SOME FIXED POINT THEOREMS FOR SINGLE AND MULTI
VALUED MAPPINGS ON ORDERED NON-ARCHIMEDEAN
FUZZY METRIC SPACES

I. ALTUN

Abstract. In the present paper, a partial order on a non- Archimedean fuzzy
metric space under the Łukasiewicz t-norm is introduced and fixed point theo-
rems for single and multivalued mappings are proved.

1. Introduction and Preliminaries

Research in the field of fixed point theory on fuzzy metric spaces ([1], [2], [3],
[6], [7], [8], [16]) has been developed following the definition such spaces [5], [10],
[20], the Generally, this theory is concerned with contractive or contractive type
mappings ([9], [11], [12], [13], [14], [17], [19]).

In this paper we introduce a partial order on a non-Archimedean fuzzy metric
space (in the sense of Kramosil and Michalek) under the Łukasiewicz t-norm and
prove a fixed point theorem for single-valued nondecreasing mappings. Similar
results are obtained for multivalued mappings.

For the sake of completeness, we first recall some notions from the theory of
fuzzy metric spaces.

Definition 1.1. [18] A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous $t$-norm if $([0, 1], *)$ is an Abelian topological monoid with the unit 1 such that
$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.2. [10] A fuzzy metric space (in the sense of Kramosil and Michalek)
is a triple $(X, M, *)$, where $X$ is a nonempty set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following properties:

(KM-1) $M(x, y, 0) = 0, \forall x \in X$
(KM-2) $M(x, y, t) = 1, \forall t > 0$ iff $x = y$
(KM-3) $M(x, y, t) = M(y, x, t), \forall x, y \in X$ and $t > 0$
(KM-4) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous, $\forall x, y \in X$
(KM-5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \forall x, y, z \in X, \forall t, s > 0$.

We will refer to such spaces as FM-spaces.

If in the above definition, the triangular inequality (KM-5) is replaced by

(NA) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s), \forall x, y, z \in X, \forall t, s > 0$

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or, equivalently,
\[ M(x, z, t) \geq M(x, y, t) \ast M(y, z, t), \forall x, y, z \in X, \forall t > 0 \]
then the triple \((X, M, \ast)\) is called a non-Archimedean fuzzy metric space. It is easy
to check that the triangular inequality (NA) implies (KM-5), that is, every non-
Archimedean fuzzy metric space is itself a fuzzy metric space.

**Definition 1.3.** [5], [18] Let \((X, M, \ast)\) be a fuzzy metric space. A sequence \(\{x_n\}\)
in \(X\) is called an \(M\)-Cauchy sequence, if for each \(\varepsilon \in (0, 1)\) and \(t > 0\) there exists
\(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for all \(m, n \geq n_0\). A sequence \(\{x_n\}\) in a fuzzy
metric space \((X, M, \ast)\) is said to be convergent to \(x \in X\) if \(\lim_{n \to \infty} M(x_n, x, t) = 1\)
for all \(t > 0\). An FM space \((X, M, \ast)\) is called \(M\)-complete if every \(M\)-Cauchy
sequence is convergent.

### 2. Fixed point Theory for Single-valued Mappings

We first prove the following lemma.

**Lemma 2.1.** Let \((X, M, \ast)\) be a non-Archimedean fuzzy metric space with \(a \ast b \geq \max\{a + b - 1, 0\}\) and \(\phi : X \times [0, \infty) \to \mathbb{R}\). Define the relation "\(\preceq\)" on \(X\) as follows:
\[ x \preceq y \iff M(x, y, t) \geq 1 + \phi(x, t) - \phi(y, t), \forall t > 0. \]
Then "\(\preceq\)" is a (partial) order on \(X\), called the partial order induced by \(\phi\).

