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Original paper

The explicit solution to fuzzy differential equation with piecewise constant arguments involving the multi-order fractional derivative in $(1,2)$

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Abstract

This paper investigates non-homogeneous linear fuzzy fractional differential equations with piecewise constant arguments involving generalized multi-order Caputo fractional derivatives of order $\mathfrak{q}_\imath \in (1,2)$. By employing generalized single-level constrained fuzzy arithmetic, explicit closed-form solutions are obtained. A fuzzy vehicle-following model is included to demonstrate the applicability of the proposed results and to illustrate the effects of fractional orders, kernel functions, and fuzzy disturbances on the solution behavior.

Keywords: Fuzzy non-homogeneous linear differential equations, piecewise constant arguments, generalized multi-order Caputo fractional derivative, explicit solution, variation of constants formula.

1 Introduction

Many real-world systems are characterized by hybrid dynamics in which continuous-time evolution interacts with discrete-time mechanisms, memory effects, and uncertainty. Differential equations with piecewise constant arguments (DEs-PCA) provide a natural mathematical framework for describing such phenomena. Since their introduction by Busenberg in the context of epidemic modeling [6], DEs-PCA have been extensively studied as dynamical systems lying at the interface between differential equations and difference equations. Parallel to these developments, fractional calculus has emerged as an effective mathematical tool for modeling systems with memory and hereditary properties. While constant-order fractional differential equations have been widely applied in various fields, recent advances have led to the development of multi-order and variable-order fractional derivatives, allowing the memory characteristics of a system to vary across different time intervals or operating regimes. This concept was initiated by Samko and Ross [22] and subsequently developed through applications in mechanics and viscoelasticity [17], approximate solutions of partial differential equations [24], and fractional derivatives defined with respect to another function [3]. These developments significantly enhance the modeling flexibility of fractional-order systems.

In many practical applications, system dynamics are further influenced by uncertainty and imprecision, which motivates the use of fuzzy fractional calculus. By representing states and parameters as fuzzy-valued quantities, fuzzy fractional differential equations provide an appropriate framework for handling incomplete or ambiguous information. Existing approaches to fuzzy fractional calculus can be broadly classified into three main frameworks. The first is based on the Hukuhara derivative and its generalizations, which played a foundational role in early developments but suffers from restrictive existence conditions for fuzzy solutions [2]. In recent years, considerable attention has been devoted to

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the qualitative analysis of fuzzy fractional differential systems under the Hukuhara derivative approach. Eidinejad and Saadati [11] investigated the Hyers-Ulam-Rassias-Kummer stability of fractional integro-differential equations, providing new insights into the stability behavior of fractional models under perturbations. Abuasbeh et al. [1] established local and global existence as well as uniqueness results for a class of fuzzy fractional functional evolution equations by employing fixed-point techniques. Khan et al. [16] further studied fuzzy delay impulsive fractional differential equations and derived existence and controllability results in credibility spaces. For fractional orders $\alpha \in (1,2)$, Shafqat et al. [23] developed existence and uniqueness theories for fuzzy fractional evolution equations with uncertainty and obtained mild solution representations under Caputo-type fractional derivatives. These contributions have significantly enriched the theoretical foundations of fuzzy fractional dynamical systems under the fractional Hukuhara derivative and stimulated further investigations into more sophisticated models involving memory effects, impulsive phenomena, and uncertainty. The second framework is based on the fuzzy interactive derivative and linearly correlated fuzzy sets, providing improved mathematical consistency and leading to the formulation of Caputo-type Fréchet fractional derivatives in fuzzy settings [13]. The third framework is associated with granular and multidimensional fuzzy arithmetic based on horizontal membership functions, as developed by Mazandarani and Piegat [18, 21], Hoa et al. [14, 15], and offers greater flexibility in fuzzy computations.

In addition to these frameworks, constrained fuzzy arithmetic (CFA) has been proposed to recover essential algebraic properties of real-number arithmetic within fuzzy interval computations [10]. CFA has demonstrated practical relevance in applied problems, such as modeling the spread of the SARS-CoV-2 virus [7]. However, arithmetic operations under CFA do not always produce well-defined fuzzy intervals, which may limit its applicability to the analysis of fuzzy dynamical systems. To address this limitation, Chalco-Cano et al. [8] introduced a generalized single-level CFA framework in which fuzzy arithmetic operations are defined directly at a single ξ-level. This formulation guarantees the well-posedness of fuzzy results while preserving the desirable algebraic properties of CFA, making it particularly suitable for the study of fuzzy fractional differential equations with hybrid characteristics.

Although the explicit solution formula established in [20] represents an important first step toward the analysis of fuzzy fractional differential equations with PCA, the obtained results are restricted to Caputo-type gH-differentiable fuzzy systems of order in $(0,1)$. Such an approach relies on the existence of the generalized Hukuhara difference, which is known to impose restrictive conditions on fuzzy-valued solutions and may exclude a broad class of dynamical behaviors. More recently, the short-memory variable-order framework developed in [19] extended the analysis to fuzzy systems with PCA involving variable fractional orders in the interval $(0,1)$. Nevertheless, the higher-order case $\mathfrak{q}_\imath \in (1,2)$ remains unexplored. From an analytical viewpoint, the solution structure associated with orders in $(1,2)$ is fundamentally different from that of the case $(0,1)$. The presence of higher-order fractional memory leads to substantially more complicated solution representations and requires new analytical techniques for handling the interaction between piecewise constant arguments, kernel-dependent fractional operators, and multi-order dynamics. To the best of our knowledge, no explicit solution theory has been established for fuzzy differential equations with PCA involving generalized multi-order Caputo-type fractional derivatives of order $\mathfrak{q}_\imath \in (1,2)$ within a mathematically consistent fuzzy arithmetic framework. The present work fills this gap by employing generalized single-level CFA, thereby avoiding the restrictive assumptions associated with Hukuhara differentiability while simultaneously incorporating kernel-dependent multi-order fractional operators and short-memory piecewise fractional dynamics. Within this setting, we investigate a class of non-homogeneous linear fuzzy fractional differential equations with piecewise constant arguments (LFFDEs-PCA). Specifically, we consider the following system:

