Iranian Journal of Fuzzy Systems

Volume 23, Number 4, (2026), pp. 171-182

Original paper

Zadeh's extension and dynamical properties with respect to d_p metrics

K. Aguirre, [1] and D. Jardón, [2]

[1,2] *Academia de Matemáticas, Universidad Autónoma de la Ciudad de México, Calz. Ermita Iztapalapa S/N, Col. Lomas de Zaragoza 09620, Mexico City, Mexico.*

kinrha@gmail.com, daniel.jardon@uacm.edu.mx

Abstract

For a given metric space (X, d), by $\mathcal{F}(X)$ we denote the family of all normal fuzzy sets of X, which are upper semicontinuous and have compact support. In this paper, firstly, we prove that, if X is compact and $f : X \to X$ is a continuous function, then its Zadeh's extension $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is continuous with respect to the d_p metric. Then we analyze some properties of Zadeh's extension with respect to d_p metrics including positive Lyapunov stability, strong sensitivity and transitivity.

Keywords: Fuzzy sets, d_p metrics, Zadeh's extension, (strong) sensitivity, positive Lyapunov stability.

1 Introduction

In 1965, Lofti Zadeh defined fuzzy sets in [15] as a class of objects with different degrees of membership. He also stated in the 70's a way to induce, from a given $f : X \to X$, a function whose domain consists of the fuzzy subsets of X: its Zadeh's extension.

In the last decades, many researchers have studied $\mathcal{F}(X)$ endowed with different metrics, among others d_∞ and Skorokhod metric (d_0), see for instance [7, 8, 12]. In 1990, Diamond and Kloeden [3] defined the metric d_p in the hyperspace of all the fuzzy sets in $\mathbb{R}^n$, for $p \in [1, \infty)$. In [1] the authors have studied separability and completeness of $\mathcal{F}(X)$ regarding d_p metrics. The d_p metrics are defined by the Lebesgue integral, for more information related to Lebesgue measure theory, see [9].

A discrete dynamical system is a pair (X, f), where X is a metric space and $f : X \to X$ is a continuous function. The collective dynamics of f can be analyzed through the induced dynamical system $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ where $\mathcal{K}(X)$ is the hyperspace of all non-empty subsets of X endowed with the Hausdorff metric, see [2, 13] for instance. Another way to study collective dynamics is through Zadeh's extension with respect to different metrics. The authors of [6, 7, 8, 11, 12] have dedicated their efforts to study Zadeh's extension with respect to d_∞, d_0 and sendograph metrics.

The aim of this work is to analyze the behavior of some dynamical properties of ,Zadeh's extension regarding the d_p metric, in the hyperspace of fuzzy sets. The d_p metric exhibits a behavior that is difficult to characterize through its a α-levels, which is why proofs of its dynamic properties have relied on techniques that are different from those used in the aforementioned metrics; moreover, some results appear in contrast with the findings reported for d_S, d_0, and d_∞. The first mandatory step is to analyze whether Zadeh's extension is continuous with respect to the d_p metrics, where we find the first discrepancy with the d_S, d_∞ and d_p metrics.

This paper is organized as follows. In Section 2, we give some definitions and some known facts about them. Section 3, is devoted to analyzing continuity of Zadeh's extension. The main result of this section is Theorem 3.5. From this, it follows the continuity the Zadeh's extension is continuous with respect to d_p metric, whenever the underlying space is compact (see Corollary 3.6). Example 3.2 shows that compactness is essential. In Section 4 we deal with stability

Corresponding Author: K. Aguirre

Received: September 2025; Revised: January 2026; Accepted: July 2026.

https://doi.org/10.22111/ijfs.2026.53211.9413

properties of Zadeh's extension. Finally, in section 5 we analyze dynamical properties. In Proposition 5.6 it is proved that strong sensitivity of a dynamical system with a compact domain is inherited by its Zadeh's extension in hereditary.

2 Preliminaries

In this paper (X, d) denotes the metric space, where X is a set endowed with the metric d, and $\mathcal{B}_d(x, \delta) = \{y \in X : d(x, y) < \delta\}$ denotes the open ball with center at $x \in X$ and radius $\delta > 0$. The intervals $[0, \infty)$ and $[0, 1]$ are denoted by $\mathbb{R}^+$ and $\mathbb{I}$, respectively.

A fuzzy set u on the metric space X is a function $u : X \to \mathbb{I}$. For any given fuzzy set u the following sets can be distinguished $u_\alpha = \{x \in X : u(x) \geq \alpha\}$, for $\alpha \in (0, 1]$ and $u_0 = \overline{\{x \in X : u(x) > 0\}} = \overline{\cup\{u_\alpha : \alpha \in (0, 1)\}}$, they are known as the α-levels and the support of u, respectively. As we have indicated above as usual, the hyperspace $\mathcal{F}(X)$ is defined as the family of all upper semicontinuous fuzzy sets u with compact support such that $u_1 \neq \emptyset$. Note that for a given $u \in \mathcal{F}(X)$ the α-level u_α is compact, for each $\alpha \in (0, 1]$. And if $A \subset X$, is a non-empty compact set, then the characteristic function $\chi_A : X \to \mathbb{I}$ belongs to $\mathcal{F}(X)$. If $A = \{x\}$, then we write χ_x instead of $\chi_{\{x\}}$.

Now, let us keep in mind that for a given metric space (X, d), the Hausdorff metric is defined on $\mathcal{K}(X)$, the hyperspace of the non-empty compact subsets of X, by $d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$. In turn, $\rho(A, B) = \max\{d(a, B) : b \in B\} = \max_{a \in A}\{\min_{b \in B} d(a, b)\}$.

The following result is well known; see [8] for instance.

Proposition 2.1. *Let (X, d) be a metric space. If $A, B, C, D, F \in \mathcal{K}(X)$, then it is true that:*

i) *If $A \subseteq B \subseteq C$, then $d_H(A, B) \leq d_H(A, C)$ and $d_H(B, C) \leq d_H(A, C)$.*

ii) *If $d_H(A, D) \leq \epsilon$ and $d_H(B, F) \leq \epsilon$, then $d_H(A \cup B, D \cup F) \leq \epsilon$.*

It is a well-known fact that if u is a fuzzy set, then u can be defined by its α-levels. The following two propositions express this condition.

Proposition 2.2. [8, Proposition 4.9] *Let X be a Hausdorff space and $u \in \mathcal{F}(X)$. If $L \colon \mathbb{I} \to (\mathcal{K}(X), d_H)$ is the function defined by $L(\alpha) = u_\alpha$ for all $\alpha \in \mathbb{I}$, then the following holds:*

i) *L is left continuous on $(0, 1]$;*

ii) $\lim\limits_{\lambda \to \alpha^+} L(\lambda) = \overline{\bigcup_{\beta > \alpha} u_\beta}$ *and* $\lim\limits_{\lambda \to \alpha^+} L(\lambda) \subseteq u_\alpha$ *for each $\alpha \in [0, 1)$;*

iii) *L is right-continuous at 0.*

Conversely, for any decreasing family $\{u_\alpha : \alpha \in \mathbb{I}\} \subseteq \mathcal{K}(X)$ satisfying i) and iii), there exists a unique $w \in \mathcal{F}(X)$ such that $w_\alpha = u_\alpha$ for every $\alpha \in \mathbb{I}$.