**Proof.** For all \(x \in X\) and \(t > 0\), \(M(x, x, t) = 1 = 1 + \phi(x, t) - \phi(x, t)\), then \(x \preceq x\), that is, "\(\preceq\)" is reflexive. Again, if \(x, y \in X\), be such that \(x \preceq y\) and \(y \preceq x\), then for all \(t > 0\),
\[ M(x, y, t) \geq 1 + \phi(x, t) - \phi(y, t) \]
and
\[ M(y, x, t) \geq 1 + \phi(y, t) - \phi(x, t). \]
This shows that \(M(x, y, t) = 1\) for all \(t > 0\), that is, \(x = y\). Thus "\(\preceq\)" is antisymmetric. Now for \(x, y, z \in X\), let \(x \preceq y\) and \(y \preceq z\), then for all \(t > 0\),
\[ M(x, y, t) \geq 1 + \phi(x, t) - \phi(y, t) \]
and
\[ M(y, z, t) \geq 1 + \phi(y, t) - \phi(z, t). \]
By (1) and (2) we have,
\[ M(x, z, t) \geq M(x, y, t) \ast M(y, z, t) \]
\[ = \max\{M(x, y, t) + M(y, z, t) - 1, 0\} \]
\[ \geq M(x, y, t) + M(y, z, t) - 1 \]
\[ \geq 1 + \phi(x, t) - \phi(z, t), \forall t > 0. \]
This shows that \(x \preceq z\). \(\square\)
Example 2.2. Let $X = (0, \infty)$, $a \ast b = ab$ and
\[ M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}, \forall t > 0. \]
Then $(X, M, \ast)$ is an M-complete non-Archimedean fuzzy metric space (see [15]).

Let $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = \frac{1}{x}$. Then for $x, y \in X$,
\[
\begin{align*}
x \preceq y & \iff M(x, y, t) \geq 1 + \phi(x, t) - \phi(y, t) \\
& \iff M(x, y, t) \geq 1 + 1 - 1 \\
& \iff 1 - M(x, y, t) \leq 1 - \frac{1}{y}.
\end{align*}
\]
It follows that $2 \preceq \frac{1}{2}, 3 \preceq 1, 1 \preceq \frac{1}{3}$ but $3 \not\preceq 5$ and $5 \not\preceq 3$. Therefore $X$ is a partially ordered space.

Example 2.3. Let $X = \mathbb{N} = \{1, 2, \cdots\}$, $a \ast b = ab$ and
\[
M(x, y, t) = \begin{cases} 
\frac{x}{y} & \text{if } x \leq y \\
\frac{y}{x} & \text{if } y \leq x
\end{cases}, \forall t > 0.
\]
Then $(X, M, \ast)$ is a non-Archimedean fuzzy metric space ([5]). Also this space is M-complete. Let $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = x$.

Then it is obvious that $x \preceq y \iff x \leq y$. Therefore $X$ is a partially ordered space. Also $X$ is a totally ordered space. If we define $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = x - \frac{1}{x}$, then it is again obvious that $x \preceq y \iff x \leq y$. Now, if $\phi : X \times [0, \infty) \to \mathbb{R}$,
\[ \phi(x, t) = 1 - \frac{5}{x}, \text{ then } 1 \preceq 2 \preceq 3 \preceq 4 \preceq 5 \preceq 6 \text{ but } 6 \not\preceq 7 \text{ and } 7 \not\preceq 6. \]
Therefore $X$ is a partially ordered space.

Definition 2.4. Let $(X, M, \ast)$ be a fuzzy metric space. If "\preceq" is an order on $X$, then the fuzzy metric space is called an ordered fuzzy metric space. Let $(X, M, \ast)$ be an ordered fuzzy metric space and let $f : X \to X$ be a mapping. If $x \preceq y$ implies that $fx \preceq fy$, then $f$ is called a nondecreasing mapping.

Theorem 2.5. Let $(X, M, \ast)$ be an M-complete non-Archimedean fuzzy metric space with $a \ast b \geq \max\{a + b - 1, 0\}$, $\phi : X \times [0, \infty) \to \mathbb{R}$ be a function bounded from above and "\preceq" the partial order induced by $\phi$. If $f : X \to X$ is a continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then $f$ has a fixed point in $X$.

