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Original paper

Characterizing idempotent uninorms on a bounded chain

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Abstract

Completeness is essential for characterizing idempotent uninorms on complete chains, as it guarantees the well-definedness of their corresponding characterization functions. In the case of a general chain, one cannot define the aforementioned characterization functions by taking the supremum or infimum of a prescribed subset. When constructing the real numbers via the Dedekind completion of the rationals, each rational number is associated with a rational cut, which forms a down-set. Inspired by this line of reasoning, this paper provides a direct characterization of idempotent uninorms defined on bounded chains via decreasing symmetric set-valued functions that map the chain to its family of down-sets.

Keywords: Aggregation operation, idempotent uninorm, bounded chain, Dedekind completion.

1 Introduction

The concept of uninorms was initially introduced in the setting of the unit interval [18] (compare also Golan's work [6] concerning special semirings). Readers are directed to [17] for the theory of the prominent families of uninorms defined on the unit interval, and to [10] for their practical applications. To streamline the exposition, we term uninorms defined on the unit interval real uninorms. Extending the scope of aggregation theory from the unit interval to bounded lattices (commonly called aggregation on lattices) has emerged as a major new branch in this area [11]. This paper focuses on idempotent uninorms. Subsequently, we review the extension of uninorms from the unit interval to bounded chains.

To the best of our knowledge, the pioneering work on real functions related to uninorms in fuzzy set theory was done by Czogała and Drewniak:

Theorem 1.1. [2, Theorem 3] *For each associative increasing idempotent function $F : [0,1]^2 \to [0,1]$ with neutral element $e \in [0,1]$, there exists a decreasing function $g : [0,1] \to [0,1]$ with a fixed point e such that*

$$F(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x), \\ \max(x,y) & \text{if } y > g(x), \\ x \text{ or } y & \text{if } y = g(x). \end{cases}$$

The first class of such operations to receive a full characterization is the class of left- or right-continuous idempotent real uninorms:

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Theorem 1.2. [4, Theorem 8] *A bivariate function* $U : [0,1]^2 \to [0,1]$ *is a left-continuous idempotent uninorm with neutral element* $e \in]0,1]$ *if and only if there exists a decreasing unary function* $g : [0,1] \to [0,1]$ *with a fixed point* e *satisfying* $g(g(x)) \geq x$ *for any* $x \leq g(0)$ *and* $g(x) = 0$ *for any* $x > g(0)$ *such that*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y \leq g(x) \text{ and } x \leq g(0), \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Theorem 1.3. [4, Theorem 9] *A bivariate function* $U : [0,1]^2 \to [0,1]$ *is a right-continuous idempotent uninorm with neutral element* $e \in [0,1[$ *if and only if there exists a decreasing unary function* $g : [0,1] \to [0,1]$ *with a fixed point* e *satisfying* $g(g(x)) \leq x$ *for any* $x \geq g(1)$ *and* $g(x) = 1$ *for any* $x < g(1)$ *such that*

$$U(x,y) = \begin{cases} \max(x,y) & \text{if } y \geq g(x) \text{ and } x \geq g(1), \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Another interesting class of idempotent uninorms was presented by Grabisch et al. [7, Example 3.98]. We recall its dual version.

Theorem 1.4. *Let* $g : [0,1] \to [0,1]$ *be a decreasing bijection and define* $U_{(g)} : [0,1]^2 \to [0,1]$ *by*

$$U_{(g)}(x,y) = \begin{cases} \max(x,y) & \text{if } g(x) + g(y) \geq 1, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Then $U_{(g)}$ *is a right-continuous disjunctive uninorm with neutral element* $g^{-1}(0.5)$.

Martín et al. [9] presented a complete characterization of locally internal, associative, and increasing functions on $[0,1]$; an improvement of the latter result restricted to idempotent real uninorms was reported in [15]:

Theorem 1.5. [15, Theorem 4] *A bivariate function* $U : [0,1]^2 \to [0,1]$ *is an idempotent uninorm with neutral element* $e \in]0,1[$ *if and only if there exists a decreasing, Id-symmetrical function* $g : [0,1] \to [0,1]$ *with a fixed point* e *such that*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g(g(x))), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g(g(x))), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x))). \end{cases}$$

being commutative on the set of points $(x, g(x))$ *such that* $x = g(g(x))$.