$$\begin{cases} {}^{C}D_{\imath^+}^{\mathfrak{q}_\imath;\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t}) = \Omega_1 \otimes w(\mathfrak{t}) + \Omega_2 \otimes w([\mathfrak{t}]) + \mathfrak{F}(\mathfrak{t}), \quad \mathfrak{t} \in [\imath, \imath+1), \quad \imath = 0,1,2,\dots,N, \\ w(0) = w_0, \quad w^{(1)}(0) = w_1, \end{cases} \tag{1}$$

where $\mathfrak{q}_\imath \in (1,2)$ for all $\imath \in 0,1,\dots,N$. The function $\mathfrak{K}$ is a real-valued, absolutely continuous, positive, and strictly increasing function satisfying $\mathfrak{K}'(\mathfrak{t}) \neq 0$ for all $\mathfrak{t} \in [0,N]$. The coefficients Ω_1 and Ω_2 are non-zero real numbers, $w(\mathfrak{t})$ is a fuzzy-valued function with initial conditions $w(0) = w_0$ and $w^{(1)}(0) = w_1$, and $\mathfrak{F} : [0,N] \to \mathbb{E}$ is a continuous fuzzy function. The assumptions imposed on $\mathfrak{K}(\mathfrak{t}), \Omega_1, \Omega_2, \mathfrak{F}(\mathfrak{t})$, and the fuzzy initial data play an essential role in the analysis. They guarantee that the generalized fractional operators are well defined and allow the variation-of-constants method to be applied in deriving an explicit solution representation on each subinterval. All algebraic operations are interpreted in the sense of single-level CFA, and the operator ${}^{C}D_{\imath^+}^{\mathfrak{q}_\imath;\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t})$ denotes the generalized multi-order Caputo fractional derivative in the sense of single-level CFA. It is worth noting that the lower terminal of the fractional derivative is updated at each discontinuity point $\imath$, which effectively suppresses the influence of the remote past on subsequent system evolution. This leads to a short-memory fractional structure that enables the proposed model to balance local dynamics and memory effects in hybrid fuzzy systems. From a modeling perspective, such a mechanism is particularly suitable for systems involving periodic updates, switching actions, repeated decision-making processes, or abrupt changes in operating conditions. In these situations, the current dynamics are primarily influenced by the most recent history after the latest update, rather than by the entire past trajectory. Therefore, the proposed framework provides a natural description of hybrid fuzzy systems whose memory is repeatedly refreshed during operation. The main objective of this paper is to derive explicit solution formulas for system (1) and to facilitate the analysis of fuzzy fractional differential equations with piecewise constant arguments involving generalized multi-order fractional dynamics. To illustrate the practical relevance of the proposed framework, an application to a fuzzy vehicle-following system is also presented.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries, definitions, and notation. In Section 3 , we construct explicit solution representations. Section 4 presents an application to a fuzzy vehicle-following system together with numerical illustrations of the theoretical results. Finally, Section 5 concludes the paper and outlines directions for future research.

2 Preliminaries

This section outlines a new framework for defining arithmetic operations in the space of fuzzy numbers, referred to as single-level constrained fuzzy arithmetic (CFA) within fuzzy interval analysis, which was originally introduced in [8]. Under this framework, arithmetic operations on fuzzy intervals are implemented through constrained interval arithmetic applied to each ξ-cut set [10]. Rather than relying on classical fuzzy interval arithmetic, CFA is designed as an extension of traditional interval arithmetic to restore essential algebraic properties of real numbers. As a result, several inherent drawbacks of conventional fuzzy interval arithmetic are resolved, including the lack of additive and multiplicative inverses and the breakdown of the distributive law.

In fuzzy calculus, a standard notation system is commonly used; see [5, 9] for details. Let $\mathbb{E}$ denote the set of all fuzzy numbers $w : \mathbb{R} \to [0, 1]$ satisfying the usual properties of normality, upper semi-continuity, fuzzy convexity, and compact support. For each $\xi \in (0, 1]$, the corresponding ξ-cut set of w is defined by $[w]^\xi = \{t_0 \in \mathbb{R} \mid w(t_0) \geq \xi\}$, whereas for $\xi = 0$ it is defined as $[w]^0 = \mathrm{cl}\{t_0 \in \mathbb{R} \mid w(t_0) > 0\}$. These ξ-cut sets are compact and admit an interval representation of the form $[w]^\xi = [\underline{w}(\xi), \overline{w}(\xi)]$.