Proof. Consider a point $x_0 \in X$ satisfying $x_0 \preceq fx_0$. We define a sequence $\{x_n\}$ in $X$ such that $x_n = fx_{n-1}$ for $n = 1, 2, \cdots$. Then, since $f$ is nondecreasing we have $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$, that is the sequence $\{x_n\}$ is nondecreasing. By the
definition of "≤" we have, ∀t > 0, φ(x₀, t) ≤ φ(x₁, t) ≤ φ(x₂, t) ≤ ···. In other words, for all t > 0, the sequence \{φ(xₙ, t)\} is nondecreasing in ℝ. Since φ is bounded from above, \{φ(xₙ, t)\} is convergent and hence it is Cauchy. So, for all ε > 0, there exists n₀ ∈ ℕ such that for all m > n > n₀ and t > 0 we have |φ(xₙ, t) − φ(xₙ⁻¹, t)| ≤ φ(xₙ⁻¹, t) − φ(xₙ⁻₂, t) < ε. Since xₙ ≤ xₙ⁻¹, it follows that

\[ M(xₙ, xₙ⁻¹, t) = 1 + φ(xₙ⁻¹, t) - φ(xₙ, t) = 1 - [φ(xₙ, t) - φ(xₙ⁻¹, t)] > 1 - ε. \]

This shows that the sequence \{xₙ\} is Cauchy in X and since X is M-complete, it converges to a point z ∈ X. Consequently, by the continuity of f, we have fz = z. □

**Example 2.6.** Let (X, M, *) be as in Example 2.3 and φ : X × [0, ∞) → ℝ, φ(x, t) = 1 − \frac{5}{x}. Define A = \{1, 2, · · · , 5\} and B = \{6, 7, · · · \}. Now if x, y ∈ A and x ≤ y, then x ≤ y. If x ∈ A and y ∈ B then x ≤ y. If x, y ∈ B, then x and y are not comparable. Now define f : X → X,

\[ fx = \begin{cases} 
   x + 1 & \text{if } x ≤ 5 \\
   6 & \text{if } x > 5
\end{cases} \]

It is clear that f is continuous and nondecreasing. Also 1 ≤ 2 = f1 and all the conditions of Theorem 2.5 are satisfied. Therefore, f has a fixed point.

3. Fixed point Theory for Multi-valued Mappings

In the following we provide multivalued versions of the preceding theorem. The results are related to those in [4].

Let X be a topological space and ≤ be a partial order on X. Let \(2^X\) denote the family of all nonempty subsets of X.

**Definition 3.1.** [4] Let A and B be two nonempty subsets of X. Then

(R-1) If for every a ∈ A, there exists b ∈ B such that a ≤ b, then A ⪯₁ B.
(R-2) If for every b ∈ B, there exists a ∈ A such that a ≤ b, then A ⪯₂ B.
(R-3) If A ⪯₁ B and A ⪯₂ B, then A ⪯ B.

**Remark 3.2.** [4] The relations ⪯₁ and ⪯₂ are different. For example, let X = ℝ, A = [\frac{1}{2}, 1], B = [0, 1], ≤ be usual order on X, then A ⪯₁ B but A ⪯₂ B; if A = [0, 1], B = [0, \frac{1}{2}], then A ⪯₂ B while A ⪯₁ B.

**Remark 3.3.** [4] ⪯₁, ⪯₂ and < are reflexive and transitive, but are not antisymmetric. For instance, let X = ℝ, A = [0, 3], B = [0, 1] ∪ [2, 3], ≤ be the usual order on X, then A < B and B < A, but A ≠ B. Hence, they are not partial orders on \(2^X\).
Definition 3.4. [4] A multi-valued operator \( T : X \to 2^X \) is called order closed if, for monotone sequences \( \{u_n\} \) and \( \{v_n\} \) in \( X \), \( u_n \to u_0, v_n \to v_0 \) and \( v_n \in Tu_n \) imply \( v_0 \in Tu_0 \).