Proposition 2.3. *Let (X, d) be a metric space and $u, v \in \mathcal{F}(X)$. Then $u_\alpha = v_\alpha$ for all $\alpha \in \mathbb{I}$ if and only if $u = v$.*

In recent decades $\mathcal{F}(X)$ has been provided with various metrics constructed from some metric d defined on X. The d_∞ is defined by $d_\infty(u, v) = \sup\{d_H(u_\alpha, v_\alpha) : \alpha \in \mathbb{I}\}$ for $u, v \in \mathcal{F}(X)$. It is easy to see that $d_H(u_\alpha, v_\alpha) \leq d_\infty(u, v)$ for each $\alpha \in \mathbb{I}$. The Skorokhod metric d_0 is defined on $\mathcal{F}(X)$ as:

$$d_0(u, v) = \inf\{\epsilon : \exists t \in T \text{ such that } \sup_{\alpha \in \mathbb{I}} : |t(\alpha) - \alpha| \leq \epsilon \text{ and } d_\infty(u, tv) \leq \epsilon\},$$

where T is the family of all increasing homeomorphisms from $\mathbb{I}$ onto itself and $tv = t(v)$. From the definition of d_0 it immediately follows that $d_0(u, w) \leq d_\infty(u, w)$ and $d_H(u_\alpha, w_\alpha) \leq d_0(u_\alpha, w_\alpha)$ for $\alpha = 0, 1$.

Another common way to provide $\mathcal{F}(X)$ with a metric is to use the sendographic metric. Consider a metric space (X, d). Take $u \in \mathcal{F}(X)$. The sendograph of u is defined as follows:

$$send(u) = \{(x, \alpha) \in u_0 \times I : u(x) \geq \alpha\}.$$

The sendograph metric d_S on $\mathcal{F}(X)$ is defined as the Hausdorff metric (on $\mathcal{K}(X \times I)$). If $u, v \in \mathcal{F}(X)$, then $d_s(u, v) = d_H(send(u), send(v))$. It is known that $d_S \leq d_0 \leq d_\infty$ [5, Proposition 2.7].

And so, we finally proceed to define the metric that is the main subject of this paper. For a given real number $p \geq 1$, the metric $d_p : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}^+$ is defined by the Lebesgue integral:

$$d_p(u,v) = \left[\int_0^1 d_H(u_\alpha, v_\alpha)^p d\alpha\right]^{\frac{1}{p}}, \text{ for } u,v \in \mathcal{F}(X),$$

from this, it follows that $d_p(u,v) \leq d_\infty(u,v)$. A proof that d_p is a metric can be found in [1]. Henceforth, the measure of a set A, denoted by $m(A)$, means the Lebesgue measure.

3 Fuzzy sets and Zadeh's extension

In this section X denotes an arbitrary metric space. If $f : X \to X$ is a continuous function, its Zadeh's extension $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is defined as follows:

$$\widehat{f}(u)(y) = \begin{cases} \sup\{u(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset. \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

The construction of Zadeh's extension guarantees the well-behaved nature of the α-levels of fuzzy sets in $\mathcal{F}(X)$.

Proposition 3.1. [8] *If $f : X \to X$ is a continuous function, then $[\widehat{f}(u)]_\alpha = f(u_\alpha)$ for each $u \in \mathcal{F}(X)$ and $\alpha \in \mathbb{I}$.*

Unlike the sendographic metric d_S, the Skorokhod metric d_0 and d_∞, for which we have that $f : (X,d) \to (X,d)$ is a continuous function if and only if its Zadeh's extension $\widehat{f} : (\mathcal{F}(X), d_\rho) \to (\mathcal{F}(X), d_\rho)$ is continuous, where $\rho = S, 0, \infty$ (see [8] and [10]). The continuity of f is not sufficient to ensure the continuity of $\widehat{f} : (\mathcal{F}(X), d_p) \to (\mathcal{F}(X), d_p)$, as the following example shows.

Example 3.2. *Let X be the interval $= [0, \infty)$ with the usual metric d and $p \geq 1$. If $f : X \to X$ is defined by $f(x) = x^2$, then we will prove that $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is not continuous with respect to d_p. Fix $u = \chi_0$, then $u_\alpha = \{0\}$, for $\alpha \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define $u_n \in \mathcal{F}(X)$ through its α-levels as follows:*

$$[u_n]_\alpha = \begin{cases} [0,n], & \text{if } \alpha \in [0, \frac{1}{n^{2p}}]. \\ \{0\}, & \text{if } \alpha \in (\frac{1}{n^{2p}}, 1]. \end{cases}$$

So:

$$d_H(u_\alpha, [u_n]_\alpha) = \begin{cases} n, & \text{if } \alpha \in \left[0, \frac{1}{n^{2p}}\right]. \\ 0, & \text{if } \alpha \in \left(\frac{1}{n^{2p}}, 1\right]. \end{cases}$$

Thus:

$$d_p(u, u_n) = \left(\int_0^{\frac{1}{n^{2p}}} n^p d\alpha\right)^{\frac{1}{p}} = \left[n^p \left(\frac{1}{n^{2p}}\right)\right]^{\frac{1}{p}} = \frac{1}{n} \text{ for all } n \in \mathbb{N},$$

this means that, $\{u_n\}_{n\in\mathbb{N}}$ converges to u with respect to d_p. On the other hand, $[\widehat{f}(u)]_\alpha = f(u_\alpha) = \{0\}$, for each $\alpha \in \mathbb{I}$. And also:

$$[\widehat{f}(u_n)]_\alpha = f([u_n]_\alpha) = [0, n^2], \text{ if } \alpha \in \left[0, \frac{1}{n^{2p}}\right],$$

$$[\widehat{f}(u_n)]_\alpha = f([u_n]_\alpha) = \{0\}, \text{ if } \alpha \in \left(\frac{1}{n^{2p}}, 1\right].$$

Therefore,

$$d_H([\widehat{f}(u)]_\alpha, [\widehat{f}(u_n)]_\alpha) = \begin{cases} n^2, & \text{if } \alpha \in \left[0, \frac{1}{n^{2p}}\right], \\ 0, & \text{if } \alpha \in \left(\frac{1}{n^{2p}}, 1\right]. \end{cases}$$

In other words, for all $n \in \mathbb{N}$ we have:

$$d_p(\widehat{f}(u), \widehat{f}(u_n)) = \left(\int_0^1 d_H([\widehat{f}(u)]_\alpha, [\widehat{f}(u_n)]_\alpha)^p d\alpha \right)^{\frac{1}{p}} = \left(\int_0^{\frac{1}{n^{2p}}} n^{2p} d\alpha \right)^{\frac{1}{p}} = 1.$$

Therefore $\{\widehat{f}(u_n)\}_{n\in\mathbb{N}}$ does not converge to $\widehat{f}(u)$ regarding d_p, that is Zadeh's extension $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is not continuous regarding d_p metric. □

The result above shows that the continuity of a function f does not guarantee the continuity of $\widehat{f}$ even for locally compact metric spaces.

In the sequel, the metric spaces $(\mathcal{F}(X), d_\infty)$, $(\mathcal{F}(X), d_0)$, $(\mathcal{F}(X), d_S)$ and $(\mathcal{F}(X), d_p)$ will be denoted respectively by $\mathcal{F}_\infty(X)$, $\mathcal{F}_0(X)$, $\mathcal{F}_S(X)$ and $\mathcal{F}_p(X)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of fuzzy sets on $\mathcal{F}(X)$, by g_n we denote, for a fixed $n \in \mathbb{N}$, the function $g_n : \mathbb{I} \to [0, \infty)$ which is defined as follows $g_n(\alpha) = d_H([u_n]_\alpha, u_\alpha)$. The next result will be helpful.