All the aforementioned results are carried out via decreasing unary functions, which are referred to as characterizing functions. The characterizations of idempotent real uninorms were extended to their counterparts on finite chains [5] and complete chains [13]:

Theorem 1.6. [5, Theorem 3] *A binary operation* U *on a finite chain* $L_n = \{0,1,2,\cdots,n\}$ *with neutral element* $0 < e < n$ *is an idempotent discrete uninorm if and only if there exists a decreasing function* $g : [0,e] \to [e,n]$ *with* $g(e) = e$ *such that*

$$U(x,y) = \begin{cases} x \wedge y & \text{if } y \leq \overline{g}(x) \text{ and } x \leq \overline{g}(0), \\ x \vee y & \text{otherwise,} \end{cases}$$

where $\overline{g}$ *is the unique symmetrical extension of* g *defined by*

$$\overline{g}(x) = \begin{cases} g(x) & \text{if } x \leq e, \\ \max\{z \in [0,e] \mid g(z) \geq x\} & \text{if } e \leq x \leq g(0), \\ 0 & \text{if } x > g(0). \end{cases}$$

Theorem 1.7. [13, Theorem 3.21] *Let* L *be a complete chain and* $e \in L \backslash \{0,1\}$. *Then* $U : L^2 \to L$ *is an idempotent uninorm on* L *if and only if there exists a decreasing function* $g : L \to L$ *with fixed point* e *and property*

$$(\forall (x,y) \in L^2)(y < f(x) \Rightarrow x \leq f(y)), \tag{S}$$

such that

$$U(x,y) = \begin{cases} x \wedge y & \text{if } (x,y) \in A_g, \\ x \vee y & \text{if } (x,y) \in B_g, \\ x \text{ or } y & \text{if } (x,y) \in C_g \end{cases} \tag{1}$$

and U is commutative on C_g, where

$$A_g = \{(x,y) \in L^2 \mid y < g(x) \text{ or } x < g(y)\},$$
$$B_g = \{(x,y) \in L^2 \mid y > g(x) \text{ or } x > g(y)\},$$
$$C_g = \{(x,y) \in L^2 \mid y = g(x) \text{ and } x = g(y)\}.$$

What unites the above studies is a common idea that, given any x in the complete chain L under consideration, there exists a dividing point $g(x)$ (which is the supremum or infimum of a specific set) with the property that $U(x,y) = x \wedge y$ whenever $y < g(x)$, and $U(x,y) = x \vee y$ whenever $y > g(x)$. Typically, the value of U at $(x, g(x))$ remains undetermined. This idea fails to carry over to general chains. For the case of general chains, Su et al. [16] proved that every idempotent uninorm on a bounded chain can be extended to an idempotent uninorm on a complete chain, such that the restriction of the extended operation to the original chain is the original uninorm:

Theorem 1.8. [16, Theorem 2] *Let $(L,\leq)$ be a bounded chain, $e \in L\backslash\{0,1\}$, U be a binary operation on L. Suppose that $\widetilde{L}$ is order-isomorphic to the Dedekind-MacNeille completion of L with $L \subseteq \widetilde{L}$. Then U is an idempotent uninorm with neutral element e if and only if there exists an idempotent uninorm $\widetilde{U}$ with neutral element e on $\widetilde{L}$ such that $\widetilde{U}|_{L^2} = U$.*

This corresponding construction relies on the rather involved Dedekind-MacNeille completion. Note that in the Dedekind completion of the rational numbers $\mathbb{Q}$ to form the real numbers $\mathbb{R}$, we define the set of Dedekind cuts on $\mathbb{Q}$, and every rational number corresponds to a rational cut, which is a down-set. For a detailed discussion of the Dedekind completion of $\mathbb{Q}$, see [14, Appendix to Chapter 1]. For a general chain, it is impossible to define the aforesaid dividing point $g(x)$ using the supremum or infimum of a given set. In the spirit of the Dedekind completion procedure, we substitute "a dividing point $g(x)$" with "a down-set $f(x)$ of the chain in question", accordingly, the condition "$U(x,y) = x \wedge y$ whenever $y < g(x)$" is replaced by "$U(x,y) = x \wedge y$ for all $y \in f(x)$", and "$U(x,y) = x \vee y$ whenever $y > g(x)$" is replaced by "$U(x,y) = x \vee y$ for all $y \notin f(x)$". This approach enables us to directly characterize idempotent uninorms on a bounded (not necessarily complete) chain by means of decreasing symmetric set-valued functions mapping this chain to its family of down-sets.

