THE URYSOHN, COMPLETELY HAUSDORFF AND COMPLETELY REGULAR AXIOMS IN $L$-FUZZY TOPOLOGICAL SPACES

C. LIANG AND F. G. SHI

Abstract. In this paper, the Urysohn, completely Hausdorff and completely regular axioms in $L$-topological spaces are generalized to $L$-fuzzy topological spaces. Each $L$-fuzzy topological space can be regarded to be Urysohn, completely Hausdorff and completely regular to some degree. Some properties of them are investigated. The relations among them and $T_2$ in $L$-fuzzy topological spaces are discussed.

1. Introduction

Since Chang [1] introduced fuzzy set theory to topology, many aspects of fuzzy topology were introduced and discussed. In a Chang $I$-topology, the open sets are fuzzy, but the topology comprising those open sets is a crisp subset of the $I$-powerset $I^X$. In 1980, Höhle [9] presented a new topological notion from a logical point of view and called it fuzzifying topology. Kubiak [11] and Šostak [21] independently extended Höhle’s fuzzy topology to $M$-subset of $L^X$ and $I$-subset of $I^X$, respectively, which is called $(L, M)$-fuzzy topology. In 1991, from a logical point of view, Ying [23] studied Höhle’s topology and called it fuzzifying topology.

As it is well known, the Urysohn, completely Hausdorff and completely regular axioms are important separation axioms in general topology. Chen and Wu generalized the Urysohn axiom to topological molecular lattice and called this generalized version the $U_2$ axiom [2]. Fang also generalized these axioms to $L$-topological space and topological molecular lattice in [4, 5]. However, none of the $L$-extended real line, the $L$-real line, and the $L$-unit interval satisfy these axioms.

In 2006, Shi [17] introduced $L$-Urysohn and $L$-completely Hausdorff axioms in $L$-topological spaces in such a way as to be compatible with the canonical topologies on the $L$-extended real line, $L$-real line, $L$-unit interval and the topology of pointwise pseudo-metric. They have been generalized to $I$-fuzzy topological spaces in [12]. Moreover, Shi [18] introduced the Urysohn and completely Hausdorff axioms to $L$-topological space such that they are compatible with pointwise metrics in $L$-set theory. In [15, 16], the completely regular axiom had been generalized to $L$-topological spaces in order to characterize pointwise uniformity.

With the development of fuzzy topology, separation axioms in $L$-fuzzy topological spaces have endowed with some degrees. In [22], Šostak firstly considered the

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graded topological (in particular, separation) properties in the context of $L$-fuzzy topological spaces. Afterwards, many researchers proposed the degrees of separation axioms. In the framework of $I$-fuzzy topologies, Yue and Fang [25] studied some low-level separation axioms defined by conditions on fuzzy points with different supports. In [19, 20], Shi introduced some separation axioms in $(L, M)$-fuzzy topology in such a way as to be compatible with $(L, M)$-fuzzy metrics.

The aim of this paper is to define the degree to which an $L$-fuzzy topological space is Urysohn, completely Hausdorff and completely regular. The relations among them are discussed.

2. Preliminaries

Throughout this paper, $I = [0, 1], I_0 = (0, 1]$ and $I_1 = [0, 1)$. $(L, \vee, \wedge, ^\prime)$ is a completely distributive De Morgan algebra, i.e., a completely distributive lattice with an order-reversing involution $\prime$. The smallest element and the largest element in $L$ are denoted by $\bot$ and $\top$, respectively. For $a, b \in L$, we say that $a$ is wedge below $b$, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \supseteq b$, there exists $d \in D$, $d \geq a$ [3]. The wedge below relation in a completely distributive lattice has the interpolation property, this means $\lambda \prec \mu \Rightarrow \exists \gamma \in L$ such that $\lambda \prec \gamma \prec \mu$. Moreover we can see that $\lambda \prec \bigvee_{\omega} \mu_i \Rightarrow \exists \mu_i$ such that $\lambda \prec \mu_i$. A complete lattice $L$ is completely distributive if and only if $b = \bigvee \{a \in L | a \prec b\}$ for each $b \in L$. An element $a$ in $L$ is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ [7]. The set of non-zero co-prime elements in $L$ is denoted by $M(L)$.

In a completely distributive De Morgan algebra $L$, there exists an implication operation $\rightarrow: L \times L \rightarrow L$ as the right adjoint for the meet operation $\land$ by

$$a \rightarrow b = \bigvee \{c \in L | a \land c \leq b\}.$$ 

We list some properties of implication operation.

Lemma 2.1. [10] Suppose that $(L, \vee, \wedge)$ is a completely distributive lattice and $\rightarrow$ is the implication operation corresponding to $\land$. Then for all $a, b, c \in L$, \{a_j\}_{j \in J}, \{b_j\}_{j \in J} \subseteq L$, the following conditions hold:

1. $(a \rightarrow b) \geq c \Leftrightarrow a \land c \leq b$;
2. $a \leq b \Leftrightarrow a \rightarrow b = \top$;
3. $a \rightarrow (b \rightarrow c) = (a \land b) \rightarrow c$;
4. $(c \rightarrow a) \land (a \rightarrow b) \leq c \rightarrow b$;
5. $c \rightarrow a \leq (a \rightarrow b) \rightarrow (c \rightarrow b)$;
6. $a \rightarrow \bigwedge_{j \in J} a_j = \bigwedge_{j \in J} (a \rightarrow a_j)$, hence $a \rightarrow b \leq a \rightarrow c$ whenever $b \leq c$;
7. $\bigvee_{j \in J} a_j \rightarrow b = \bigvee_{j \in J} (a_j \rightarrow b)$, hence $a \rightarrow c \geq b \rightarrow c$ whenever $a \leq b$.

For a completely distributive lattice $L$ and a nonempty set $X$, $L^X$ denotes the set of all $L$-fuzzy sets on $X$. $L^X$ is also a completely distributive lattice when it inherits the structure of lattice $L$ in a natural way, by defining $\vee, \wedge, \leq$ pointwisely. The set of non-zero co-prime elements in $L^X$ is denoted by $M(L^X)$. Each member
in $M(L^X)$ is also called a point. It is easy to see that $M(L^X)$ is exactly the set of all fuzzy points $x_\lambda (\lambda \in M(L))$.

An $L$-topological space is a pair $(X, \delta)$, where $\delta(\subseteq L^X)$ contains $\bot, \top$ and is closed under arbitrary suprema and finite infimums. Elements in $\delta$ are called open $L$-sets and their quasi-complements are called closed $L$-sets. A closed $L$-set $P$ is called a remote-neighborhood (or R-neighborhood for short) of $e \in M(L^X)$ if $e \notin P$. The set of all closed R-neighborhoods of $e$ is denoted by $\eta^-(e)$.

**Definition 2.2.** [11, 21] An $L$-fuzzy topology on a set $X$ is a map $\mathcal{T} : L^X \to L$ such that

1. $\mathcal{T}(\bot) = \mathcal{T}(-) = \top$;
2. $\forall U, V \in L^X, \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$;
3. $\forall \{U_j | j \in J\} \subseteq L^X, \mathcal{T}(\bigvee_{j \in J} U_j) \geq \bigwedge_{j \in J} \mathcal{T}(U_j)$.