Definition 2.1. [9] *Let $\mathscr{L}, \mathscr{R} : [0, 1] \to [0, 1]$ be two continuous and increasing functions satisfying the boundary conditions $\mathscr{L}(0) = \mathscr{R}(0) = 0$ and $\mathscr{L}(1) = \mathscr{R}(1) = 1$. Consider real numbers $a_1 \leq a_2 \leq a_3 \leq a_4$. A fuzzy set $w : \mathbb{R} \to [0, 1]$ is called an $\mathscr{L}$–$\mathscr{R}$ fuzzy number if it admits the following representation:*

$$w(x) = \begin{cases} \mathscr{L}\left(\dfrac{x - a_1}{a_2 - a_1}\right), & \text{if} \quad a_1 \leq x < a_2 \\ 1, & \text{if} \quad a_2 \leq x \leq a_3 \\ \mathscr{R}\left(\dfrac{a_4 - x}{a_4 - a_3}\right), & \text{if} \quad a_3 < x \leq a_4 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

The fuzzy number w is symbolically represented as $w = (a_1, a_2, a_3, a_4)_{\mathscr{L},\mathscr{R}}$. For an $\mathscr{L}$–$\mathscr{R}$ fuzzy number w, the corresponding ξ-cut sets are determined by the formulas given below:

$$[w]^\xi = \left[a_1 + (a_2 - a_1)\mathscr{L}^{-1}(\xi), a_4 - (a_4 - a_3)\mathscr{R}^{-1}(\xi)\right]. \tag{3}$$

When the functions $\mathscr{L}$ and $\mathscr{R}$ are linear, a notable special case emerges, namely the class of trapezoidal fuzzy numbers. In this case, the trapezoidal fuzzy number w is specified by a quadruple $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ satisfying $a_1 \leq a_2 \leq a_3 \leq a_4$. Moreover, if $a_2 = a_3$, the fuzzy number w is called a triangular fuzzy number.

In what follows, we briefly review several fundamental concepts of fuzzy interval analysis based on the single-level CFA operations approach. The necessary theoretical background is drawn from [8].

Definition 2.2. *Consider a fuzzy number $w \in \mathbb{E}$ whose ξ-cut sets take the form $[w]^\xi = [\underline{w}(\xi), \overline{w}(\xi)]$ for every $\xi \in [0, 1]$. In line with the approach of* [8]*, w can be written in terms of its lower and upper bounds as shown below:*

$$\begin{aligned} &w^{cf} : [0, 1] \times [0, 1] \to \mathbb{R} \\ &(\xi, \alpha) \mapsto w^{cf}(\xi, \alpha) = \underline{w}(\xi) + (\overline{w}(\xi) - \underline{w}(\xi))\alpha, \end{aligned} \tag{4}$$

or

$$\begin{aligned} &w^{cf}:[0,1]\times[0,1]\to\mathbb{R} \\ &(\xi,\alpha)\mapsto w^{cf}(\xi,\alpha)=\overline{w}(\xi)+(\underline{w}(\xi)-\overline{w}(\xi))\alpha. \end{aligned} \tag{5}$$

The formulas (4) and (5) are referred to as the increasing convex constraint function and the decreasing convex constraint function of $w\in\mathbb{E}$, respectively. It is important to note that these definitions of $w\in\mathbb{E}$ correspond to real-valued functions of two variables $(\xi,\alpha)\in[0,1]\times[0,1]$. Each fuzzy interval is associated with one unique decreasing convex constraint function and one unique increasing convex constraint function. Nevertheless, different constraint functions can describe the same fuzzy interval.

For convenience, we denote the constraint function associated with the fuzzy number w by $\mathscr{C}(w)=w^{\mathrm{cf}}(\xi,\alpha)$. Moreover, the following formula will be employed to reconstruct the ξ-cut sets of $w\in\mathbb{E}$ from its constraint function for any $\xi\in[0,1]$ (see Theorem 4 in [8]):

$$\mathscr{C}^{-1}\left(w^{\mathrm{cf}}(\xi,\alpha)\right):=[w]^{\xi}=\left[\inf_{\xi_*\geqslant\xi}\min_{\alpha\in[0,1]}w^{\mathrm{cf}}(\xi_*,\alpha),\sup_{\xi_*\geqslant\xi}\max_{\alpha\in[0,1]}w^{\mathrm{cf}}(\xi_*,\alpha)\right]. \tag{6}$$

Theorem 5 in [8] establishes that the level sets $\mathscr{C}^{-1}\left(w^{\mathrm{cf}}(\xi,\alpha)\right)$ form a valid fuzzy interval, that is, $\mathscr{C}^{-1}\left(w^{\mathrm{cf}}(\xi,\alpha)\right)$ defined by (6) is well defined. It has also been shown that decreasing and increasing convex constraint functions are dual to one another. In the present work, arithmetic operations are developed using decreasing convex constraint functions, although an equivalent construction can be obtained from increasing convex constraint functions. Based on this idea, a fuzzy interval arithmetic framework, called single-level constrained fuzzy interval arithmetic, is employed. The framework uses the constraint function associated with a fuzzy number and performs all interval operations at the same parameter level $\alpha\in[0,1]$. This approach is particularly suitable for modeling situations in which the values within a fuzzy interval evolve simultaneously from the lower bound to the upper bound.