Theorem 3.5. Let \((X,M,\ast)\) be an \( M \)-complete non-\( \ast \)-Archimedean fuzzy metric space with \( a \ast b \geq \max\{a+b-1,0\} \), \( \phi : X \times [0,\infty) \to \mathbb{R} \) a function bounded from above and \( \preceq \) a partial order induced by \( \phi \). Suppose \( F : X \to 2^X \) is an order closed operator with \( \{x_0\} \prec_1 Fx_0 \) for some \( x_0 \in X \). If \( \forall x,y \in X, x \preceq y \implies Fx \prec_1 Fy \) (that is, \( F \) is nondecreasing with respect to \( \prec_1 \)), then \( F \) has a fixed point in \( X \).

Proof. Since \( Fx \) is nonempty for all \( x \in X \), there exists \( x_1 \in Fx_0 \) such that \( x_0 \preceq x_1 \).

Now since \( Fx_0 \prec_1 Fx_1 \), there exists \( x_2 \in Fx_1 \) such that \( x_1 \preceq x_2 \). Continuing this process, we get an increasing sequence \( \{x_n\} \), which satisfies \( x_{n+1} \in Fx_n \). By the definition of \( \preceq \), we have \( \phi(x_n,t) \leq \phi(x_{n+1},t) \leq \phi(x_2,t) \leq \cdots \forall t > 0 \). In other words, for all \( t > 0 \) the sequence \( \{\phi(x_n,t)\} \) is nondecreasing in \( \mathbb{R} \). Since \( \phi \) is bounded from above, \( \{\phi(x_n,t)\} \) is convergent and hence Cauchy. So, for all \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for all \( m > n > n_0 \) and \( t > 0 \) we have \( |\phi(x_m,t) - \phi(x_n,t)| = \phi(x_m,t) - \phi(x_n,t) < \varepsilon \). Therefore, since \( x_n \preceq x_m \),

\[
M(x_n,x_m,t) \geq 1 + \phi(x_n,t) - \phi(x_m,t) \\
= 1 - [\phi(x_m,t) - \phi(x_n,t)] \\
> 1 - \varepsilon.
\]

This shows that the sequence \( \{x_n\} \) is Cauchy in \( X \) and since \( X \) is \( M \)-complete, it converges to a point \( z \in X \). Consequently, since \( F \) is order closed, we have \( z \in Fz \) and \( x_{n+1} \in Fx_n \).

We can similarly prove the following theorem.

Theorem 3.6. Let \((X,M,\ast)\) be an \( M \)-complete non-\( \ast \)-Archimedean fuzzy metric space with \( a \ast b \geq \max\{a+b-1,0\} \), \( \phi : X \times [0,\infty) \to \mathbb{R} \) a function bounded from below and \( \preceq \) the partial order induced by \( \phi \). Suppose \( F : X \to 2^X \) is an order closed operator with \( Fx_0 \prec_2 \{x_0\} \) for some \( x_0 \in X \). If \( \forall x,y \in X, x \preceq y \implies Fx \prec_2 Fy \) (\( F \) is nondecreasing with respect to \( \prec_2 \)), then \( F \) has a fixed point in \( X \).

Example 3.7. Let \((X,M,\ast)\) be as in Example 2.3 and \( \phi : X \times [0,\infty) \to \mathbb{R} \), \( \phi(x,t) = 1 - \frac{x}{x} \). Define \( A = \{1,2,\cdots,5\} \) and \( B = \{6,7,\cdots\} \). Now if \( x,y \in A \) and \( x \preceq y \), then \( x \preceq y \). If \( x \in A \) and \( y \in B \) then \( x \preceq y \). If \( x,y \in B \), then \( x \) and \( y \) are not comparable.

Now define \( F : X \to 2^X \),

\[
Fx = \{6,x+1\}.
\]

It is clear that \( F \) is order closed and \( \{1\} \prec_1 \{2,6\} = F1 \). Also, if \( x \preceq y \) then \( Fx \prec_1 Fy \), and all the conditions of Theorem 3.5 are satisfied. Therefore, \( F \) has a fixed point.

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References


Ishak Altun, Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey
E-mail address: ialtun@kku.edu.tr, ishakaltun@yahoo.com