$\mathcal{T}(U)$ can be interpreted as the degree to which $U$ is an open set. $\mathcal{T}^*(U) = \mathcal{T}(U^{\bot})$ will be called the degree of closedness of $U$. The pair $(X, \mathcal{T})$ is called an $L$-fuzzy topological space.

A map $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called continuous if $\mathcal{T}(f^\leftarrow(B)) \geq \mathcal{U}(B)$ holds for all $B \in L^Y$, where $f^\leftarrow$ is defined by $f^\leftarrow(B)(x) = B(f(x))$. A map $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called open if $\mathcal{T}(B) \leq \mathcal{U}(f^\leftarrow(B))$ holds for all $B \in L^X$, where $f^\leftarrow(B)(y) = \bigvee_{f(x) = y} B(x)$. A map $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called a homeomorphism if and only if $f$ is bijective and $f$ is both continuous and open.

**Definition 2.3.** [6, 24] (1) Let $\mathcal{T}$ be an $L$-fuzzy topology on $X$. A map $B : L^X \to L$ is called a base of $\mathcal{T}$ if $B$ satisfies the following condition:

$$\mathcal{T}(A) = \bigvee_{\bigwedge_{j \in J} B_j = A \land j \in J} \bigwedge B(B_j).$$

(2) Let $\phi : L^X \to L$ be a map. Then $\phi$ is called a subbase of $\mathcal{T}$ if and only if $\phi^{(\cap)} : L^X \to L$ is a base of $\mathcal{T}$, where

$$\phi^{(\cap)}(A) = \bigwedge_{\bigcap_{j \in J} B_j = A \land j \in J} \bigvee \phi(B_j)$$

with $\cap$ standing for “finite intersection”.

(3) Let $\{(X_j, \mathcal{T}_j)\}_{j \in J}$ be a collection of $L$-fuzzy topological spaces and $P_t : \prod_{j \in J} X_j \to X_i$ be the projection. Then the $L$-fuzzy topology on $\prod_{j \in J} X_j$ determined by the subbase

$$\forall A \in L^{\prod_{j \in J} X_j}, \phi(A) = \bigvee_{j \in J \land P_t(U) = A} \mathcal{T}_j(U)$$

is called the product $L$-fuzzy topology of $\{\mathcal{T}_j | j \in J\}$, denoted by $\prod_{j \in J} \mathcal{T}_j$, and $\left(\prod_{j \in J} X_j, \prod_{j \in J} \mathcal{T}_j\right)$ is called the product space of $\{(X_j, \mathcal{T}_j)\}_{j \in J}$.

**Definition 2.4.** [13] Given an $L$-fuzzy topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we call $(Y, \mathcal{T}|_Y)$ (where $\mathcal{T}|_Y(U) = \bigvee_{V \in L^X, V|_Y = U} \mathcal{T}(V)$) a subspace of $(X, \mathcal{T})$. 

Definition 2.5. [19] For an \((L, M)\)-fuzzy topological space \((X, \mathcal{T})\), define the degree \(T_1(X, \mathcal{T})\) to which \((X, \mathcal{T})\) is \(T_1\) as follows:

\[
T_1(X, \mathcal{T}) = \bigwedge_{a \nleq b} \bigvee_{a \nleq b}^{P \geq b} \mathcal{T}(P).
\]

Definition 2.6. [19] For an \((L, M)\)-fuzzy topological space \((X, \mathcal{T})\), define the degree \(T_2(X, \mathcal{T})\) to which \((X, \mathcal{T})\) is \(T_2\) as follows:

\[
T_2(X, \mathcal{T}) = \bigwedge_{a \nleq b} \bigvee \left\{ \mathcal{T}(P') \land \mathcal{T}(Q) \mid a \nleq P \land P \geq Q \geq b \right\}.
\]

Definition 2.7. [8] The \(L\)-fuzzy unit interval \(I(L)\) is defined as the set of all equivalence classes \([\lambda]_*\) of antitone maps \(\lambda : \mathbb{R} \to L\), for any \(t < 0\), \(\lambda(t) = \top\) and for any \(t > 1\), \(\lambda(t) = \bot\). The equivalence identifies two such maps \(\lambda, \mu\) iff \(\lambda(t+) = \mu(t+), \forall t \in \mathbb{R}\). The canonical \(L\)-topology is generated from the subbase \(\{\mathcal{L}_t, \mathcal{R}_t \mid t \in \mathbb{R}\}\), where

\[
\mathcal{L}_t : I(L) \to L \quad \text{by} \quad \mathcal{L}_t(\lambda) = \lambda(t-)',
\]

\[
\mathcal{R}_t : I(L) \to L \quad \text{by} \quad \mathcal{R}_t(\lambda) = \lambda(t+).
\]

Definition 2.8. [15] An \(L\)-topological space \((X, \delta)\) is said to be pointwise completely regular if for each \(e \in M(L^X)\) and each \(B \in \eta^{-}(e)\), there exists a family of sets \(\{A(t) : t \in (0, 1)\} \subseteq L^X\) such that for all \(e \nleq A(s) \geq A(s)^\circ \geq A(r)^- \geq A(r) \geq B\) for all \(r, s \in (0, 1)\) with \(s < r\).

Lemma 2.9. [14] An \(L\)-topological space \((X, \delta)\) is pointwise completely regular if and only if for each \(e \in M(L^X)\) and each \(B \in \eta^{-}(e)\), there exists \(\{A(t) : t \in (0, 1)\} \subseteq L^X\) such that for all \(s < r\), \(e \nleq \bigvee_{t \leq 0} A(t), B \leq \bigwedge_{t < 1} A(t)\), and \(A(s)^\circ \geq A(r)^-\).

The following definitions and two theorems were presented for an \(I\)-fuzzy topology. They can be easily transformed to an \(L\)-fuzzy topology as follows:

Definition 2.10. [12] Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space and \(A \in L^X\). The degree to which \(D \in L^X\) is the interior of \(A\) is defined by

\[
\text{Int}(A, D) = \left\{ \begin{array}{ll}
\bot, & \exists B \nleq D \text{ with } B \leq A \text{ such that } \mathcal{T}(B) \geq \mathcal{T}(D); \\
\bigwedge_{x \preceq D} \mathcal{T}(B), & \text{otherwise.}
\end{array} \right.
\]

The degree to which \(D \in L^X\) is the closure of \(A\) is defined by

\[
\text{Cl}(A, D) = \left\{ \begin{array}{ll}
\bot, & \exists B \nleq D \text{ with } B \geq A \text{ such that } \mathcal{T}^*(B) \geq \mathcal{T}^*(D); \\
\bigwedge_{x \preceq D'} \mathcal{T}^*(B), & \text{otherwise.}
\end{array} \right.
\]

Theorem 2.11. [12] Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space and \(A, P, Q \in L^X\). Then the following facts are valid.

1. If \(\text{Int}(A, P) \neq \bot\), then \(\text{Int}(A, P) \leq \mathcal{T}(P)\), \(P \leq A\) and for all \(B \nleq P\) with \(B \leq A\), \(\mathcal{T}(B) \nleq \mathcal{T}(P)\).
(2) \( \text{Int}(A, A) = T(A) \);
(3) For all \( P \leq A \), there exists a \( Q \in L^X \) such that \( P \leq Q \leq A \), \( \text{Int}(A, Q) \geq T(P) \);
(4) If \( \text{Int}(A, P) = a \neq \bot \), then for all \( b \prec a \), there exists a \( Q \in L^X \) such that \( P \leq Q \leq A, T(Q) \geq b \) and \( \text{Int}(A, Q) \geq b \).