Definition 2.3. *The ordering relation on the space $\mathbb{E}$ is defined as follows:*

(i) *A fuzzy number w_1 is said to be less than or equal to another fuzzy number w_2, denoted by $w_1\preceq_{cf}w_2$, if the condition $\mathscr{C}(w_1)\le\mathscr{C}(w_2)$ holds for every $\alpha\in[0,1]$.*

(ii) *Two fuzzy numbers w_1 and w_2 are regarded as equal, written as $w_1=w_2$, if and only if both $w_1\preceq_{cf}w_2$ and $w_2\preceq_{cf}w_1$ are satisfied. Equivalently, this is the case when $\mathscr{C}(w_1)=\mathscr{C}(w_2)$ for all $\alpha\in[0,1]$.*

Definition 2.4. *Let $*$ be a binary arithmetic operation on $\mathbb{R}$. For two fuzzy numbers w_1 and w_2 with increasing convex constraint functions $\mathscr{C}(w_1)$ and $\mathscr{C}(w_2)$, the induced arithmetic operations on $\mathbb{E}$ are defined symbolically as follows:*

$$\mathscr{C}(w_1\circledast w_2)\triangleq\mathscr{C}(w_1)*\mathscr{C}(w_2). \tag{7}$$

Here, the symbol $\circledast$ denotes arithmetic operations on $\mathbb{E}$, including addition, subtraction, multiplication, and division. When $\circledast$ corresponds to division, it is assumed that $\mathscr{C}(w_2):=w_2^{cf}(\xi,\alpha)\neq0$ for all $(\xi,\alpha)\in[0,1]^2$.

This fuzzy interval arithmetic operates in a levelwise manner. Let w_1 and w_2 be two fuzzy numbers with ξ-cut sets $[w_1]^\xi=[\underline{w_1}(\xi),\overline{w_1}(\xi)]$ and $[w_2]^\xi=[\underline{w_2}(\xi),\overline{w_2}(\xi)]$ for all $\xi\in[0,1]$. For each $\alpha\in[0,1]$, the corresponding values are taken from the increasing convex constraint functions $\mathscr{C}(w_1)$ and $\mathscr{C}(w_2)$, and the operation is carried out in $\mathbb{R}$. Repeating this process over all $\alpha\in[0,1]$ allows the extreme values of the resulting fuzzy interval to be recovered via formula (6).

Remark 2.1. *Arithmetic operations between fuzzy intervals performed through their constraint functions are referred to as single-level CFA operations. The resulting fuzzy number, denoted by $\mathscr{C}(w_1\circledast w_2)$, requires the application of formula (6) to recover the ξ-cut sets of $E:=w_1\circledast w_2$. Equivalently,*

$$[w_1\circledast w_2]^{\xi}=\left[\inf_{\xi_*\geqslant\xi}\min_{\alpha\in[0,1]}E^{cf}(\xi_*,\alpha),\sup_{\xi_*\geqslant\xi}\max_{\alpha\in[0,1]}E^{cf}(\xi_*,\alpha)\right]. \tag{8}$$

When reconstructed using equation (8), the results of these arithmetic operations always satisfy the requirements of a valid fuzzy interval (see [8]). For convenience, the symbols $+,\ominus,\otimes$, and $\oslash$ are used to denote addition, subtraction, multiplication, and division, respectively, in the space of fuzzy numbers under single-level CFA operations.

Example 2.5. *To demonstrate the single-level CFA operations, we introduce the $\mathscr{L}$–$\mathscr{R}$ fuzzy numbers u_1, u_2, and u_3 as follows:*

$$u_1(x)=\begin{cases} x-1, & \text{if } x\in[1,2)\\ 1, & \text{if } x\in[2,3]\\ 4-x, & \text{if } x\in(3,4]\\ 0, & \text{otherwise}\end{cases},\ u_2(x)=\begin{cases}\dfrac{x-1}{2}, & \text{if } x\in[1,3)\\ \dfrac{5-x}{2}, & \text{if } x\in[3,5]\\ 0, & \text{if } x\notin[1,5]\end{cases},\ u_3(x)=\begin{cases}(x-1)^2, & \text{if } x\in[1,2)\\ (3-x)^2, & \text{if } x\in[2,3]\\ 0, & \text{if } x\notin[1,3]\end{cases}. \tag{9}$$

From the formula (3), one obtains the ξ-cut sets of u_1, u_2, and u_3 for all $\xi\in[0,1]$ as follows:

$$[u_1]^\xi=[1+\xi,4-\xi],\quad [u_2]^\xi=[1+2\xi,5-2\xi],\quad [u_3]^\xi=[1+\sqrt{\xi},3-\sqrt{\xi}]. \tag{10}$$

Using the formula (4) of Definition 2.2, the increasing convex constraint functions of u_1, u_2, and u_3 are given by

$$\mathscr{C}(u_1)=1+\xi+(3-2\xi)\alpha,\ \mathscr{C}(u_2)=1+2\xi+(4-4\xi)\alpha,\ \mathscr{C}(u_3)=1+\sqrt{\xi}+(2-2\sqrt{\xi})\alpha, \tag{11}$$

where $(\xi,\alpha)\in[0,1]^2$. According to Definition 2.4, the arithmetic operations of fuzzy numbers u_1, u_2, u_3 can be computed directly from their increasing convex constraint function. In particular, we obtain:

$$\mathscr{C}(u_1+u_2)=\mathscr{C}(u_1)+\mathscr{C}(u_2)=2+3\xi+(7-6\xi)\alpha, \tag{12}$$
$$\mathscr{C}(u_2\ominus u_3)=\mathscr{C}(u_2)-\mathscr{C}(u_3)=(2\xi-\sqrt{\xi})+(2-4\xi+2\sqrt{\xi})\alpha, \tag{13}$$
$$\mathscr{C}(-2\otimes u_3)=-2\mathscr{C}(u_3)=-2-2\sqrt{\xi}-(4-4\sqrt{\xi})\alpha, \tag{14}$$
$$\mathscr{C}(u_1\otimes u_2)=\mathscr{C}(u_1)\mathscr{C}(u_2)=1+3\xi+2\xi^2+(7+4\xi-8\xi^2)\alpha+(12-20\xi+8\xi^2)\alpha^2, \tag{15}$$