Lemma 3.3. *Let (X, d) be a metric space, $\{u_n\}_{n\in\mathbb{N}}$, a sequence of fuzzy sets in $\mathcal{F}(X)$, $u \in \mathcal{F}(X)$ and $p \geq 1$. If $\{u_n\}_{n\in\mathbb{N}}$ converges to u with respect to d_p, then $\{g_n(\alpha)\}_{n\in\mathbb{N}}$ converges to zero in measure.*

Proof. Fix $\epsilon > 0$. For each $n \in \mathbb{N}$ and $\delta > 0$ consider the following sets $E_n(\delta) = \{\alpha \in \mathbb{I} : g_n(\alpha) \geq \delta\}$. For a given $\delta_0 > 0$ choose $\epsilon_1 = \epsilon^{\frac{1}{p}}\delta_0 > 0$. Since $\{u_n\}_{n\in\mathbb{N}}$ converges to u with respect to the metric d_p, there exists $N \in \mathbb{N}$, such that for each $n \geq N$, we have that:

$\epsilon^{\frac{1}{p}}\delta_0 > d_p(u_n, u) = \left(\int_0^1 d_H([u_n]_\alpha, u_\alpha)^p d\alpha\right)^{\frac{1}{p}} \geq \left(\int_{E_n(\delta_0)} \delta_0^p d\alpha\right)^{\frac{1}{p}}$,

$\epsilon^{\frac{1}{p}}\delta_0 > (\delta_0^p m(E_n(\delta_0))^{\frac{1}{p}}$.

Therefore $\epsilon > m(E_n(\delta_0))$. That is $\lim_{n\to\infty} m(E_n(\delta_0)) = 0$ which concludes the proof. □

In order to attain the next result we may argue similarly:

Corollary 3.4. *Let (X, d) be a metric space, $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be two sequences of fuzzy sets on $\mathcal{F}(X)$ and $p \geq 1$. If $\lim_{n\to\infty} d_p(u_n, v_n) = 0$, then $\{d_H([u_n]_\alpha, [v_n]_\alpha)\}_{n\in\mathbb{N}}$ converges to zero in measure.*

Lemma 3.3 and Corollary 3.4 are aimed at establishing a statement that specifies sufficient conditions to deduce the continuity of $\widehat{f}$, with respect to d_p, from continuity of f. In fact we will prove a stronger theorem.

A popular way for constructing fuzzy sets is precisely to use known fuzzy sets. With that intention the Hutchinson operator is defined. Let (X, d) be a metric space and $f_1, f_2, \cdots, f_k$, with $k \in \mathbb{N}$, be a family of continuous functions from X into itself. In [8] the Hutchinson operator $F : \mathcal{F}(X) \to \mathcal{F}(X)$ was defined levelwise by $[F(u)]_\alpha = [\widehat{f_1}(u)]_\alpha \cup [\widehat{f_2}(u)]_\alpha \cup \cdots \cup [\widehat{f_k}(u)]_\alpha$, for $\alpha \in \mathbb{I}$. In [8] mit was proven that F is well defined and that it is continuous regarding d_∞ and d_0. We now proceed to prove the similar result for d_p metrics.

Theorem 3.5. *Let (X, d) be a compact metric space, $p \geq 1$ and $k \in \mathbb{N}$. If $f_1, f_2, \cdots, f_k : X \to X$ are continuous functions, then Hutchinson operator $F : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is continuous as well.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}(X)$ be a sequence that converges to $u \in \mathcal{F}(X)$ with respect to d_p. Fix $\epsilon > 0$. Since (X, d) is compact, the metric space $(\mathcal{K}(X), d_H)$ is also compact, so $\bar{f} : (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ is uniformly continuous. Thus, there is $\delta > 0$ such that from $d_H(A, B) < \delta$ it follows that $d_H(\overline{f}_i(A), \overline{f}_i(B)) < \frac{\epsilon}{2^{1/p}}$ for each $i = 1, 2, \cdots, k$. In consequence, Proposition 2.1 tells us that $d_H(A, B) < \delta$ implies:

$$d_H\left(\bar{f}_1(A) \cup \bar{f}_2(A) \cup \cdots \cup \bar{f}_k(A), \bar{f}_1(B) \cup \bar{f}_2(B) \cup \cdots \cup \bar{f}_k(B)\right) < \frac{\epsilon}{2^{\frac{1}{p}}}. \tag{1}$$

By the Lemma 3.3, $\{g_n\}_{n\in\mathbb{N}}$ converges to zero in measure. Thus, there exists $N \in \mathbb{N}$, such that if $n \geq N$, then

$$m(\{\alpha \in \mathbb{I} : g_n(\alpha) \geq \delta\}) < \frac{\epsilon^p}{2L^p}, \tag{2}$$

where L is the diameter of X. From this follows that $d_H(f_1([u_n]_\alpha) \cup \cdots \cup f_k([u_n]_\alpha), f_1(u_\alpha) \cup \cdots \cup f_k(u_\alpha)) \leq L$, for each $n \in \mathbb{N}$. We define, for each $n \in \mathbb{N}$, the set $E_n = \{\alpha \in \mathbb{I} : g_n(\alpha) \geq \delta\}$. Fix $n_0 \geq N$ and observe that according 2 we have that:

$$\int_{E_{n_0}} d_H\left(f_1([u_{n_0}]_\alpha \cup \cdots \cup f_k([u_{n_0}]_\alpha), f_1(u_\alpha) \cup \cdots \cup f_k(u_\alpha)\right)^p d\alpha \leq$$

$$\int_{E_{n_0}} L^p d\alpha = m(E_{n_0})L^p < \frac{\epsilon^p}{2L^p} L^p = \frac{\epsilon^p}{2}. \tag{3}$$

On the other hand if $\alpha \in \mathbb{I} \setminus E_{n_0}$, then $d_H([u_{n_0}]_\alpha, u_\alpha) < \delta$. Thus 1 implies that $d_H(\bar{f}_1([u_{n_0}]_\alpha) \cup \cdots \cup \bar{f}_k([u_{n_0}]_\alpha), \bar{f}_1(u_\alpha) \cup \cdots \cup \bar{f}_k(u_\alpha)) < \frac{\epsilon}{2^{\frac{1}{p}}}$ and

$$\int_{\mathbb{I} \setminus E_{n_0}} d_H(\bar{f}_1([u_{n_0}]_\alpha) \cup \cdots \cup \bar{f}_k([u_{n_0}]_\alpha), \bar{f}_1(u_\alpha) \cup \cdots \cup \bar{f}_k(u_\alpha))^p d\alpha < \frac{\epsilon^p}{2}. \tag{4}$$

So (3) and (4) imply that $d_p^p(F(u_{n_0}), F(u)) < \epsilon^p$. Therefore $\{F(u_n)\}_{n \in \mathbb{N}}$ converges to $F(u)$ with respect to d_p, which implies that $F : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is continuous. □

Corollary 3.6. *Let (X, d) be a compact metric space, $p \geq 1$ and $f : X \to X$ a continuous function. Then $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is continuous.*

The proof of the next fact is essentially the same as the previous one it is enough to replace Lemma 3.3 with Corollary 3.4.