**Theorem 2.12.** [12] Let \( (X, T) \) be an \( L \)-fuzzy topological space and \( A, P, Q \in L^X \). Then the following statements are true:

(1) If \( \text{Cl}(A, P) \neq \bot \), then \( \text{Cl}(A, P) \geq T^*(P), P \geq A \) and for all \( P \not\leq B \) with \( A \leq B \), \( T^*(B) \not\leq T^*(P) \);
(2) \( \text{Cl}(A, A) = T^*(A) \);
(3) For all \( P \geq A \), there exists a \( Q \in L^X \) such that \( P \geq Q \geq A \), \( \text{Cl}(A, Q) \geq T^*(P) \);
(4) If \( \text{Cl}(A, P) = a \neq \bot \), then for all \( b \prec a \), there exists a \( Q \in L^X \) such that \( P \geq Q \geq A, T^*(Q) \geq b \) and \( \text{Cl}(A, Q) \geq b \).

3. The Urysohn Axiom

In this section, we define the degree to which an \( L \)-fuzzy topological space is Urysohn separated, and discuss its relation with \( T_2 \) in \( L \)-fuzzy topological space.

**Definition 3.1.** For an \( L \)-fuzzy topological space \( (X, T) \), we define the degree \( U(a, b) \) to which two points \( a, b \) with \( a \not\leq b \) are Urysohn separated as follows:

\[
U(a, b) = \bigvee_{P_2 \leq Q_2, b \leq Q_1, a \not\leq P_1} \left( T(P_1') \wedge T(Q_1) \wedge \text{Cl}(Q_1, P_2) \wedge \text{Int}(P_1, Q_2) \right).
\]

The degree to which \( (X, T) \) is Urysohn separated is defined by

\[
U(X, T) = \bigwedge_{a \not\leq b} U(a, b).
\]

**Theorem 3.2.** Let \( (X, T) \) be an \( L \)-fuzzy topological space. Then

\[
U(X, T) = \bigwedge_{a \not\leq b} \bigvee_{P_2 \leq Q_2, b \leq Q_1, a \not\leq P_1} \left( T(P_1') \wedge T(Q_1) \wedge \text{Cl}(Q_1, P_2) \wedge \text{Int}(P_1, Q_2) \right).
\]

**Proof.** It is obvious that

\[
U(X, T) \geq \bigwedge_{a \not\leq b \leq Q_1, P_2 \leq Q_2, P_1 \not\leq a} \left( T(P_1') \wedge T(Q_1) \wedge \text{Cl}(Q_1, P_2) \wedge \text{Int}(P_1, Q_2) \right).
\]

Now we prove

\[
U(X, T) \leq \bigwedge_{a \not\leq b \leq Q_1, P_2 \leq Q_2, P_1 \not\leq a} \left( T(P_1') \wedge T(Q_1) \wedge \text{Cl}(Q_1, P_2) \wedge \text{Int}(P_1, Q_2) \right).
\]

Let \( a, b \in M(L^X), P_1, P_2, Q_1, Q_2 \in L^X \), satisfying \( a \not\leq P_1, b \leq Q_1 \) and \( P_2 \leq Q_2 \). If \( Q_1 \not\leq P_2 \), then \( \text{Cl}(Q_1, P_2) = \bot \). If \( Q_2 \not\leq P_1 \), then \( \text{Int}(P_1, Q_2) = \bot \). This implies
First we check \( \prec T \gamma P \). Then there exist \( \gamma \) such that
\[
\left( T(P'_1) \land T(Q_1) \land Cl(Q_1, P_2) \land Int(P_1, Q_2) \right).
\]
Theorem 3.3. Let \((X, \mathcal{T})\) be an \( L \)-fuzzy topological space. Then for any points \( a, b \) with \( a \not\prec b \), it follows that
\[
U(a, b) = \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\prec a} \left( T(P'_1) \land T(Q_1) \land P_2 \land T(Q_2) \right).
\]
Proof. First we check
\[
U(a, b) \geq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\prec a} \left( T(P'_1) \land T(Q_1) \land P_2 \land T(Q_2) \right).
\]
Suppose that \( \gamma \in L \) and \( \gamma \prec \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\prec a} \left( T(P'_1) \land T(Q_1) \land P_2 \land T(Q_2) \right).
Then there exist \( P_1, P_2, Q_1, Q_2 \in L^X \) such that \( b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\prec a \) and
\( \gamma \prec T(P'_1) \land T(Q_1) \land T(P_2) \land T(Q_2). \)
By Theorem 2.11 (3), we know that there exists a \( Q_0 \in L^X \) such that \( Q_2 \leq Q_0 \leq P_1 \) and \( Int(P_0, Q_0) \geq T(Q_2). \) By Theorem 2.12 (3), we know that there exists a \( P_0 \in L^X \) such that \( Q_1 \leq P_0 \leq P_2 \) and \( Cl(Q_1, P_0) \geq Q_2. \)
Therefore we can obtain \( b \leq Q_1 \leq P_0 \leq P_2 \leq Q_2 \leq Q_0 \leq P_1 \not\prec a \) and
\( \gamma \prec T(P'_1) \land T(Q_1) \land Cl(Q_1, P_0) \land Int(P_1, Q_0). \) Then we have
\[
\gamma \prec \bigvee_{P_0 \leq Q_0, b \leq Q_1, a \not\prec P_1} \left( T(P'_1) \land T(Q_1) \land Cl(Q_1, P_0) \land Int(P_1, Q_0) \right) = U(a, b).
\]
Since \( \gamma \) is arbitrary, we know
\[
U(a, b) \geq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\prec a} \left( T(P'_1) \land T(Q_1) \land P_2 \land T(Q_2) \right).
\]
Secondly, we check
\[
U(a, b) \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\prec a} \left( T(P'_1) \land T(Q_1) \land P_2 \land T(Q_2) \right).
\]
Let $P_1, P_2, Q_1, Q_2 \in L^X$ with $a \notin P_1, b \leq Q_1, P_2 \leq Q_2$. Now we prove that

$$\mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \mathcal{Cl}(Q_1, P_2) \land \text{Int}(P_1, Q_2) \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1} \left( \mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \mathcal{T}(P_2') \land \mathcal{T}(Q_2) \right).$$

Let $\text{Int}(P_1, Q_2) = g \neq \perp, \mathcal{Cl}(Q_1, P_2) = h \neq \perp$. By Theorem 2.11 (4), for all $c \prec g$, there exists a $Q_c \in L^X$ such that $Q_2 \leq Q_c \leq P_1, \mathcal{T}(Q_c) \geq c$ and $\text{Int}(P_1, Q_c) \geq c$. By Theorem 2.12 (4), for all $d \prec h$, there exists a $P_d \in L^X$ such that $Q_1 \leq P_d \leq P_2, \mathcal{T}(P_d') \geq d$ and $\mathcal{Cl}(Q_1, P_d) \geq d$. Therefore $b \leq Q_1 \leq P_d \leq P_2 \leq Q_2 \leq Q_c \leq P_1 \neq a$, then we have

$$\mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \mathcal{Cl}(Q_1, P_2) \land \text{Int}(P_1, Q_2) \leq \mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \left( \bigvee_{d \prec h} \mathcal{T}(P_d') \right) \land \left( \bigvee_{c \prec g} \mathcal{T}(Q_c) \right) \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \neq a} \left( \mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \mathcal{T}(P_2') \land \mathcal{T}(Q_2) \right).$$

This implies

$$\mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \mathcal{Cl}(Q_1, P_2) \land \text{Int}(P_1, Q_2) \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \neq a} \left( \mathcal{T}(P_1') \land \mathcal{T}(Q_1) \land \mathcal{T}(P_2') \land \mathcal{T}(Q_2) \right).$$

The proof is completed. \(\square\)

By Theorem 3.3, we have the following theorems.