where $(\xi,\alpha)\in[0,1]^2$. By solving an optimization problem with the variable $\alpha\in[0,1]$ and a fixed $\xi\in[0,1]$, and by applying the formula (8) in Remark 2.1, one obtains the following ξ-cut sets:

$$[u_1+u_2]^\xi=[2+3\xi,9-3\xi], \tag{16}$$
$$[-2\otimes u_3]^\xi=[-6+2\sqrt{\xi},-2-2\sqrt{\xi}], \tag{17}$$
$$[u_1\otimes u_2]^\xi=[1+3\xi+2\xi^2,20-13\xi+2\xi^2], \tag{18}$$
$$[u_2\ominus u_3]^\xi=\begin{cases}[-1/8,17/8], & 0\le\xi\le 1/16\\ [2\xi-\sqrt{\xi},2-2\xi+\sqrt{\xi}], & 1/16\le\xi\le 1\end{cases}. \tag{19}$$

Furthermore, the fuzzy numbers (u_1+u_2), $(u_2\ominus u_3)$, $(-2\otimes u_3)$, and $(u_1\otimes u_2)$, with their ξ-cut sets given in Eqs. (16)–(19), can be expressed as membership functions in the following form:

$$(u_1+u_2)(x)=\begin{cases}\dfrac{x-2}{3}, & \text{if } x\in[2,5)\\ 1, & \text{if } x\in[5,6]\\ \dfrac{9-x}{3}, & \text{if } x\in(6,9]\\ 0, & \text{if } x\notin[2,9]\end{cases},\ (u_2\ominus u_3)(x)=\begin{cases}0, & \text{if } x<-1/8\\ \left(\dfrac{1+\sqrt{1+8x}}{4}\right)^2, & \text{if } x\in[-1/8,1]\\ \left(\dfrac{1+\sqrt{17-8x}}{4}\right)^2, & \text{if } x\in[1/8,17/8]\\ 0, & \text{if } x>17/8\end{cases},$$

$$(-2\otimes u_3)(x)=\begin{cases}\dfrac{(x+6)^2}{4}, & \text{if } x\in[-6,-4)\\ \dfrac{(-x-2)^2}{4}, & \text{if } x\in[-4,-2]\\ 0, & \text{if } x\notin[-6,-2]\end{cases},\ (u_1\otimes u_2)(x)=\begin{cases}\dfrac{\sqrt{8x+1}-3}{4}, & \text{if } x\in[1,6)\\ 1, & \text{if } x\in[6,9]\\ \dfrac{13-\sqrt{8x+9}}{4}, & \text{if } x\in(9,20]\\ 0, & \text{if } x\notin[1,20]\end{cases}.$$

Proposition 2.6. [8, Proposition 17] *Let $u_1,u_2,u_3\in\mathbb{E}$. Then, the following properties hold:*

(i) $u_1\ominus u_1=\hat{0}$, *where $\hat{0}$ is the zero fuzzy number;*

(ii) $u_1+(-1)\otimes u_2=u_1\ominus u_2$ *and* $u_1\ominus(-1)\otimes u_2=u_1+u_2$;

(iii) $\hat{0} \ominus (u_1 \ominus u_2) = u_2 \ominus u_1$ *and* $u_1 \ominus u_2 = (-1) \otimes (u_2 \ominus u_1) = (-1) \otimes u_2 \ominus (-1) \otimes u_1$;

(iv) $(u_1 + u_2) \ominus u_2 = u_1$ *and* $(u_1 \ominus u_2) \ominus u_1 = (-1) \otimes u_2$;

(v) $u_1 \oslash u_1$, *with* $0 \notin [u_1]^0$, *always exists and* $u_1 \oslash u_1 = \chi_{\{1\}}$, *where* $\chi_{\{A\}}$ *denotes the characteristic function of set A;*

(vi) $u_1 \otimes (\chi_{\{1\}} \oslash u_1) = \chi_{\{1\}}$ *and* $u_2 \otimes (\chi_{\{1\}} \oslash u_1) = u_2 \oslash u_1$, *with* $0 \notin [u_1]^0$;

(vii) $(u_2 \ominus u_1) \oslash u_1 = (u_2 \oslash u_1) \ominus \chi_{\{1\}}$ *and* $(u_1 \otimes u_2) \oslash u_1 = u_2$, *with* $0 \notin [u_1]^0$;

(viii) $u_1 \otimes (u_2 + u_3) = u_1 \otimes u_2 + u_1 \otimes u_3$ *and* $u_1 \otimes (u_2 \ominus u_3) = u_1 \otimes u_2 \ominus u_1 \otimes u_3$;

(ix) $u_2 \oslash u_1 + u_3 \oslash u_1 = (u_2 + u_3) \oslash u_1$.

Definition 2.7. *The distance between* $u_1, u_2 \in \mathbb{E}$ *is defined by the mapping* $\|\cdot\|_C : \mathbb{E} \times \mathbb{E} \to \mathbb{R}^+ \cup \{0\}$ *as follows:*

$$d_C[u_1, u_2] := \|u_1 \ominus u_2\|_C = \sup_{\xi \in [0,1]} \max_{\alpha \in [0,1]} \Big| \mathscr{C}(u_1) - \mathscr{C}(u_2) \Big|. \tag{20}$$

Here, $\mathscr{C}(u_i) := u_i^{cf}(\xi, \alpha)$, *for* $i = 1, 2$, *denotes the corresponding increasing convex constraint function.*

Equipped with the metric d_C, the space $\mathbb{E}$ is a complete metric space. Within this framework, continuity and convergence of a fuzzy-valued function $w : [a, b] \to \mathbb{E}$ are defined via the metric (20). We denote by $\mathbb{C}([a, b], \mathbb{E})$ the set of all continuous fuzzy functions from $[a, b]$ into $\mathbb{E}$.