Proposition 3.7. *Let (X, d) be a compact metric space, $p \geq 1$ and $f : (X, d) \to (X, d)$ a be continuous function. Then $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is uniformly continuous.*

However, compactness is not a necessary condition in establishing the continuity of $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ as evidenced by the identity function.

4 Stability for $(\mathcal{F}_p(X), \widehat{f})$

A discrete dynamical system is a pair (X, f) where X is equipped with a metric d and $f : X \to X$ is a continuous function. Let us recall that $f^0(x) = Id(x)$, $f^1(x) = f(x)$, $f^2(x) = f(f^1(x))$, $f^3(x) = f(f^2(x))$ and for each $k \in \mathbb{N}$, $f^k(x) = f(f^{k-1}(x))$. As usual for $x_0 \in X$ the orbit of a point $x_0 \in X$ is the sequence:

$$x_0, f(x_0), f^2(x_0), \cdots, f^n(x_0), \cdots$$

which is denoted by $O(x_0, f)$. By virtue of Corollary 3.6, Zadeh's extension $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is a continuous function, as long as $f : X \to X$ is a continuous function and (X, d) is a compact metric space, all of which brings to mind the discrete fuzzy dynamical system $(\mathcal{F}_p(X), \widehat{f})$.

Let (X, f) be a discrete dynamical system if the orbits of neighboring points remain close, it is considered to have stable behavior. The following property captures the aforementioned idea.

Definition 4.1. *Let (X, d) be a metric space, $f : X \to X$ and $x_0 \in X$. We say that $x_0 \in X$ is a positively Lyapunov stable (PLS) point if for each $\epsilon > 0$, there exists $\delta > 0$ such that, if $d(x_0, y) < \delta$, then $d(f^n(x_0), f^n(y)) < \epsilon$ for all $n \geq 0$. We say that (X, f) is a positively Lyapunov stable (PLS) function, if each $x \in X$ is PLS.*

It is a well-known fact that for each open cover $\mathcal{C}$ of a compact metric space X, there exists $\delta > 0$ such that, if $x, y \in X$ and $d(x, y) < \delta$, there is $C \in \mathcal{C}$ with $\{x, y\} \subseteq C$.

Lemma 4.2. *Let (X, d) be a compact metric space and $f : X \to X$ a PLS function, then for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \epsilon$, for all $n \geq 0$.*

Proof. Fix $\epsilon > 0$. Since f is PLS, for each $x \in X$, there exists $\delta_x > 0$ such that $f^n(\mathcal{B}_d(x, \delta_x)) \subseteq \mathcal{B}_d(f^n(x), \frac{\epsilon}{2})$ for all $n \geq 0$. Note that $\{\mathcal{B}_d(x, \delta_x) : x \in X\}$ is an open cover of X hence, there exists $\delta > 0$ such that, if $d(x, y) < \delta$, then there exists x_0, with $\{x, y\} \subseteq \mathcal{B}_d(x_0, \delta_{x_0})$. So $d(x, x_0) < \delta_{x_0}$ and $d(y, x_0) < \delta_{x_0}$, therefore $d(f^n(x), f^n(x_0)) < \frac{\epsilon}{2}$ and $d(f^n(y), f^n(x_0)) < \frac{\epsilon}{2}$ for all $n \geq 0$, hence $d(f^n(x), f^n(y)) < \epsilon$ for all $n \geq 0$. □

It is a well known fact that if $f : (X, d) \to (X, d)$ is a continuous function, then $\left(\widehat{f}\right)^n = \widehat{f^n}$, we refer the reader to [12] for instance. So we can write $\widehat{f^n}$ instead of $\left(\widehat{f}\right)^n$. Henceforth, we suppose that $f : X \to X$ is a continuous function.

Proposition 4.3. *If (X, d) is a compact metric space and $p \geq 1$, then the following statements are equivalent.*

i) *$f : X \to X$ is PLS;*

ii) $\bar{f}:\mathcal{K}(X)\to\mathcal{K}(X)$ *is PLS;*

iii) $\widehat{f}:\mathcal{F}_p(X)\to\mathcal{F}_p(X)$ *is PLS.*

Proof. To prove i)→ ii), let us assume that f is PLS. We fix $A\in\mathcal{K}(X)$ and $\epsilon>0$. By Lemma 4.2, there exists $\delta>0$ such that, $d(x,y)<\delta$, implies $d(f^n(x),f^n(y))<\epsilon$, for all $n\geq 0$. Take $B\in\mathcal{K}(X)$ such that $d_H(A,B)<\delta$, by definition of PLS and Lemma 4.2, we have that for each one pair $a\in A$ and $b\in B$ it holds that $d(f^n(a),f^n(B))<\epsilon$ and $d(f^n(b),f^n(A))<\epsilon$ Therefore $d_H(f^n(A),f^n(B))<\epsilon$.

Now, let us assume that ii) is true and fix $\epsilon>0$. Notice that $\mathcal{K}(X)$ is a compact metric space. As $\bar{f}:\mathcal{K}(X)\to\mathcal{K}(X)$ is PLS, Lemma 4.2 implies the existence of $\delta>0$ such that, if $d(A,B)<\delta$ then $d(\bar{f}^n(A),\bar{f}^n(B))<\frac{\epsilon}{4^{\frac{1}{p}}}$ for each $n\in\mathbb{N}$.

Take $u\in\mathcal{F}(X)$ and $n\in\mathbb{N}$. For each $v\in\mathcal{F}(X)$ such that $d(u,v)<\frac{\epsilon\delta}{4^{\frac{1}{p}}L}$, where L is the diameter of X. We define $E_v=\{\alpha\in\mathbb{I}:d_H(u_\alpha,v_\alpha)\geq\delta\}$, by an argument similar to the proof of Lemma 3.3, we have that $\frac{\epsilon^p}{4L^p}>m(E_v)$, so for each $n\in\mathbb{N}$ is satisfied that

$$d_p^p(\widehat{f}^n(u),\widehat{f}^n(v))\leq\int_{E_v}L^p d_\alpha+\int_{(0,1]\setminus E_v}\left(\frac{\epsilon}{4^{\frac{1}{p}}}\right)^p d\alpha\leq L^p m(E_v)+\frac{\epsilon^p}{4}=\frac{\epsilon^p}{2}<\epsilon^p.$$

We can get proof of implication iii)→ i) by using $d(x,y)=d_p(\chi_x,\chi_y)$ and $d(f(x),f(y))=d_p(\widehat{f}(\chi_x),\widehat{f}(\chi_y))=d_p(\chi_{f(x)},\chi_{f(y)})$ for all $x,y\in X$. $\square$

We can argue analogously to prove the following result.

Corollary 4.4. *Let (X,d) be a compact metric space and $p\geq 1$. Then:*

i) *If $x\in X$ is PLS of (X,f), then χ_x is PLS in $(\mathcal{F}_p(X),\widehat{f})$.*

ii) *If $A\in\mathcal{K}(X)$ is PLS of $(\mathcal{K}(X),\bar{f})$, then χ_A is PLS in $(\mathcal{F}_p(X),\widehat{f})$.*

Now we will address to contractions, these functions are relevant in many areas of mathematics.

Definition 4.5. *Let (X,d) be a metric space. A function $f:X\to X$ is a Lipschitz function, if there exists a $0\leq\lambda$ such that, for all $x,y\in X$ we have that:*

$$d(f(x),f(y))\leq\lambda d(x,y).$$

If $\lambda<1$, then f is said to be a contraction.