**Theorem 3.4.** If $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a homeomorphism, then for all $a \neq b$, $U^X \hookrightarrow (a, b) = U^Y \hookrightarrow (f^\rightarrow (a), f^\rightarrow (b)).$

**Theorem 3.5.** Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $(Y, \mathcal{T}|_Y)$ be a subspace of $(X, \mathcal{T})$. Then $U (X, \mathcal{T}) \leq U (Y, \mathcal{T}|_Y)$.

**Theorem 3.6.** Let $(X, \mathcal{T})$ be the product of L-fuzzy topological spaces $\{(X_j, \mathcal{T}_j)\}_{j \in J}$. Then $\bigwedge_{j \in J} U (X_j, \mathcal{T}_j) \leq U (X, \mathcal{T})$. In addition, if $(X_j, \mathcal{T}_j)$ is stratified for all $j \in J$, then $\bigwedge_{j \in J} U (X_j, \mathcal{T}_j) = U (X, \mathcal{T})$.

**Proof.** Suppose that $\gamma \prec \bigwedge_{j \in J} U (X_j, \mathcal{T}_j)$. Then for any $j \in J$, $\gamma \prec U (X_j, \mathcal{T}_j)$. Let $a, b \in M(L^X)$ with $a \neq b$. Then there exists a $k \in J$ such that $P_k^\gamma (a) \neq P_k^\gamma (b)$.
Let \( X \) be parallel to \( T \) and homeomorphic to a subspace \( U \) discussed. The Urysohn axiom and Definition 4.1. Theorem 3.7.

By Theorem 3.3 we have
\[
\gamma < U(X_k, T_k) = \bigwedge_{a \leq b \leq c \leq d} \bigvee_{a \in a_k} \left( T_k(D_k) \land T_k(A_k) \land T_k(B_k) \land T_k(C_k) \right)
\]
\[
\leq \bigvee_{P_{k}^{-}(b) \leq a \leq b \leq c \leq d} \left( T_k(D_k) \land T_k(A_k) \land T_k(B_k) \land T_k(C_k) \right).
\]
This implies that there exist \( A_k, B_k, C_k, D_k \in L^X \), such that \( P_{k}^{-}(b) \leq A_k \leq B_k \leq C_k \leq D_k \neq P_{k}^{-}(a) \) and
\[
\gamma \leq T_k(D_k) \land T_k(A_k) \land T_k(B_k) \land T_k(C_k)
\]
\[
\leq T(P_{k}^{-}(D_k)) \land T(P_{k}^{-}(A_k)) \land T(P_{k}^{-}(B_k)) \land T(P_{k}^{-}(C_k)).
\]

By \( b \leq P_{k}^{-}(A_k) \leq P_{k}^{-}(B_k) \leq P_{k}^{-}(C_k) \leq P_{k}^{-}(D_k) \neq a \), we obtain
\[
\gamma \leq \bigvee_{b \leq A \leq B \leq C \leq D} \left( T(D') \land T(A) \land T(B) \land T(C) \right).
\]
Therefore,
\[
\gamma \leq \bigwedge_{a \leq b \leq A \leq B \leq C \leq D} \left( T(D') \land T(A) \land T(B') \land T(C) \right).
\]
Since \( \gamma \) is arbitrary, we know \( \bigwedge_{j \in J} U(X_j, T_j) \leq U(X, T) \).

In addition, assume that \( (X_j, T_j) \) is stratified for all \( j \in J \), then \( (X_j, T_j) \) is homeomorphic to a subspace \( (X_j, T|_{X_j}) \) of \( (X, T) \), where \( X_j \) is a subset of \( X \) parallel to \( X_j \) through \( x = (x_j)_{j \in J} \). From Theorem 3.4 and 3.5, we have
\[
\bigwedge_{j \in J} U(X_j, T_j) = \bigwedge_{j \in J} U(X_j, T|_{X_j}) \geq U(X, T).
\]

By Definition 2.6 and Theorem 3.3, we can easily obtain the relation between the Urysohn axiom and \( T_2 \) axiom.

**Theorem 3.7.** Let \( (X, T) \) be an \( L \)-fuzzy topological space. Then \( U(X, T) \leq T_2(X, T) \).

## 4. The Completely Hausdorff Axiom

In this section, the degree to which an \( L \)-fuzzy topological space is completely Hausdorff is introduced and its relation with the degree of Urysohn separation is discussed.

**Definition 4.1.** For an \( L \)-fuzzy topological space \( (X, T) \), we define the degree \( H(a, b) \) to which two points \( a, b \) with \( a \leq b \) are completely Hausdorff separated as follows:
\[
H(a, b) = \bigvee_{F \in \mathcal{F}(a, b)} \bigwedge_{s < r} \bigvee_{P \geq Q} (\text{Cl}(F(r, Q)) \land \text{Int}(F(s, P))).
\]
where \( F(a, b) = \{ F : (0, 1) \to L^X \mid \forall s < r, a \not\leq F(s) \geq F(r) \geq b \} \). The degree to which \((X, \mathcal{T})\) is completely Hausdorff separated is defined by:

\[
H(X, \mathcal{T}) = \bigwedge_{a \not\leq b} H(a, b).
\]

**Theorem 4.2.** Let \((X, \mathcal{T})\) be an L-fuzzy topological space. Then for all points \(a, b \in M(L^X)\) with \(a \not\leq b\),

\[
H(a, b) = \bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T} (G(t+)) \right),
\]

where \( \mathcal{G}(a, b) = \{ G : (0, 1) \to L^X \mid G \text{ is an antitone map with } a \not\leq G(0^+), b \leq G(1^-) \} \).

**Proof.** Suppose that \( \lambda \prec H(a, b) \). Then there exists an \( F \in F(a, b) \) such that for any \( s, r \in (0, 1) \) with \( s < r \), there exist \( P, Q \) with \( P \geq Q \) such that \( \lambda \prec Cl(F(r), Q) \land Int(F(s), P) \). It is easy to check that \( F \in \mathcal{G}(a, b) \).