Definition 2.8. *Let* $w : (a, b) \to \mathbb{E}$ *be a fuzzy-valued function. We say that w is differentiable in the sense of single-level CFA (abbreviated as cf-differentiable) on* (a, b) *if, for every* $\mathfrak{t} \in (a, b)$*, the following conditions hold:*

(i) *(First-order cf-derivative) There exists a fuzzy number* $w'(\mathfrak{t}) \in \mathbb{E}$*, and it is defined by*

$$\lim_{l \to 0} \frac{1}{l} \otimes (w(\mathfrak{t} + l) \ominus w(\mathfrak{t})) = w^{(1)}(\mathfrak{t}). \tag{21}$$

(ii) *(Second-order cf-derivative) There exists a fuzzy number* $w^{(2)}(\mathfrak{t}) \in \mathbb{E}$*, and it is defined by*

$$\lim_{l \to 0} \frac{1}{l} \otimes \Big(w^{(1)}(\mathfrak{t} + l) \ominus w^{(1)}(\mathfrak{t}) \Big) = w^{(2)}(\mathfrak{t}). \tag{22}$$

A fuzzy function w is cf-differentiable of order $k \in \{1, 2\}$ on $[a, b]$ if it is cf-differentiable of order k at every $\mathfrak{t} \in (a, b)$, right cf-differentiable of order k at a, and left cf-differentiable of order k at b. We denote by $\mathbb{C}^k([a, b], \mathbb{E})$ the space of all fuzzy functions that are continuously cf-differentiable of order k on $[a, b]$, and use the notation $\dfrac{d^k}{d\mathfrak{t}^k} w(\mathfrak{t}) = w^{(k)}(\mathfrak{t})$ for $k \in \{1, 2\}$. Moreover, cf-differentiability of order k of a fuzzy function $w : (a, b) \to \mathbb{E}$ at $\mathfrak{t} \in (a, b)$ is equivalent to the differentiability of order k of its associated constraint function $w^{\mathrm{cf}}(\xi, \mathfrak{t}, \alpha)$ with respect to $\mathfrak{t}$. Consequently, the following relation holds

$$\mathscr{C}(w^{(k)}(\mathfrak{t})) = \frac{\partial^k w^{\mathrm{cf}}(\xi, \mathfrak{t}, \alpha)}{\partial^k \mathfrak{t}}, \qquad k \in \{1, 2\}. \tag{23}$$

For notational convenience, we denote by $\mathcal{K}_{\mathfrak{K}}$ the class of continuously differentiable functions that are nonnegative, strictly increasing, and possess a nonzero derivative for all $\mathfrak{t} \in [a, \infty)$. To introduce the general Caputo fractional derivative of order $\mathfrak{q} \in (1, 2)$ in the sense of single-level CFA operations, we first recall several function spaces from general fractional calculus within the fuzzy framework. Let $\mathfrak{K} \in \mathcal{K}_{\mathfrak{K}}$, and define

$$\mathbb{C}_{\epsilon,\mathfrak{K}}([a, b], \mathbb{E}) = \{w : (a, b] \to \mathbb{E} \text{ such that } (\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(a))^{\epsilon} \otimes w(\mathfrak{t}) \in \mathbb{C}([a, b], \mathbb{E})\}, \tag{24}$$

where $\mathbb{C}_{0,\mathfrak{K}}([a, b], \mathbb{E}) = \mathbb{C}([a, b], \mathbb{E})$; and

$$\mathbb{C}^2_{\epsilon,\mathfrak{K}}([a, b], \mathbb{E}) = \Big\{ w : (a, b] \to \mathbb{E} \text{ such that } w^{[2]} \in C_{\epsilon,\mathfrak{K}}([a, b], \mathbb{E}) \Big\}, \tag{25}$$

where $w^{[k]} = \left(\dfrac{1}{\mathfrak{K}'(\mathfrak{t})} \otimes \dfrac{d}{d\mathfrak{t}} \right)^k w$, for $k \in \{1, 2\}$, and $C^2_{0,\mathfrak{K}}([a, b], \mathbb{E}) = C^2([a, b], \mathbb{E})$.

Definition 2.9. *Let* $\mathfrak{K} \in \mathcal{K}_{\mathfrak{K}}$, $\mathfrak{q} > 0$, *and* $w \in \mathbb{C}_{\epsilon,\mathfrak{K}}([a,b],\mathbb{E})$. *The general fractional integral of* w, *defined in the sense of single-level CFA operations, is given by*

$$I_{a^+}^{\mathfrak{q};\mathfrak{K}} w(\mathfrak{t}) = \frac{1}{\Gamma(\mathfrak{q})} \int_a^{\mathfrak{t}} \mathscr{G}_{\mathfrak{q}-1}^{\mathfrak{K}}(\mathfrak{t};s) \otimes w(s)ds, \tag{26}$$

where $\mathscr{G}_{\mathfrak{q}-1}^{\mathfrak{K}}(\mathfrak{t};s) := \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}-1}$. *Moreover, the constraint function of* $I_{a^+}^{\mathfrak{q};\mathfrak{K}} w(\mathfrak{t})$ *is given:*

$$\mathscr{C}\left(I_{a^+}^{\mathfrak{q};\mathfrak{K}} w(\mathfrak{t})\right) = \frac{1}{\Gamma(\mathfrak{q})} \int_a^{\mathfrak{t}} \mathscr{G}_{\mathfrak{q}-1}^{\mathfrak{K}}(\mathfrak{t};s)\mathscr{C}(w(s))ds = \frac{1}{\Gamma(\mathfrak{q})} \int_a^{\mathfrak{t}} \mathscr{G}_{\mathfrak{q}-1}^{\mathfrak{K}}(\mathfrak{t};s) w^{cf}(\xi,s,\alpha)ds, \tag{27}$$

where $(\xi,\mathfrak{t},\alpha) \in [0,1] \times (a,b] \times [0,1]$.