Note that, if (X,d) is a metric space and $f:X\to X$ is a contraction with factor $\lambda\in[0,1)$, then $d(f^n(x),f^n(y))\leq\lambda^n d(x,y)$. In particular f is also PLS, because $\lim\limits_{n\to\infty}\lambda^n=0$.

Let (X,d) be a metric space. If $f:(X,d)\to(X,d)$ is a contraction, then its Zadeh's extension $\widehat{f}:(\mathcal{F}(X),d_\infty)\to(\mathcal{F}(X),d_\infty)$ is a contraction (see [7]), but, particularly if $X=\mathbb{I}$, in [14] there exists a contraction f such that its Zadeh's extension $\widehat{f}:(\mathcal{F}(X),d_0)\to(\mathcal{F}(X),d_0)$ is not a contraction.

Proposition 4.6. *Let (X,d) be a metric space, $p\geq 1$ and $k\in\mathbb{N}$. If $f_1,f_2,\cdots,f_k:X\to X$ are contractions, then the Hutchinson operator $F:\mathcal{F}_p(X)\to\mathcal{F}_p(X)$ is a contraction.*

Proof. Since $f_1,f_2,\cdots,f_k$ are contractions, then there are $\lambda_1,\lambda_2,\cdots,\lambda_k\in[0,1)$ such that $d(f_i(x),f_i(y))\leq\lambda_i d(x,y)$, for anyone $x,y\in X$ and $1\leq i\leq k$. We define $\lambda=\max\{\lambda_1,\cdots,\lambda_k\}$, note that $\lambda<1$. If $u,v\in\mathcal{F}(X)$, by Proposition 2.1 we have that:

$$d_H(f_1(u_\alpha)\cup\cdots\cup f_k(u_\alpha),f_1(v_\alpha)\cup\cdots\cup f_k(v_\alpha))\leq\lambda d_H(u_\alpha,v_\alpha).$$

Therefore:

$$\int_0^1 d_H(f_1(u_\alpha)\cup\cdots\cup f_k(u_\alpha),f_1(v_\alpha)\cup\cdots\cup f_k(v_\alpha))^p d_\alpha\leq\lambda^p d_p^p(u,v),$$

this means that $d_p(F(u),F(v))\leq\lambda d_p(u,v)$. Hence $F:\mathcal{F}_p(X)\to\mathcal{F}_p(X)$ is a contraction. $\square$

Corollary 4.7. *Let (X,d) be a metric space and $p\geq 1$. Then $f:(X,d)\to(X,d)$ is a contraction, if and only if its Zadeh's extension $\widehat{f}:\mathcal{F}_p(X)\to\mathcal{F}_p(X)$ is a contraction.*

Proof. Suppose that $f : (X, d) \to (X, d)$ is a contraction; to prove that $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is a contraction, it is enough to consider $F = \widehat{f}$ in the previous proposition. For the inverse implication, notice that $d(x, y) = d_p(\chi_x, \chi_y)$ and $d(f(x), f(y)) = d_p(\widehat{f}(\chi_x), \widehat{f}(\chi_y)) = d_p(\chi_{f(x)}, \chi_{f(y)})$ for all $x, y \in X$. □

If in Proposition 4.6 and Corollary 4.7, the word contraction is replaced by the Lipschitz function, the results remain correct.

It is well known that if a function f is a contraction (Lipschitz function), then it is continuous and PLS. According to the previous result, if f is a contraction (Lipschitz function), then $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is a contraction (Lipschitz function), hence it is continuous and PLS, regardless of the compactness of the space X. It is worth mentioning that in [1] it was proven that if X is a compact metric space, then $\mathcal{F}_p(X)$ is complete for any $p \geq 1$. This implies that if f is a contraction on X then its Zadeh's extension has a unique fixed point.

5 Sensitivity, transitivity and periodic density

A dynamical system is Devaney chaotic if it is transitive, sensitive and has a dense set of periodic points. However, it is known that any transitive system with a dense set of periodic points is also sensitive. The properties related to Devaney chaos is studied below.

Note that at the opposite extreme of the PLS property, the sensitivity to initial conditions is defined for the discrete dynamical system (X, f), which shows has instability at any point.

Definition 5.1. *Let (X, f) be a dynamical system.*

i) *A function $f : X \to X$ is sensitive* [4] *(or sensitive to initial conditions) if there exists $\delta > 0$ such that for all $x \in X$ and for all $\epsilon > 0$ there are $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$.*

ii) *A function $f : X \to X$ is said to be strongly sensitive if we can take $\lambda > 0$ such that for each $x \in X$ and $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there is $y \in X$ with $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \lambda$.*

In other words, f is sensitive if each $x \in X$ is not PLS. Note that strong sensitivity implies sensitivity to initial conditions, but the converse is not always true (see [13, Example 1.2]).

In [6] it is shown that if (X, d) is a metric space and $(\mathcal{F}(X), d_\rho)$ is (strongly) sensitive for $\rho = \infty, 0, S$, then (X, d) is also (strongly) sensitive. The proof is based on the fact that $d_S(u, \chi_x) = d_\infty(u, \chi_x) = d_0(u, \chi_x) = d_H(u_0, \{x\})$, for all $u \in \mathcal{F}(X)$ and $x \in X$. However, it does not always occur that $d_p(u, \chi_x) = d_H(u, \{x\})$, so the proofs follow a different path.

Example 5.2. *Let $X = \mathbb{R}$, $p = 1$ and d the usual metric. Take $3 \in X$, and u as follows:*

$$u(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 2]. \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then $u_\alpha = [2\alpha, 2]$, for each $\alpha \in \mathbb{I}$ and therefore $d_H(u_\alpha, \{3\}) = 3 - 2\alpha$. Thus:

$$d_p(u, \chi_3) = \int_0^1 (3 - 2\alpha) d\alpha = 2.$$

Whereas $d_H(u_0, \{3\}) = d([0, 2], \{3\}) = 3$.

Proposition 5.3. *Suppose that (X, d) is a compact metric space and $p \geq 1$. If $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is sensitive, then:*

i) *$f : X \to X$ is sensitive.*

ii) *$\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is sensitive.*

Proof. Suppose that $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is sensitive and that f is not sensitive ($\bar{f}$ is not sensitive), then there exists $x_0 \in X$ ($A_0 \in \mathcal{K}(X)$) which is PLS in (X, f) $((\mathcal{K}(X), \bar{f}))$, then Corollary 4.4 implies that χ_{x_0} (χ_{A_0}) is PLS in $(\mathcal{F}_p(X), \widehat{f})$, which is a contradiction. Therefore we obtain i) and ii). □

Since that strong sensitivity implies sensitivity, the next result follows from Proposition 5.3

Corollary 5.4. *Suppose that (X,d) is a compact metric space and $p \geq 1$. If $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is strongly sensitive, then:*

i) *$f : X \to X$ is sensitive.*

ii) *$\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is sensitive.*

We emphasize the compactness hypothesis in Proposition 5.3. In general, it is not always true that, if $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is sensitive, then $f : X \to X$ is sensitive, as may be verified in the following example:

Example 5.5. *Let $X = [0,\infty)$ with the usual metric d and $p \geq 1$. We define $f : X \to X$ as follows:*