Since \( \lambda \prec Cl(F(r), Q) \land Int(F(s), P) \), it follows that \( \lambda \prec Cl(F(r), Q) \) and \( \lambda \prec Int(F(s), P) \). By Theorems 2.11 and 2.12, we know that there exist \( Q_{(r,s)}, P_{(r,s)} \) such that \( F(r) \leq Q_{(r,s)} \leq Q \leq P \leq P_{(r,s)} \leq F(s) \) and \( \lambda \leq T(P_{(r,s)}) \). From

\[
F(t-) \leq \bigwedge_{t \leq t' \leq l} F(t') \leq \bigwedge_{t \leq t' \leq l} \bigwedge_{a \leq a'} Q_{(t, a')} \leq \bigwedge_{t \leq t' \leq l} \bigwedge_{a \leq a'} F(a) = F(t-), \quad (t \in I_0)
\]

and

\[
F(t+) \geq \bigvee_{t \leq t' \geq l} F(t') \geq \bigvee_{t \leq t' \geq l} \bigvee_{a \leq a'} P_{(t, a')} \geq \bigvee_{t \leq t' \geq l} \bigvee_{a \leq a'} F(a) = F(t+), \quad (t \in I_1)
\]

we know that

\[
T^*(F(t-)) = T^* \left( \bigwedge_{t \leq t' \leq l} \bigwedge_{a \leq a'} Q_{(t, a')} \right) \geq \bigwedge_{t \leq t' \leq l} \bigwedge_{a \leq a'} T^* (Q_{(t, a)}) \geq \lambda, \quad (t \in I_0)
\]

and

\[
T(F(t+)) = T \left( \bigvee_{t \leq t' \geq l} \bigvee_{a \leq a'} P_{(t, a')} \right) \geq \bigvee_{t \leq t' \geq l} \bigwedge_{a \leq a'} T (P_{(t, a)}) \geq \lambda, \quad (t \in I_1)
\]

Hence

\[
\lambda \leq \bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T} (G(t+)) \right).
\]

This means that

\[
H(a, b) \leq \bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T} (G(t+)) \right).
\]
On the other hand, let $\mu < \bigvee_{G \in \mathcal{G}(a,b)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right)$. Then there exists a $G \in \mathcal{G}(a,b)$ such that $\mu < \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+))$. Then for all $t \in I_0$, $\mu < \mathcal{T}^*(G(t-))$ and for all $t \in I_1$, $\mu < \mathcal{T}(G(t+))$. It is easy to check that $G \in \mathcal{F}(a,b)$.

For all $s < r$, from $G(s+) \leq G(s)$ and $G(r) \leq G(r-)$, we know that there exist $Q_{(r,s)}, P_{(r,s)}$ such that $G(r) \leq Q_{(r,s)} \leq G(r-) \leq G(s+) \leq P_{(r,s)} \leq G(s)$ and $\text{Int}(G(s), P_{(r,s)}) \geq \mathcal{T}(G(s+)) \geq \mu, \text{Cl}(G(r), Q_{(r,s)}) \geq \mathcal{T}^*(G(r-)) \geq \mu$. Therefore,

$$\mu \leq \bigvee_{F \in \mathcal{F}(a,b), s < r} \bigvee_{P \geq Q} (\text{Cl}(F(r), Q) \land \text{Int}(F(s), P)).$$

This means that

$$H(a, b) \geq \bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right).$$

The proof is completed. \(\square\)

By Theorem 4.2, we can obtain the following theorem.

**Theorem 4.3.** If $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a homeomorphism, then for all $a \not\leq b$, $H_{\mathcal{U}}^Y(a, b) = H_{\mathcal{U}}^Y(f^{-1}(a), f^{-1}(b))$.

**Theorem 4.4.** Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $(Y, \mathcal{T}|_Y)$ be a subspace of $(X, \mathcal{T})$. Then $H(X, \mathcal{T}) \leq H(Y, \mathcal{T}|_Y)$.

**Proof.** By Theorem 4.2, we need to prove

$$\bigwedge_{e \not\leq d} \bigvee_{G \in \mathcal{G}(e,d)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right) \leq \bigwedge_{a \not\leq b} \bigvee_{G \in \mathcal{G}(a,b)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*_Y(G_Y(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}_Y(G_Y(t+)) \right).$$

Suppose that $\lambda \prec \bigwedge_{e \not\leq d} \bigvee_{G \in \mathcal{G}(e,d)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right)$. Then for any $e, d \in M(L^X)$ with $e \not\leq d$, there exists $G \in \mathcal{G}(e,d)$ such that $\lambda \prec \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+))$.

For any $a, b \in M(L^Y)$ with $a \not\leq b$, we have $a, b \in M(L^X)$. Then there exists $G \in \mathcal{G}(a, b)$ such that $\lambda \prec \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+))$.\[118\]
Let \( \lambda \in a, b \). Then \( G \). Therefore, \( \lambda \) is arbitrary, it follows that \( \gamma \). Let \( \gamma < H(X_k, T_k) \)

\[
\begin{align*}
\lambda & < \bigwedge_{t \in I_0} T^* (G(t-)) \wedge \bigwedge_{t \in I_1} T (G(t+)) \\
& \leq \bigvee_{G|Y \in G(a, b)} \left( \bigwedge_{t \in I_0} T^*_Y (G(t-)) \wedge \bigwedge_{t \in I_1} T (G(t+)) \right)
\end{align*}
\]

Therefore,

\[
\lambda \leq \bigwedge_{a \not\in b, G_Y \in G_Y(a, b)} \left( \bigwedge_{t \in I_0} T^*_Y (G(t-)) \wedge \bigwedge_{t \in I_1} T (G(t+)) \right).
\]

Since \( \lambda \) is arbitrary, it follows that

\[
\bigwedge_{a \not\in b, G_Y \in G_Y(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \wedge \bigwedge_{t \in I_1} T (G(t+)) \right)
\]

\[
\leq \bigwedge_{a \not\in b, G_Y \in G_Y(a, b)} \left( \bigwedge_{t \in I_0} T^*_Y (G(t-)) \wedge \bigwedge_{t \in I_1} T (G(t+)) \right).
\]

**Theorem 4.5.** Let \((X, \mathcal{T})\) be the product of \(L\)-fuzzy topological spaces \(\{(X_j, T_j)\}_{j \in J}\). Then \( \bigwedge_{j \in J} H(X_j, T_j) \leq H(X, \mathcal{T}) \). In addition, if \((X_j, T_j)\) is stratified for all \(j \in J\), then \( \bigwedge_{j \in J} H(X_j, T_j) = H(X, \mathcal{T}) \).

**Proof.** Suppose that \( \gamma < \bigwedge_{j \in J} H(X_j, T_j) \). Then for any \(j \in J\), \( \gamma < H(X_j, T_j) \). Let \(a, b \in M(L^X)\) with \(a \not\in b\). Then there exists a \(k \in J\) such that \(P_k^{-}(a) \not\in P_k^{-}(b)\). Then

\[
\begin{align*}
\gamma & < H(X_k, T_k) \\
& = \bigwedge_{a \not\in b, G_k \in G_k(a, b)} \left( \bigwedge_{t \in I_0} T^*_k (G_k(t-)) \wedge \bigwedge_{t \in I_1} T_k (G_k(t+)) \right)
\end{align*}
\]

\[
\leq \bigvee_{G_k \in G_k(P_k^{-}(a), P_k^{-}(b))} \left( \bigwedge_{t \in I_0} T^*_k (G_k(t-)) \wedge \bigwedge_{t \in I_1} T_k (G_k(t+)) \right).
\]

This implies that there exists \(G_k \in G_k(P_k^{-}(a), P_k^{-}(b))\) such that

\[
\gamma < \bigwedge_{t \in I_0} T^*_k (G_k(t-)) \wedge \bigwedge_{t \in I_1} T_k (G_k(t+))
\]
For any \( t \in (0, 1) \), let \( G(t) = P_k^v(G_k(t)) \). Then \( G \in G(a, b) \).