Definition 2.10. *Let* $\mathfrak{q} \in (1,2)$, $\mathfrak{K} \in \mathcal{K}_{\mathfrak{K}}$, *and let* $w : [a,b] \to \mathbb{E}$ *satisfy* $w \in \mathbb{C}^2_{\epsilon,\mathfrak{K}}([a,b],\mathbb{E})$. *The general Caputo fractional derivative of* w, *defined in the sense of single-level CFA operations (cf-Caputo FD), is given:*

$${}^C D_{a^+}^{\mathfrak{q};\mathfrak{K}} w(\mathfrak{t}) = \frac{1}{\Gamma(2-\mathfrak{q})} \int_a^{\mathfrak{t}} \mathscr{G}_{1-\mathfrak{q}}^{\mathfrak{K}}(\mathfrak{t};s) \left(\frac{1}{\mathfrak{K}'(s)} \otimes \frac{d}{ds}\right)^2 w(s)ds, \quad \mathfrak{t} > a. \tag{28}$$

Here, $\left(\frac{1}{\mathfrak{K}'(s)} \otimes \frac{d}{ds}\right)^2 z(s)$ *denotes the repeated application of the operator* $\left(\frac{1}{\mathfrak{K}'(s)} \otimes \frac{d}{ds}\right)$ *to* $z(s)$. *Moreover, the constraint function associated with* ${}^C D_{a^+}^{\mathfrak{q};\mathfrak{K}} w(\mathfrak{t})$ *is given by*

$$\mathscr{C}\left({}^C D_{a^+}^{\mathfrak{q};\mathfrak{K}} w(\mathfrak{t})\right) = \frac{1}{\Gamma(2-\mathfrak{q})} \int_a^{\mathfrak{t}} \mathscr{G}_{1-\mathfrak{q}}^{\mathfrak{K}}(\mathfrak{t};s) \left(\frac{1}{\mathfrak{K}'(s)} \frac{d}{ds}\right)^2 w(\xi,s,\alpha)ds, \tag{29}$$

where $\mathfrak{t} > a$, *and* $(\xi,\alpha) \in [0,1]^2$.

Remark 2.2. [4, Lemma 2] *Consider the following initial value problem:*

$$\begin{cases} {}^C D_{0^+}^{\mathfrak{q};\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t}) = \lambda \otimes w(\mathfrak{t}) + f(\mathfrak{t}), & \mathfrak{t} \in [0,\infty), \\ w(0) = w_0, \quad w^{(1)}(0) = w_1, \end{cases} \tag{30}$$

where $\lambda \in \mathbb{R}$, $\mathfrak{q} \in (1,2)$, $\mathfrak{K}(\mathfrak{t}) \in \mathcal{K}_{\mathfrak{K}}$, $f(\mathfrak{t}) \in \mathbb{E}$ *is a continuous fuzzy function, and* ${}^C D_{0^+}^{\mathfrak{q};\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t})$ *denotes the general Caputo FD of a fuzzy function* w. *The solution of the problem* (30) *is expressed as:*

$$\begin{aligned} w(\mathfrak{t}) = {} & E_{\mathfrak{q},1}(\lambda(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(0))^{\mathfrak{q}}) \otimes w_0 + \frac{1}{\mathfrak{K}'(0)}(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(0)) E_{\mathfrak{q},2}(\lambda(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(0))^{\mathfrak{q}}) \otimes w_1 \\ & + \int_0^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}-1} E_{\mathfrak{q},\mathfrak{q}}(\lambda(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}}) \otimes f(s)\, ds, \end{aligned} \tag{31}$$

where $E_{\mathfrak{q},\beta}(z)$ *is the two-parameter Mittag-Leffler function, with positive parameters* $\mathfrak{q}$ *and* β.

3 Representation of a solution to problem (1)

In this section, we examine the fuzzy fractional differential equations with piecewise constant arguments in the sense of single-level CFA operations, as described in problem (1).

Lemma 3.1. [12] *Consider the nonhomogeneous linear fuzzy difference system*

$$y(n+1) = A(n) \otimes y(n) + g(n), \quad y(n_0) = y_0, \tag{32}$$

where $A(n) = (a_{ij}(n))$ is a nonsingular 2×2 real-valued matrix function, $y_0 = (y_{10}, y_{20})^T$ and $g(n) = (g_1(n), g_2(n))^T$, where $y_{10}, y_{20}, g_1, g_2 \in \mathbb{E}$, for $n \geq n_0 \geq 0$, $n \in \mathbb{N}$. Then this system admits a unique solution given explicitly by

$$y(n) = \left(\prod_{i=n_0}^{n-1} A(i)\right) \otimes y_0 + \sum_{r=n_0}^{n-1} \left(\prod_{i=r+1}^{n-1} A(i)\right) \otimes g(r), \quad n \geq n_0. \tag{33}$$

Here, we adopt the standard conventions

$$\prod_{i=p+1}^{p} A(i) = I_2, \qquad \sum_{i=p+1}^{p} B(i) = 0_2, \tag{34}$$

where I_2 and 0_2 denote the identity matrix and the zero matrix of order 2, respectively.