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1]. \\ 2x-1, & \text{if } x \in (1,\infty). \end{cases}$$

Observe that f is a Lipschitz map. Take two different points $x, y \in [0,\infty)$. If x and y belong to $[0,1]$, then $|f(x)-f(y)| = |x-y|$. If $x, y \in (1,\infty)$, then $|f(x)-f(y)| = 2|x-y|$. Finally if, $x \in [0,1]$ and $y \in (1,\infty)$, which implies $0 \leq y-1 < y-x$. Thus $d(f(x), f(y)) = |x-(2y-1)| = |x-y| + |y-1| < |x-y| + |x-y| = 2|x-y|$, hence f is a Lipschitz map with constant $\lambda = 2$. In order to prove that Zadeh's extension is a Lipschitz map, we can argue as in the proof of Proposition 4.6 with only one map and prove that $\widehat{f}$ is Lipschitz. It is also continuous, because so is any Lipschitz map.
Let $u \in \mathcal{F}(X)$ and $0 < \epsilon < 1$. We define $a_0 = \max u_0$, and $x_0 = a_0 + 2$. Note that $x_0 \in (1,\infty)$. We define the fuzzy set ϕ as follows:

$$\phi(x) = \begin{cases} u(x), & \text{if } x \in u_0. \\ \frac{\epsilon^p}{2^p x_0^p}, & \text{if } x = x_0. \\ 0, & \text{otherwise.} \end{cases}$$

Note that $u \neq \phi$ and $\phi_\alpha = u_\alpha \cup \{x_0\}$ for all $\alpha \leq \frac{\epsilon^p}{2^p x_0^p}$, hence $d_H(\phi_\alpha, u_\alpha) = d(x_0, u_\alpha) \leq x_0$. If $\frac{\epsilon^p}{2^p x_0^p} < \alpha \leq 1$, then $u_\alpha = \phi_\alpha$, thus $d_H(\phi_\alpha, u_\alpha) = 0$. Therefore:

$$d_p(u,\phi) = \left[\int_0^{\frac{\epsilon^p}{2^p x_0^p}} d_H(u_\alpha, \phi_\alpha)^p d_\alpha \right]^{\frac{1}{p}} \leq \left[\int_0^{\frac{\epsilon^p}{2^p x_0^p}} x_0^p d\alpha \right]^{\frac{1}{p}} = \frac{\epsilon}{2} < \epsilon.$$

On the other hand, if $\alpha \leq \frac{\epsilon^p}{2^p x_0^p}$, we have two possibilities $a_0 \in [0,1]$ or $a_0 \in (1,\infty)$. If $a_0 \in [0,1]$, then $u_\alpha \subseteq [0,1]$, and $\max u_\alpha \leq a_0$, hence $d_H(f^n(\phi_\alpha), f^n(u_\alpha)) = d(f^n(x_0), f^n(u_\alpha)) = d(2^n x_0 - (2^n-1), u_\alpha) \geq d(2^n x_0 - (2^n-1), a_0) = |2^n x_0 - 2^n + 1 - a_0| = |2^n(a_0+2) - 2^n + 1 - a_0| = |(2^n a_0 - a_0) + 1 + 2^n| \geq 2^n$. If $a_0 \in (1,\infty)$ then $d_H(f^n(\phi_\alpha), f^n(u_\alpha)) \geq d(2^n x_0 - (2^n-1), 2^n a_0 - (2^n-1)) = |2^n(a_0+2) - 2^n + 1 - 2^n a_0 + 2^n - 1| \geq 2^n$. In the another case, when $\frac{\epsilon^p}{2^p x_0^p} < \alpha \leq 1$, we have $d_H(f^n(\phi_\alpha), f^n(u_\alpha)) = 0$. Therefore:

$$d_p(\widehat{f}^n(u), \widehat{f}^n(\phi)) = \left[\int_0^{\frac{\epsilon^p}{2^p x_0^p}} d_H(f^n(u_\alpha), f^n(\phi_\alpha))^p d\alpha \right]^{\frac{1}{p}} \geq \frac{\epsilon}{2x_0} 2^n.$$

Due to the Archimedean property, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have that $\frac{\epsilon}{2x_0} 2^n > 1$, that is, $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is strongly sensitive (and therefore sensitive) with constant $\lambda = 1$. However, f is not sensitive (and therefore, it is not strongly sensitive neither) since for each $\delta > 0$ we can consider $x_0 = 1/2$ and $\epsilon = \min\{1/4, \delta\}$ for which we have that for each $n \in \mathbb{N}$ from $|y - x_0| < \epsilon$ it follows that $|f^n(y) - f^n(x_0)| = |y - x_0| < \epsilon \leq \delta$

In the opposite direction, in [5] it is shown that if (X,d) is strongly sensitive, then$(\mathcal{F}(X), d_\rho)$ is also strongly sensitive, for $\rho = \infty, 0, S$. The proof uses the fact that for every $u, v \in \mathcal{F}(X)$, we have $d_H(u_0, v_0) \leq d_0(u,v) \leq d_\infty(u,v)$; however, as Example 5.2 shows, in the d_p metric, it is not always true that $d_H(u_0, v_0) \leq d_p(u,v)$.

Proposition 5.6. *Suppose that (X,d) is a compact metric space, $p \geq 1$. If the continuous function $f : X \to X$ is strongly sensitive, then $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is strongly sensitive as well.*

Proof. Fix $\lambda > 0$ which witnesses strong sensitivity of f. Take $u \in \mathcal{F}(X)$ and $\epsilon > 0$. Since the set of all elements in $\mathcal{F}(X)$ with finite support is dense in $\mathcal{F}_p(X)$ ([1], Theorem 3.7, claim 1), without loss of generality we can consider that $|u_0| = k$ for some $k \in \mathbb{N}$. Suppose that $u_0 = \{x_1, x_2, \cdots, x_k\}$. For each $i = 1, 2, \cdots, k$, take $N_i \in \mathbb{N}$ which witnesses strong sensitivity regarding x_i and $\frac{\epsilon}{2}$. Put $N = \max\{N_1, N_2, \cdots, N_k\}$ and fix $n \geq N$. For each $i = 1, 2, \cdots, k$ there is $y_i \in X$ such that $d(x_i, y_i) < \frac{\epsilon}{2}$ and $d(f^n(x_i), f^n(y_i)) > \lambda$. Take $x_j \in u_1 \neq \emptyset$, thus $x_j \in u_\alpha$ for each $\alpha \in \mathbb{I}$. Observe that for each $i \in \{1, 2, \cdots, k\}$ such that $i \neq j$ only one of the following conditions can be true $d(f^n(x_j), f^n(y_i)) \leq \lambda/2$ or $d(f^n(x_j), f^n(x_i)) \leq \lambda/2$. Put $c_j = y_j$. Moreover, for each $i \neq j$ define $c_i = y_i$ whenever $d(f^n(x_j), f^n(x_i)) \leq \lambda/2$ and $c_i = x_i$ otherwise. Note that for each $A \subset u_0$ such that $x_j \in A$ we have that $d_H(A, B_A) < \frac{\epsilon}{2}$ and $d_H(f^n(A), f^n(B_A)) > \lambda/2$, where $B_A = \{c_r : r \in \text{Index}(A)\}$ and $\text{Index}(A)$ is the set of indexes of all the elements of A. So, by Proposition 2.2, the family $\{B_{u_\alpha} : \alpha \in \mathbb{I}\}$ defines a fuzzy set w such that $w_\alpha = B_{u_\alpha}$. Observe that $d_H(u_\alpha, w_\alpha) < \frac{\epsilon}{2}$ and $d_H(f^n(u_\alpha), f^n(w_\alpha)) > \lambda/2$, for each $\alpha \in \mathbb{I}$, this implies that:

$$d_p(u, w) = \left(\int_{\mathbb{I}} d_H(u_\alpha, w_\alpha)^p d\alpha\right)^{1/p} \leq \left(\int_{\mathbb{I}} \left(\frac{\epsilon}{2}\right)^p d\alpha\right)^{1/p} = \frac{\epsilon}{2} < \epsilon,$$

moreover

$$d_p(\widehat{f}^n(u), \widehat{f}^n(w)) = \left(\int_{\mathbb{I}} d_H\left([\widehat{f}^n(u)]_\alpha, [\widehat{f}^n(w)]_\alpha\right)^p d\alpha\right)^{1/p} =$$

$$\left(\int_{\mathbb{I}} d_H\left(f^n(u_\alpha), f^n(w_\alpha)\right)^p d\alpha\right)^{1/p} \geq \left(\int_{\mathbb{I}} (\lambda/2)^p d\alpha\right)^{1/p} = \frac{\lambda}{2} > \frac{\lambda}{4}.$$

Therefore $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is strongly sensitive. □

Now let us move on to transitivity properties.

Definition 5.7. *Let X be a topological space and $f : X \to X$ a continuous function.*

i) *A discrete dynamical system (X, f) is transitive, if for any two non empty open subsets U, V of X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.*

ii) *A discrete dynamical system (X, f) is weakly mixing, if $f \times f : X \times X \to X \times X$ is transitive.*

The following fact referred to transitivity regarding d_∞, d_0 and d_S metrics was proven in [7].

Theorem 5.8. [7] *Let (X, d) be a metric space. Then, the following conditions are equivalent:*

i) *(X, f) is weakly mixing;*

ii) *$(\mathcal{K}(X), \bar{f})$ is transitive;*

iii) *$(\mathcal{F}_S(X), \widehat{f})$ is transitive.*

iv) *$(\mathcal{F}_0(X), \widehat{f})$ is transitive.*

v) *$(\mathcal{F}_\infty(X), \widehat{f})$ is transitive.*

Let us recall that $d_p(u, v) \leq d_\infty(u, v)$, for all $u, v \in \mathcal{F}(x)$, from which the next two corollaries follow.

Corollary 5.9. *Let (X, d) be a compact metric space and $p \geq 1$. If (X, f) is weakly mixing, then $(\mathcal{F}_p(X), \widehat{f})$ is transitive.*

Corollary 5.10. *Let (X, d) be a compact metric space and $p \geq 1$. If $(\mathcal{K}(X), \bar{f})$ is transitive, then $(\mathcal{F}_p(X), \widehat{f})$ is transitive.*

In the other direction, if we assume that f is transitive, we have the following.

Proposition 5.11. *Let (X, d) be a compact metric space, $p \geq 1$ and $f : X \to X$ a continuous function. If $(\mathcal{F}_p(X), \widehat{f})$ is transitive, then (X, f) is also transitive.*

Proof. Take $a, b \in X$, $\delta > 0$ and $\epsilon > 0$. There exist $w \in \mathcal{F}(X)$ and $n \in \mathbb{N}$, such that $d_p(\chi_a, w) < \delta$ and $d_p(\chi_b, \widehat{f}^n(w)) < \epsilon$. We define $g_a(\alpha) = d_H([\chi_a]_\alpha, w_\alpha) = d_H(\{a\}, w_\alpha)$. Note that, if $\alpha < \beta$, then $d_H(\{a\}, w_\beta) \leq d_H(\{a\}, w_\alpha)$, i.e. $g_a(\alpha)$ is a decreasing function. As

$$d_p(\chi_a, w) = \left(\int_0^1 d_H([\chi_a]_\alpha, w_\alpha)^p d\alpha \right)^{\frac{1}{p}} < \delta,$$

there exists $\alpha \in (0, 1]$, for which $d_H(\{a\}, w_\alpha) < \delta$, thus $d_H(\{a\}, w_1) < \delta$. Analogously prove we can that $d_H(\{b\}, f^n(w_1)) < \epsilon$.

If $x_0 \in w_1$, then $d(a, x_0) \leq d_H(\{a\}, w_1) < \delta$. Since $f^n(x_0) \in f^n(w_1)$ we have that $d(b, f^n(x_0)) \leq d_H(\{b\}, f^n(w_1)) < \epsilon$. Hence $d(a, x_0) < \delta$ and $d(b, f^n(x_0)) < \epsilon$ which proves that (X, f) is transitive. □

In the sequel we will focus on the periodic density of the induced map $\widehat{f}$. Let us recall that $x \in X$ is a periodic point of $f : X \to X$, if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. Note that, if $u \in \mathcal{F}(X)$ is a periodic point, that is, there exists $n \in \mathbb{N}$, then by Proposition 2.3, $f^n(u_\alpha) = u_\alpha$ for each $\alpha \in \mathbb{I}$.

Definition 5.12. *Let (X, d) be a metric space. A dynamical system (X, f) has periodic density, if set $\{x \in X : f^n(x) = x \text{ for some } n \in \mathbb{N}\}$ is dense in X.*

Proposition 5.13. *Let (X, d) be a metric space. Suppose that r is a metric on $\mathcal{F}(X)$, such that for each compact $B \subset X$ and $v \in \mathcal{F}(X)$ there exists $\alpha \in \mathbb{I}$ such that $d_H(B, v_\alpha) \leq d_r(\chi_B, v)$. If $\widehat{f}$ has periodic density on $\mathcal{F}_r(X)$, then $\bar{f}$ has periodic density on $\mathcal{K}(X)$.*

Proof. Suppose that $U = \{v \in \mathcal{F}(X) : \widehat{f^n}(v) = v \text{ for some } n \in \mathbb{N}\}$ is dense in $\mathcal{F}_r(X)$. Let $A \in \mathcal{K}(X)$ and $\epsilon > 0$, then there exists $u \in U$ such that $d_r(\chi_A, u) < \epsilon$. Note that we can find $\alpha \in \mathbb{I}$ such that $d_H(A, u_\alpha) \leq d_r(\chi_A, u) < \epsilon$. Since $u \in U$, there exists $m \in \mathbb{N}$ such that $\widehat{f^m}(u) = u$, this implies that $\overline{f^m}(u_\alpha) = u_\alpha$. Thus $\{u_\beta : \beta \in \mathbb{I} \text{ and } u \in U\}$ is dense in $\mathcal{K}(X)$ and contains only periodic points. □

Bearing in mind that for each pair $u, w \in \mathcal{F}(X)$ it hold $d_H(u_0, w_0) \leq d_S(u, w) \leq d_0(u, w) \leq d_\infty(u, w)$. Note that for each $A \in \mathcal{K}(X)$ and every $u \in \mathcal{F}(X)$ there exists $\alpha \in \mathbb{I}$ such that $d_H(A, u_\alpha) \leq d_p(\chi_A, u)$. The previous proposition implies the following result.