From

\[
T^*(G(t-)) = T^* \left( \bigwedge_{s < t} P_k^v(G_k(s)) \right) \\
= T^* \left( P_k^v \left( \bigwedge_{s < t} G_k(s) \right) \right) \geq T_k^* (G_k(t-)), \ (t \in I_1)
\]

and

\[
T(G(t+)) = T \left( \bigvee_{s > t} P_k^v(G_k(s)) \right) \\
= T \left( P_k^v \left( \bigvee_{s > t} G_k(s) \right) \right) \geq T_k^* (G_k(t+)), \ (t \in I_0)
\]

we know

\[
\gamma \triangleq \bigwedge_{t \in I_0} T_k^* (G_k(t-)) \land \bigwedge_{t \in I_1} T_k (G_k(t+)) \\
\leq \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \\
\leq \bigvee_{G \in G(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \right).
\]

Hence

\[
\gamma \leq \bigwedge_{a \notin b, G \in G(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \right) = H(X, T).
\]

Since \( \gamma \) is arbitrary, we know \( \bigwedge_{j \in J} H(X_j, T_j) \leq H(X, T) \).

In addition, assume that \((X_j, T_j)\) is stratified for all \( j \in J \), then \((X_j, T_j)\) is homeomorphic to a subspace \((\tilde{X}_j, T|_{\tilde{X}_j})\) of \((X, T)\), where \(\tilde{X}_j\) is a subset of \(X\) parallel to \(X_j\) through \(x = (x_j)_{j \in J}\). From Theorems 4.3 and 4.4, we have

\[
\bigwedge_{j \in J} H(X_j, T_j) = \bigwedge_{j \in J} H(\tilde{X}_j, T|_{\tilde{X}_j}) \geq H(X, T).
\]

Now we study the relation between the Urysohn axiom and the completely Hausdorff axiom.

**Theorem 4.6.** Let \((X, T)\) be an \(L\)-fuzzy topological space. Then \(H(X, T) \leq U(X, T)\).

**Proof.** By Theorems 3.3 and 4.2, we only need to prove that for all \(a, b \in M(L^X)\),

\[
\bigvee_{G \in G(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \right) \\
\leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1, x \notin a} \left( T(P'_1) \land T(Q_1) \land T(P'_2) \land T(Q_2) \right).
\]
For each \( \lambda \in L \) and \( \lambda < \bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \right) \), there exists a \( G \in \mathcal{G}(a, b) \) such that \( \lambda < \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \). Since \( G \in \mathcal{G}(a, b) \), we have \( a \notin G(0+) \), and \( b \leq G(1-) \). Then there exist \( m, s \) with \( 0 < m < s < 1 \) such that \( a \notin G(m+) \geq G(s-) \geq G(1-) \geq b \). Let \( P_1 = Q_2 = G(m+) \), \( Q_1 = P_2 = G(s-) \). Then \( b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a \) and

\[
\lambda \leq \quad T^*(G(m+)) \land T(G(m+)) \land T^*(G(s-)) \land T(G(s-)) = T(P_1') \land T(Q_1) \land T(P_2') \land T(Q_2).
\]

Then

\[
\lambda \leq \quad \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \left( T(P_1') \land T(Q_1) \land T(P_2') \land T(Q_2) \right).
\]

Since \( \lambda \) is arbitrary, we have

\[
\bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} T^* (G(t-)) \land \bigwedge_{t \in I_1} T (G(t+)) \right) \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \left( T(P_1') \land T(Q_1) \land T(P_2') \land T(Q_2) \right).
\]

Therefore \( H(X, \mathcal{T}) \leq U(X, \mathcal{T}) \). \( \Box \)

5. The Completely Regular Axiom

In this section, the completely regular axiom in \( L \)-topological space is generalized to \( L \)-fuzzy topological space. Each \( L \)-fuzzy topological space can be regarded to be completely regular to some degree. Moreover, the relation between \( T_{\frac{3}{4}} \) and the degree of completely Hausdorff separation is investigated.

**Definition 5.1.** Let \((X, \mathcal{T})\) be an \( L \)-fuzzy topological space. The degree to which \((X, \mathcal{T})\) is completely regular is defined by:

\[
C_{reg}(X, \mathcal{T}) = \bigwedge_{e \in M(X^L)} \bigwedge_{e \notin B \in L^X} \left\{ T^*(B) \rightarrow \bigvee_{A \in K(e, B), s < r} \bigvee_{P \geq Q} \left( Cl(A(r), Q) \land Int(A(s), P) \right) \right\},
\]

where \( K(e, B) = \left\{ A : (0, 1) \rightarrow L^X \mid e \notin \bigvee_{t > 0} A(t), \ B \leq \bigwedge_{t < 1} A(t) \right\} \).

Moreover define

\[
T_{\frac{3}{4}}(X, \mathcal{T}) = C_{reg}(X, \mathcal{T}) \land T_1(X, \mathcal{T}).
\]
Theorem 5.2. Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space. Then
\[
Creg(X, \mathcal{T}) = \bigwedge_{e \in M(L^X)} \bigwedge_{B \in L^X} \left\{ \mathcal{T}^*(B) \rightarrow \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H(\lambda + t)) \right) \right\},
\]
where
\[
\mathcal{H}(e, B) = \{ H : (0, 1) \rightarrow L^X \mid H \text{ is an antitone map with } e \leq H(0+), B \leq H(1-) \}.
\]
Proof. The proof is similar to that of Theorem 4.2. \(\square\)

Theorem 5.3. Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space and \((Y, \mathcal{T}|_Y)\) be a subspace of \((X, \mathcal{T})\). Then \(Creg(X, \mathcal{T}) \leq Creg(Y, \mathcal{T}|_Y)\).

