0^t q(t,s,\widehat{x}_s)ds\Big)\Big]^{\beta}\Big)\\
&=d_L\Big(\prod_{i=1}^{2}\Big(\Big[(\alpha+1)t(x^{\alpha}_{il})^2_t+\int_0^t(t-s)(x^{\alpha}_{il})_s ds-(x^{\alpha}_{il})_t,\\
&\qquad\qquad(3-\alpha)t(x^{\alpha}_{ir})^2_t+\int_0^t(t-s)(x^{\alpha}_{ir})_s ds-(x^{\alpha}_{ir})_t\Big],\\
&\qquad\Big[(\beta+1)t(x^{\beta}_{il})^2_t+\int_0^t(t-s)(x^{\beta}_{il})_s ds-(x^{\beta}_{il})_t,\\
&\qquad\qquad(3-\beta)t(x^{\beta}_{ir})^2_t+\int_0^t(t-s)(x^{\beta}_{ir})_s ds-(x^{\beta}_{ir})_t\Big]\Big)\Big)\\
&\le t\max_{1\le i\le 2}\{|(\alpha+1)(x^{\alpha}_{il})^2_t-(\beta+1)(x^{\beta}_{il})^2_t|,|(3-\alpha)(x^{\alpha}_{ir})^2_t-(3-\beta)(x^{\beta}_{ir})^2_t|\}\\
&+\int_0^t(t-s)\max_{1\le i\le 2}\{|(x^{\alpha}_{il})_s-(x^{\beta}_{il})_s|,|(x^{\alpha}_{ir})_s-(x^{\beta}_{ir})_s|\}ds\\
&+\max_{1\le i\le 2}\{|(x^{\alpha}_{il})_t-(x^{\beta}_{il})_t|,|(x^{\alpha}_{ir})_t-(x^{\beta}_{ir})_t|\}\\
&\le 3T\max_{1\le i\le 2}\{|(x^{\alpha}_{il})_t-(x^{\beta}_{il})_t||(x^{\alpha}_{il})_t+(x^{\beta}_{il})_t|,|(x^{\alpha}_{ir})_t-(x^{\beta}_{ir})_t||(x^{\alpha}_{ir})_t+(x^{\beta}_{ir})_t|\}\\
&+\Big(\frac{T^2}{2}+1\Big)\max_{1\le i\le 2}\{|(x^{\alpha}_{il})_t-(x^{\beta}_{il})_t|,|(x^{\alpha}_{ir})_t-(x^{\beta}_{ir})_t|\}\\
&\le 3T|(x^{\alpha}_{ir})_t+(x^{\beta}_{ir})_t|d_L([x_t]^{\alpha},[x_t]^{\beta})+\Big(\frac{T^2}{2}+1\Big)d_L([x_t]^{\alpha},[x_t]^{\beta})\\
&=\Big(3T|(x^{\alpha}_{ir})_t+(x^{\beta}_{ir})_t|+\frac{T^2}{2}+1\Big)d_L([x_t]^{\alpha},[x_t]^{\beta})\\
&\le\Big(3T|(x^{\alpha}_{ir})_t+(x^{\beta}_{ir})_t|+\frac{T^2}{2}+1\Big)\frac{1}{3cT}\delta_3\\
&\le\frac{\varepsilon}{3cT}.
\end{aligned}$$

Then all conditions stated in Theorem 3.7 are satisfied, so the problem (5) has a unique extremal solution.

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