Proposition 5.14. *Suppose that (X, d) is a metric space, $p \geq 1$ and $r \in \{\infty, 0, S, p\}$. If $\widehat{f}$ has periodic density on $\mathcal{F}_r(X) = (\mathcal{F}(X), d_r)$, then $\bar{f}$ has periodic density on $\mathcal{K}(X)$.*

It is well known that if $f : X \to X$ has periodic density, then $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ and $\widehat{f} : \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$ also have periodic density. On the other hand, in [6] it has been proved that if $f : X \to X$ has periodic density, then $\widehat{f} : \mathcal{F}_0(X) \to \mathcal{F}_0(X)$ also has periodic density. Regarding the question about whether the periodic density of $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ implies periodic density of $f : X \to X$, in [2] there is an example which shows that this fact is not always true. In contrast we have the next result.

Proposition 5.15. *Let (X, d) be a metric space and $p \geq 1$. If f has periodic density on (X, d), then $\widehat{f}$ has periodic density on $\mathcal{F}_p(X)$.*

Proof. Suppose that f has periodic density on (X, d) and that D is the set of all periodic points of f. In ([1], Theorem 3.7, claim 1) it is proved that, if D is dense in X, then $\Phi = \{\varphi \in \mathcal{F}(X) : |\varphi_0| < \infty, \varphi_0 \subseteq D\}$ is dense in $\mathcal{F}_p(X)$. Fix $\varphi \in \Phi$, then there exists $k \in \mathbb{N}$ such that $\varphi_0 = \{x_1, x_2, \cdots, x_k\} \subseteq D$. For each $x_i \in \varphi_0$ with $1 \leq i \leq k$, there exists $n_i \in \mathbb{N}$, such that $f^{n_i}(x_i) = x_i$, so $m = n_1 \times n_2 \times \cdots \times n_k$ satisfies that $f^m(x_i) = x_i$, for all $x_i \in \varphi_0$. Hence $f^m(\varphi_\alpha) = \varphi_\alpha$ for all $\alpha \in \mathbb{I}$ which implies $\widehat{f^m}(\varphi) = \varphi$. Therefore Φ is a dense set of periodic points of $\widehat{f}$ in $\mathcal{F}_p(X)$. □

Additionally, in ([1], Theorem 3.7) the authors prove that if D is a dense countable in (X, d), then $\Gamma = \{\varphi \in \mathcal{F} : |\varphi_0| < \infty, \varphi_0 \subseteq D \text{ and } \varphi_0(X) \subseteq \mathbb{Q}\}$ is a dense countable in $\mathcal{F}_p(X)$. Therefore Γ is a countable set of periodic points of $\widehat{f}$.

Corollary 5.16. *Let (X, d) be a metric space and $p \geq 1$. If f has a dense countable set of (X, d), which consists only of periodic points, then $\widehat{f}$ has a dense countable set of $\mathcal{F}_p(X)$, which consists only of periodic points of $\widehat{f}$.*

The above result is consistent with [6], where it is proven that if f has a dense countable subset of periodic points of (X, f), then there exists a dense countable subset of periodic points of $(\mathcal{F}_0(X), f)$ and a dense countable subset of periodic points of $(\mathcal{F}_S(X), f)$.

6 Conclusions

In this work, it has been proven that if (X, d) is a compact metric space and $f : X \to X$ is a continuous function, then $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ is also continuous for each $p \geq 1$. However, Identity function and Corollary 4.7 attest that compactness is not a necessary condition. It is important to note that the conditions for obtaining the continuity of $\widehat{f}$, in Corollary 4.7, are over f and not over X.

On the other hand, Example 3.2 shows that local compactness of X is not sufficient to ensure the continuity of $\widehat{f}$. So the question on which properties on the underlying space guarantee the continuity of the Zadeh's extension, regarding d_p metrics, remains an interesting subject for future research.

Moreover in this work, some properties have been analyzed, such as positive Lyapunov stability, contractility, (strong) sensitivity and transitivity for a dynamical system $f : X \to X$ and its Zadeh's extension $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$. For PLS and sensitivity properties, the compactness of X was used, not only to guarantee the continuity of $\widehat{f}$. Regarding the question: is it true that the sensitivity of $\widehat{f} : \mathcal{F}_p(X) \to \mathcal{F}_p(X)$ implies the sensitivity of $f : X \to X$? Example 3 shows that in general this is not true. However, we have a positive answer whenever the underlying space X is compact, see Proposition 5.3.

Acknowledgement

This research paper was supported by UACM and CONAHCyT (SECIHTI): Grant Convocatoria 2023(1) Estancias Posdoctorales por México.

References

[1] K. Aguirre, I. Sánchez, *Some properties of the d_p-metrics on fuzzy sets*, Topology and Its Applications, **322** (2022), 108323. http://dx.doi.org/10.1016/j.topol.2022.108323

[2] J. Banks, *Chaos for induced hyperspace maps*, Chaos, Solitons and Fractals, **25**(3) (2005), 681-685. https://doi.org/10.1016/j.chaos.2004.11.089

[3] P. Diamond, P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems, **35** (1990), 241-249. https://doi.org/10.1016/0165-0114(90)90197-E

[4] J. Guckenheimer, *Sensitive dependence to initial conditions for one dimensional maps*, Communications in Mathematical Physics, **70** (1979), 133-160. https://doi.org/10.1007/BF01982351

[5] D. Jardón, I. Sánchez, *Expansive properties of induced dynamical systems*, Fuzzy Sets and Systems, **425**(4) (2020). https://doi.org/10.1016/j.fss.2020.11.023

[6] D. Jardón, I. Sánchez, *Sensitivity and strong sensitivity on induced dynamical systems*, Iranian Journal of Fuzzy Systems, **18**(4) (2021), 69-78. http://dx.doi.org/10.22111/ijfs.2021.6177

[7] D. Jardón. I. Sánchez, M. Sanchis, *Transitivity in fuzzy hyperspaces*, Mathematics, **8**(11) (2020), 1862. https://doi.org/10.3390/math8111862

[8] D. Jardón, I. Sánchez, M. Sanchis, *Some questions about Zadeh's extesion on metric spaces*, Fuzzy Sets and System, **379** (2020), 115-124. https://doi.org/10.1016/j.fss.2018.10.019

[9] W. Johnston, *The Lebesgue integral for undergraduates*, The Mathematical Association of America, (2015). https://doi.org/10.1090/text/027

[10] J. Kupka, *On fuzzifications of discrete dynamical systems*, Information Sciences, **181**(13) (2011), 2858-2872. https://doi.org/10.1016/j.ins.2011.02.024

[11] H. Román-Flores, L. C. Barros, R. C. Bassanezi, *A note on Zadeh's extensions*, Fuzzy Set and Systems, **117**(3) (2001), 327-331. https://doi.org/10.1016/S0165-0114(98)00408-4

[12] H. Román-Flores, Y. Chalco-Cano, *Some chaotic of Zadeh's extensions*, Chaos, Solitons and Fractals, **35**(3) (2008), 452-459. https://doi.org/10.1016/j.chaos.2006.05.036

[13] P. Sharma, A. Nagar, *Inducing sensitivity on hyperspaces*, Topology and Its Applications, **157**(13) (2010), 2052-2058. https://doi.org/10.1016/j.topol.2010.05.002

[14] X. Wu, X. Zhang, G. Chen, *Answers to some questions about Zadeh's extension principle on metric spaces*, Fuzzy Sets and Systems, **387** (2020), 174-180. https://doi.org/10.1016/j.fss.2019.03.011

[15] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X