Proof. By Theorem 5.2, we only need to prove
\[
\bigwedge_{e \in M(L^X)} \bigwedge_{B \in L^X} \left\{ \mathcal{T}^*(B) \rightarrow \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H(\lambda + t)) \right) \right\}
\]
\[
\leq \bigwedge_{a \in M(L^Y)} \bigwedge_{A \in L^Y} \left\{ \mathcal{T}^* (A) \rightarrow \bigvee_{H_Y \in \mathcal{H}_Y(a, A)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H_Y(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H_Y(\lambda + t)) \right) \right\}.
\]
Let \(\lambda \prec \bigwedge_{e \in M(L^X)} \bigwedge_{B \in L^X} \left\{ \mathcal{T}^*(B) \rightarrow \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(H(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H(\lambda + t)) \right) \right\} \).
Then for any \(e \in M(L^X), B \in L^X\) with \(e \preceq B\) we have
\[
\lambda \wedge \mathcal{T}^*(B) \leq \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H(\lambda + t)) \right).
\]
Let \(a \in M(L^Y), A \in L^Y\) with \(a \preceq A\) and \(V \in L^X\) with \(V|_Y = A\). For any \(H \in \mathcal{H}(a, V)\), define \(\mathcal{H}_Y|_Y (a, V|_Y) = \{ H|_Y : (0, 1) \rightarrow L^Y \mid H \in \mathcal{H}(a, V) \}\), where \(H|_Y (t)(y) = H(t)(y)\) for any \(t \in (0, 1)\) and \(y \in Y\). Then we have
\[
\lambda \wedge \mathcal{T}^*(V) \leq \bigvee_{H \in \mathcal{H}(a, V)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H(\lambda + t)) \right)
\]
\[
\leq \bigvee_{H_Y \in \mathcal{H}_Y(a, V|_Y)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H_Y(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H_Y(\lambda + t)) \right)
\]
\[
\leq \bigvee_{H_Y \in \mathcal{H}_Y(a, A)} \left( \bigwedge_{t \in I_0} \mathcal{T}^* (H_Y(\lambda)) \wedge \bigwedge_{t \in I_1} \mathcal{T} (H_Y(\lambda + t)) \right).
\]
Hence
\[ \lambda \land T^*_Y (A) = \lambda \land \left( \bigvee_{V \in L^X, \gamma = A} T^* (V) \right) \leq \bigvee_{H_Y \in H_Y (a, A)} \left( \bigwedge_{I_0} T^*_Y (H_Y (t-)) \land \bigwedge_{I_1} T^*_Y (H_Y (t+)) \right). \]

This implies

\[ \lambda \leq T^*_Y (A) \rightarrow \bigvee_{H_Y \in H_Y (a, A)} \left( \bigwedge_{I_0} T^*_Y (H_Y (t-)) \land \bigwedge_{I_1} T^*_Y (H_Y (t+)) \right). \]

Therefore,

\[ \lambda \leq \bigwedge_{a \in M (L^X) : a \notin A} \bigwedge_{a \in L^Y} \left\{ T^*_Y (A) \rightarrow \bigvee_{H_Y \in H_Y (a, A)} \left( \bigwedge_{I_0} T^*_Y (H_Y (t-)) \land \bigwedge_{I_1} T^*_Y (H_Y (t+)) \right) \right\}. \]

Since \( \lambda \) is arbitrary, we know that

\[ \bigwedge_{a \in M (L^X) : a \notin A} \bigwedge_{a \in L^Y} \left\{ T^*_Y (A) \rightarrow \bigvee_{H_Y \in H_Y (a, A)} \left( \bigwedge_{I_0} T^*_Y (H_Y (t-)) \land \bigwedge_{I_1} T^*_Y (H_Y (t+)) \right) \right\} \]

\[ \leq \bigwedge_{a \in M (L^X) : a \notin A} \bigwedge_{a \in L^Y} \left\{ T^*_Y (A) \rightarrow \bigvee_{H_Y \in H_Y (a, A)} \left( \bigwedge_{I_0} T^*_Y (H_Y (t-)) \land \bigwedge_{I_1} T^*_Y (H_Y (t+)) \right) \right\}. \]

\[ \square \]

**Theorem 5.4.** If \( f : (X, T) \rightarrow (Y, U) \) is a homeomorphism, then \( Creg (X, T) = Creg (Y, U) \).

**Proof.** By Theorem 5.2, we can easily obtain this theorem. \( \square \)

**Theorem 5.5.** Let \( (X, T) \) be the product of \( L \)-fuzzy topological spaces \( \{(X_j, T_j)\}_{j \in J} \). Then \( \bigwedge_{j \in J} Creg (X_j, T_j) \leq Creg (X, T) \). In addition, if \( (X_j, T_j) \) is stratified for all \( j \in J \), then \( \bigwedge_{j \in J} Creg (X_j, T_j) = Creg (X, T) \).

**Proof.** Let \( \gamma \prec \bigwedge_{j \in J} Creg (X_j, T_j) \). Then for any \( j \in J, \gamma \prec Creg (X_j, T_j) \). This implies for any \( e_j \in M (L^{X_j}), A_j \in L^{X_j} \) with \( e_j \not\equiv A_j \), that

\[ \gamma \land T^*_j (A_j) \leq \bigvee_{H_j \in H_j (e_j, A_j)} \left( \bigwedge_{I_0} T^*_j (H_j (t-)) \land \bigwedge_{I_1} T^*_j (H_j (t+)) \right). \]
Now we prove for any \( e \in M(\mathbb{L}^X) \), \( A \in \mathbb{L}^X \) with \( e \nless A \), that
\[
\gamma \land T^*(A) \leq \bigvee_{H \in H(e,A)} \left( \bigwedge_{t \in I_0} T^*(H(t-)) \land \bigwedge_{t \in I_1} T(H(t+)) \right).
\]

Let \( \mu \prec \gamma \land T^*(A) \). Then \( \mu \prec \gamma \) and \( \mu \prec T^*(A) \). Therefore, there exists a finite set \( \Omega \subseteq J \) and \( A_k \in \mathbb{L}^X \) \((k \in \Omega)\) such that \( e \nless \bigvee_{k \in \Omega} P_{k^*}^- (A_k) \geq A \) and \( \mu \prec T_k(A_k) \).

Obviously, for any \( k \in \Omega, P_{k^*}^- (e) \nless A_k \), we have
\[
\gamma \land T_k^*(A_k) \leq \bigvee_{H \in H_k(P_{k^*}^-(e),A_k)} \left( \bigwedge_{t \in I_0} T_k^*(H_k(t-)) \land \bigwedge_{t \in I_1} T_k(H_k(t+)) \right).
\]

By \( \mu \prec \gamma \) and \( \mu \prec T_k^*(A_k) \), we know for any \( k \in \Omega \) that
\[
\mu \leq \gamma \land T_k^*(A_k) \leq \bigvee_{H_k \in H_k(P_{k^*}^-(e),A_k)} \left( \bigwedge_{t \in I_0} T_k^*(H_k(t-)) \land \bigwedge_{t \in I_1} T_k(H_k(t+)) \right).
\]

Let \( \alpha \prec \mu \). Then for any \( k \in \Omega \), there exists an \( H_k \in H_k(P_{k^*}^-(e),A_k) \) such that \( \alpha \prec \bigwedge_{t \in I_0} T_k^*(H_k(t-)) \land \bigwedge_{t \in I_1} T_k(H_k(t+)) \). Then
\[
\alpha \leq \bigwedge_{k \in \Omega} \left( \bigwedge_{t \in I_0} T_k^*(H_k(t-)) \land \bigwedge_{t \in I_1} T_k(H_k(t+)) \right).
\]

We define a map \( H : (0, 1) \rightarrow \mathbb{L}^X \) as follows:
\[
H(t) = \bigvee_{k \in \Omega} P_k^-(H_k(t)), \quad \forall t \in (0, 1).
\]

Now we prove \( H \in H(e,A) \). It is obvious that \( H \) is an antitone map. From \( P_{k^*}^-(e) \nless \bigvee_{t > 0} H_k(t) \) \((\forall k \in \Omega)\), we know that \( e \nless \bigvee_{t > 0} H(t) \). In fact, if \( e \leq \bigvee_{t > 0} H(t) = \bigvee_{t > 0, k \in \Omega} P_k^-(H_k(t)) \), then there exists \( k_0 \in \Omega \) such that \( e \leq \bigvee_{t > 0} P_{k_0}^-(H_{k_0}(t)) \).