2 Preliminaries

In this section, we recall some basic notions and results concerning posets [3] and idempotent uninorms on a bounded chain [16].

A *poset* $(P,\leq)$ refers to a nonempty set P equipped with a reflexive, antisymmetric and transitive binary relation $\leq$. A poset $(P,\leq)$ is called *bounded* if it has the greatest element 1_P and the least element 0_P. By a *down-set* of a poset $(P,\leq)$ we shall mean a subset D of P with the property that if $x \in D$ and $y \in P$ is such that $y \leq x$, then $y \in D$; thus, $\emptyset$ is a down-set. The family of all down-sets of P is denoted by $\mathcal{O}(P)$. For any $x \in P$, the subset $\downarrow x = \{y \in P \mid y \leq x\}$ is called the principal filter at x w.r.t. $(P,\leq)$. A poset $(P,\leq)$ is called a *chain* if a and b are comparable for all $a, b \in P$, i.e., either $a \leq b$ or $b \leq a$ for all $a, b \in P$. In a chain, the join and meet operations are denoted by $\vee$ and $\wedge$, respectively. A *Cuesta-Dutari cut* in a chain C is an ordered pair $(L,R) \subseteq C \times C$ such that $L \cup R = C$ and $x < y$ for all $x \in L$ and $y \in R$. Note: $(\emptyset, C)$ and $(C, \emptyset)$ are Cuesta-Dutari cuts in C; thus C always has a Cuesta-Dutari cut. For any Cuesta-Dutari cut (L,R) in a chain C, L forms a down-set and uniquely determines the corresponding Cuesta-Dutari cut. A Cuesta-Dutari cut (L,R) in C will be called a *Dedekind cut* if L and R are non-empty sets.

Let $(P,\leq)$ and $(Q,\preceq)$ be posets and $f : P \to Q$ be a function. We say that f is *order-reversing* or *decreasing* if for all $x, y \in P$, $x \leq y$ implies $f(x) \succeq f(y)$.

Let $(C,\leq)$ be a bounded chain. An operation $U : C^2 \to C$ is called a *uninorm* [8] if it is commutative, associative, increasing with respect to both variables and has a neutral element $e \in C$. A uninorm U on C is *idempotent* if $U(x,x) = x$ for all $x \in C$. We write $\mathbf{Ide}(C,e)$ for all idempotent uninorms on a chain C having a neutral element $e \in C \setminus \{0_C, 1_C\}$. For each $U \in \mathbf{Ide}(C,e)$, it holds that $U(x,y) = x \wedge y$ whenever $x, y \leq e$ and $U(x,y) = x \vee y$ whenever $x, y \geq e$. As established in [13], any increasing and associative binary operation U admitting a neutral element on a chain C satisfies $U(x,y) \in \{x,y\}$ for all $x, y \in C$, i.e., it is *internal*.

3 Main result

Let C be a complete chain and $U \in \mathbf{Ide}(C,e)$. In [13], the set A_x is defined for any $x \in C$:

$$A_x = \{y \in C \mid U(x,z) = \min(x,z) \text{ for all } z \in \downarrow y\}.$$

The set A_x can be equivalently written as

$$A_x = \begin{cases} \{y \in C \mid U(x,y) = \min(x,y)\} & \text{if } x \le e, \\ \{y \in [0,e[\mid U(x,y) = \min(x,y) = y\} & \text{if } x > e. \end{cases}$$

As a matter of fact, A_x possesses a more concise representation, that is,

$$A_x = \{y \in C \mid U(x,y) \le e\}. \tag{2}$$

Clearly, the pair $(A_x, C \setminus A_x)$ is a Cuesta-Dutari cut in C for all $x \in C$, so A_x is a down-set for all $x \in C$. A_x may be an empty set (for details, see Example 3.6(ii) below); thus $(A_x, C \setminus A_x)$ is not a Dedekind cut in C. Let us define a set-valued function $f : C \to \mathcal{O}(C)$ by

$$f(x) = \{y \in C \mid U(x,y) \le e\}. \tag{3}$$

In what follows, we introduce a fundamental fact and a related notion to characterize the above set-valued function.