Therefore, \( P_{k_0}^-(e) \nless \bigvee_{t > 0} H_{k_0}(t) \), which is a contradiction.

From \( A_k \leq \bigwedge_{s < 1} H_k(s) \), we know that
\[
\bigwedge_{s < 1} H(s) = \bigwedge_{s < 1, k \in \Omega} P_k^-(H_k(s)) = \bigwedge_{k \in \Omega} \left( \bigwedge_{s < 1} P_k^-(H_k(s)) \right) \geq \bigvee_{k \in \Omega} P_{k^*}^- (A_k) \geq A.
\]

Therefore, \( H \in H(e,A) \).
For any \( t \in I_0 \), we have
\[
T^* (H (t^-)) = T^* \left( \bigwedge_{s < t} H (s) \right) = T^* \left( \bigwedge_{s < t} P^{-}_{t_k} (H_k (s)) \right)
\geq \bigwedge_{k \in \Omega} T^* \left( \bigwedge_{s < t} P^{-}_{t_k} (H_k (s)) \right) = \bigwedge_{k \in \Omega} T^* (H_k (t^-))
\]

For any \( t \in I_1 \), we have
\[
T (H (t^+)) = T \left( \bigvee_{s > t} H (s) \right) = T \left( \bigvee_{s > t} P^{+}_{t_k} (H_k (s)) \right)
\geq \bigwedge_{k \in \Omega} T \left( \bigvee_{s > t} P^{+}_{t_k} (H_k (s)) \right) = \bigwedge_{k \in \Omega} T_k (H_k (t^+))
\]

Therefore,
\[
\alpha \leq \bigwedge_{k \in \Omega} \left( \bigwedge_{t \in I_0} T^* (H_k (t^-)) \wedge \bigwedge_{t \in I_1} T_k (H_k (t^+)) \right)
\leq \bigwedge_{t \in I_0} T^* (H (t^-)) \wedge \bigwedge_{t \in I_1} T (H (t^+))
\leq \bigvee_{H \in H(e, A)} \left( \bigwedge_{t \in I_0} T^* (H (t^-)) \wedge \bigwedge_{t \in I_1} T (H (t^+)) \right).
\]

Since \( \alpha \) is arbitrary, we know that
\[
\mu \leq \bigvee_{H \in H(e, A)} \left( \bigwedge_{t \in I_0} T^* (H (t^-)) \wedge \bigwedge_{t \in I_1} T (H (t^+)) \right).
\]

Since \( \mu \) is arbitrary, it follows that
\[
\gamma \wedge T^* (A) \leq \bigvee_{H \in H(e, A)} \left( \bigwedge_{t \in I_0} T^* (H (t^-)) \wedge \bigwedge_{t \in I_1} T (H (t^+)) \right).
\]

Therefore, \( \gamma \leq Creg (X, T) \). Since \( \gamma \) is arbitrary, we have \( \bigwedge_{j \in J} Creg (X_j, T_j) \leq Creg (X, T) \).

In addition, assume that \( (X_j, T_j) \) is stratified for all \( j \in J \), then \( (X_j, T_j) \) is homeomorphic to a subspace \( \tilde{X}_j, T |_{\tilde{X}_j} \) of \( (X, T) \), where \( \tilde{X}_j \) is a subset of \( X \) parallel to \( X_j \) through \( x = (x_j)_{j \in J} \). From Theorems 5.3 and 5.4, we have
\[
\bigwedge_{j \in J} Creg (X_j, T_j) = \bigwedge_{j \in J} Creg \left( \tilde{X}_j, T |_{\tilde{X}_j} \right) \geq Creg (X, T).
\]

\( \square \)
Theorem 5.6. Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space. Then \(T_{3\frac{1}{2}}(X, \mathcal{T}) \leq H(X, \mathcal{T})\).

Proof. By Theorems 4.2 and Theorem 5.2, we need to prove
\[
\bigwedge_{e \in M(L^X)} \bigwedge_{e \notin B \subseteq L^X} \left\{ \mathcal{T}^*(B) \rightarrow \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(H(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(H(t+)) \right) \right\} 
\]
\[
\land \bigvee_{a \notin b \in P \geq b} \mathcal{T}(P')
\]
\[
\leq \bigwedge_{a \notin b \in P \geq b} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right).
\]

Let \(\lambda \prec \bigwedge_{e \in M(L^X)} \bigwedge_{e \notin B \subseteq L^X} \left\{ \mathcal{T}^*(B) \rightarrow \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(H(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(H(t+)) \right) \right\} \land \bigvee_{a \notin b \in P \geq b} \mathcal{T}(P')\). Then for any \(e \in M(L^X), B \subseteq L^X\) with \(e \notin B\) we have
\[
\lambda \land \mathcal{T}^*(B) \leq \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(H(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(H(t+)) \right).
\]

For any \(a, b \in M(L^X)\) with \(a \notin b\), there exists \(P \in L^X\) such that \(a \notin P \geq b\) and \(\lambda \prec \mathcal{T}^*(P)\). From \(\mathcal{H}(a, P) \subseteq \mathcal{G}(a, b)\), we know that
\[
\lambda = \lambda \land \mathcal{T}^*(P)
\]
\[
\leq \bigvee_{H \in \mathcal{H}(a, P)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(H(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(H(t+)) \right)
\]
\[
\leq \bigvee_{G \in \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right).
\]

Hence
\[
\lambda \leq \bigwedge_{a \notin b \in G \subseteq \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right).
\]

Since \(\lambda\) is arbitrary, we know that
\[
\bigwedge_{e \in M(L^X)} \bigwedge_{e \notin B \subseteq L^X} \left\{ \mathcal{T}^*(B) \rightarrow \bigvee_{H \in \mathcal{H}(e, B)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(H(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(H(t+)) \right) \right\} \land \bigvee_{a \notin b \in P \geq b} \mathcal{T}(P')
\]
\[
\leq \bigwedge_{a \notin b \in G \subseteq \mathcal{G}(a, b)} \left( \bigwedge_{t \in I_0} \mathcal{T}^*(G(t-)) \land \bigwedge_{t \in I_1} \mathcal{T}(G(t+)) \right).
\]

Therefore, \(T_{3\frac{1}{2}}(X, \mathcal{T}) \leq H(X, \mathcal{T})\). \(\square\)
By Theorems 3.7, 4.6 and 5.6, we can obtain the following theorem.

**Theorem 5.7.** Let \((X, T)\) be an \(L\)-fuzzy topological space. Then
\[
T_{3\frac{1}{2}}(X, T) \leq H(X, T) \leq U(X, T) \leq T_2(X, T).
\]

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**References**


Chengyu Liang*, College of Science, North China University of Technology, No.5 Jinuaizhuang Road, Shijingshan District, 100144 Beijing, P.R. China
E-mail address: liangchengyu87@163.com

Fu-Gui Shi, School of Mathematics and Statistics, Beijing Institute of Technology, 5 South Zhongguancun Street, Haidian District, 100081 Beijing, P.R. China
E-mail address: fuguishi@bit.edu.cn

* Corresponding author