Note that for any chain $(C, \le)$, $(\mathcal{O}(C), \subseteq)$ forms a chain (In fact, for all $A, B \in \mathcal{O}(C)$ with $A \ne B$, we may assume without loss of generality that there exists $a \in A \setminus B$. Since $b \in B$ and $a \le b$ would imply $a \in B$, we conclude $b < a$ for every $b \in B$, so $B \subseteq A$.) A set-valued function $f : C \to \mathcal{O}(C)$ is called *symmetric* [12, Definition 1] if $x \in f(y)$ if and only if $y \in f(x)$ for all $x, y \in C$.

In [13], the properties of A_x are investigated in the setting of complete chains C. One can readily verify that all these properties still hold for arbitrary chains. We now enumerate several of these properties for subsequent use. Note that these properties are restated in terms of the function f, rather than being formulated in set-theoretic language.

Lemma 3.1. *Let $U \in \mathbf{Ide}(C, e)$.*

(i) *If f is defined by* (3)*, then f is a decreasing and symmetric set-valued function with $f(e) = [0, e]$.*

(ii) *Then there exists some decreasing and symmetric function $f : C \to \mathcal{O}(C)$ with $f(e) = [0, e]$ such that*

$$U(x,y) = \begin{cases} x \wedge y & \text{if } y \in f(x), \\ x \vee y & \text{if } y \notin f(x). \end{cases} \tag{4}$$

Ouyang et al. [13] analyzed the properties of A_x and proved that any $U \in \mathbf{Ide}(C, e)$ can be written as in (4). However, they overlooked the converse statement! The mapping $U : C^2 \to C$ defined by (4) yields a uninorm.

Lemma 3.2. [1] *Let X be a set and let $U : X^2 \to X$ be an internal operation, i.e., $U(x,y) \in \{x, y\}$ for all $x, y \in X$. Then, the following statements are equivalent:*

(i) *The operation U is not associative.*

(ii) *There exist pairwise distinct $x, y, z \in X$ such that $U(x,y)$, $U(x,z)$ and $U(y,z)$ are pairwise distinct.*

Proposition 3.3. *Let C be a bounded chain, $e \in C \setminus \{0_C, 1_C\}$, $f : C \to \mathcal{O}(C)$ be a decreasing and symmetric function with $f(e) = [0, e]$ and $U : C^2 \to C$ be defined by* (4)*. Then $U \in \mathbf{Ide}(C, e)$.*

Proof. **Commutativity**: Since $U(x,x) = x$ for all $x \in C$, we only need to consider the case $x \ne y$. If $U(x,y) = x \wedge y$, then $y \in f(x)$ and thus $x \in f(y)$, so $U(y,x) = x \wedge y$; if $U(x,y) = x \vee y$, then $y \notin f(x)$ and thus $x \notin f(y)$, so $U(y,x) = x \vee y$.

Monotonicity: Assume that $x, y, z \in C$ with $y < z$. If $z \notin f(x)$, then $U(x,z) = x \vee z \ge x \vee y \ge U(x,y)$. If $z \in f(x)$ then $y \in f(x)$ as $f(x) \in \mathcal{O}(C)$, so $U(x,y) = x \wedge y \le x \wedge z = U(x,z)$.

Associativity: Assume $*$ is not associative. From Lemma 3.2, it follows that there exist pairwise distinct $x, y, z \in C$ such that $U(x,y)$, $U(x,z)$ and $U(y,z)$ are pairwise distinct. Suppose, without loss of generality, that $x < y < z$. If $z \in f(x)$, then $y \in f(x)$ as $f(x) \in \mathcal{O}(C)$, which gives $U(x,y) = x = U(x,z)$, a contradiction. As a consequence, $z \notin f(x)$. It then follows that $U(x,z) = z$ and $U(y,z) \ge U(x,z) = z$, implying $U(y,z) = U(x,z)$, another contradiction. Hence, U is associative.

Neutral element: This follows immediately from $f(e) = [0, e]$. □

From Lemma 3.1 and Proposition 3.3, we deduce the following result.

Theorem 3.4. *Let C be a bounded chain, $e \in C \backslash \{0_C, 1_C\}$ and $U : C^2 \to C$ be an operation. Then $U \in \mathbf{Ide}(C, e)$ if and only if there exists some decreasing and symmetric function with $f : C \to \mathcal{O}(C)$ with $f(e) = [0, e]$ such that U can be represented as* (4)*.*

We may interpret Theorem 9 in the following manner. Working within complete chains, for any element x in the complete chain C under discussion, we introduce the characterizing function $g(x) = \sup f(x)$. This function $g(x)$ obeys the subsequent properties: $U(x, y) = x \wedge y$ whenever $y < g(x)$, and $U(x, y) = x \vee y$ whenever $y > g(x)$. For a general chain, such a characterizing function $g(x)$ fails to be well-defined. Following the construction of the Dedekind completion for real numbers, we adopt a down-set $f(x)$ (or equivalently, a Cuesta-Dutari cut $(f(x), C \setminus f(x))$ to replace the ill-defined dividing point $g(x)$). In this way, idempotent uninorms on a bounded chain admit a direct characterization via decreasing symmetric set-valued functions mapping this chain to its family of all its down-sets.

Remark 3.5. (i) *By the virtue of the proof of Lemma 3.1 and Proposition 3.3, we can see that Theorem 3.4 remains valid for an arbitrary chain that need not be bounded.*

(ii) *As opposed to Theorems 4 and 6, U takes definite values at all points within its domain.*

(iii) *Let C be a complete chain, $e \in C \backslash \{0_C, 1_C\}$, $U \in \mathbf{Ide}(C, e)$ and f be define by* (3)*. Define $g : C \to C$ by $g(x) = \sup f(x)$. Suppose that $y < g(x)$ for some $x, y \in C$. Then $y \in f(x)$. The symmetry of f implies that $x \in f(y)$, so $x \leq \sup f(y) = g(y)$. Therefore, g satisfies property **(S)**. Consider any fixed element $x \in C$. Given that $f(x)$ is a down-set, we have $U(x, y) = x \wedge y$ for all $y < g(x)$, $U(x, y) = x \vee y$ for all $y > g(x)$ and $U(x, y) \in \{x, y\}$ for all $y = g(x)$. Based on the foregoing arguments, we conclude that U can be represented as* (1)*.*

We illustrate the above result with several examples given in [16, Example 1].

Example 3.6. (i) *Let $C_1 = \mathbb{Q} \cap [0, 1]$, i.e., C_1 is the set of all rational numbers from $[0, 1]$, and $e_1 = 0.5$. Consider the set-valued function $f : C_1 \to \mathcal{O}(C_1)$ defined by $f_1(x) = [0, 1 - x]$. Applying Theorem 3.4, we derive the following uninorm U_1 on C_1, which also appears in* [16, Example 1 (iii)]:

$$U_1(x, y) = \begin{cases} \min(x, y) & \text{if } y \in f_1(x), \\ \max(x, y) & \text{otherwise.} \end{cases}$$

However, the set-valued function $f_2 : C_1 \to \mathcal{O}(C_1)$ defined by $f_2(x) = [0, 1 - x[$ induces the following uninorm U_2 on C_1:

$$U_2(x, y) = \begin{cases} \min(x, y) & \text{if } y \in f_2(x), \\ \max(x, y) & \text{otherwise..} \end{cases}$$

(ii) *Consider the set $\mathbb{N}$ of all positive integers with the standard order $\leq$ and the uninorm U_3 on $\mathbb{N}$ with neutral element 2 defined by*

$$U_3(x, y) = \begin{cases} \min(x, y) & \text{if } y \in f_3(x), \\ \max(x, y) & \text{otherwise.} \end{cases}$$

This uninorm can be found in [16, Example 1 (iv)]*. The corresponding set-valued function $f_3 : \mathbb{N} \to \mathcal{O}(\mathbb{N})$ is given by*

$$f_3(x) = \begin{cases} \{1, 2\} & \text{if } x \leq 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

4 Conclusions

It was proven by Su et al. [16] that any idempotent uninorm on a bounded chain admits an extension to an idempotent uninorm on a complete chain, and the restriction of this extension to the original chain equals the original operation. To carry out this construction, we need to employ the highly elaborate Dedekind-MacNeille completion. Drawing on the Dedekind completion of rational numbers, this paper has presented a direct characterization of idempotent uninorms on a bounded (not necessarily complete) chain by means of decreasing symmetric set-valued functions mapping this chain to the collection of all its down-sets.

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