Proof. For $n \geq m \geq n_0$, we set a real-valued matrix function

$$\Phi(n,m) := \prod_{i=m}^{n-1} A(i) = A(n-1)A(n-2)\cdots A(m), \tag{35}$$

where $A(n) = (a_{ij}(n))$ is a nonsingular 2×2 real-valued matrix function. Then, it follows from the standard convention (34) and the definition (35) that

$$\Phi(m,m) = I_2,\ \Phi(n+1,m) = A(n)\Phi(n,m),\ \Phi(n,m) = \Phi(n,l)\Phi(l,m), \quad \text{for } n \geq l \geq m. \tag{36}$$

Since the matrices $A(i)$ is invertible, $\Phi(n,m)$ is also invertible; moreover, one has

$$\Phi^{-1}(n,m) = \prod_{i=m}^{n-1} A^{-1}(i) = A^{-1}(m)A^{-1}(m+1)\cdots A^{-1}(n-1). \tag{37}$$

+ First, we seek the solution of Eq. (32) with the form:

$$y(n) = \Phi(n,n_0) \otimes z(n), \quad n \geq n_0, \tag{38}$$

where $z(n) = (z_1(n), z_2(n)) \in \mathbb{E} \times \mathbb{E}$ is a fuzzy function to be found. Substituting (38) into the nonhomogeneous linear fuzzy difference system (32) yields

$$y(n+1) = A(n) \otimes y(n) + g(n) = A(n)\Phi(n,n_0) \otimes z(n) + g(n) = \Phi(n+1,n_0) \otimes z(n) + g(n), \tag{39}$$

where we use the second property of (36). This is equivalent to

$$\Phi(n+1,n_0) \otimes z(n+1) = \Phi(n+1,n_0) \otimes z(n) + g(n). \tag{40}$$

Since $\Phi(n+1,n_0)$ is invertible, we multiply (40) by $\Phi^{-1}(n+1,n_0)$ and obtain

$$z(n+1) = z(n) + \Phi^{-1}(n+1,n_0) \otimes g(n). \tag{41}$$

From (41), we observe that for each specific value of n from n_0 to $N-1$

$$\begin{cases} z(n_0+1) = z(n_0) + \Phi^{-1}(n_0+1,n_0) \otimes g(n_0), \\ z(n_0+2) = z(n_0+1) + \Phi^{-1}(n_0+2,n_0) \otimes g(n_0+1), \\ \vdots \\ z(N-1) = z(N-2) + \Phi^{-1}(N-1,n_0) \otimes g(N-2), \\ z(N) = z(N-1) + \Phi^{-1}(N,n_0) \otimes g(N-1). \end{cases} \tag{42}$$

Summing Eq. (42) from $n = n_0$ to $n = N-1$ yields

$$z(N) + \sum_{r=n_0+1}^{N-1} z(r) = z(n_0) + \sum_{r=n_0+1}^{N-1} z(r) + \sum_{r=n_0}^{N-1} \Phi^{-1}(r+1,n_0) \otimes g(r). \tag{43}$$

Multiplying both sides of (43) by $\Phi(N,n_0)$, we get

$$\begin{aligned} y(N) &= \Phi(N,n_0)\otimes z(n_0) + \sum_{r=n_0}^{N-1}\Phi(N,n_0)\Phi^{-1}(r+1,n_0)\otimes g(r) \\ &= \Phi(N,n_0)\otimes z(n_0) + \sum_{r=n_0}^{N-1}\Phi(N,r+1)\Phi(r+1,n_0)\Phi^{-1}(r+1,n_0)\otimes g(r) \\ &= \Phi(N,n_0)\otimes z(n_0) + \sum_{r=n_0}^{N-1}\Phi(N,r+1)\otimes g(r), \end{aligned} \tag{44}$$

where we utilize the third property of (36), the definitions (35) and (37). In addition, by $y(n_0) = y_0$ and $\Phi(n_0, n_0) = I_2$ one implies from (38) that $z(n_0) = y_0$. Then, by (35), we can rewrite (45) in the following form:

$$y(N) = \left(\prod_{i=n_0}^{N-1} A(i)\right)\otimes y_0 + \sum_{r=n_0}^{N-1}\left(\prod_{i=r+1}^{N-1} A(i)\right)\otimes g(r). \tag{45}$$

By changing index N to n, we obtain the solution formula (33) for the nonhomogeneous linear fuzzy difference system (32).

+ Second, we verify that the formula (33) satisfies Eq. (32). Specifically, we show that the function $y(n) = Y_1(n)+Y_2(n)$ is a solution of Eq. (32), where we set

$$Y_1(n) = \left(\prod_{i=n_0}^{n-1} A(i)\right)\otimes y_0, \qquad Y_2(n) = \sum_{r=n_0}^{n-1}\left(\prod_{i=r+1}^{n-1} A(i)\right)\otimes g(r). \tag{46}$$

Observe that

$$Y_1(n+1) = \left(\prod_{i=n_0}^{n} A(i)\right)\otimes y_0 = A(n)\left(\prod_{i=n_0}^{n-1} A(i)\right)\otimes y_0 = A(n)\otimes Y_1(n), \tag{47}$$

and

$$\begin{aligned} Y_2(n+1) &= \sum_{r=n_0}^{n}\left(\prod_{i=r+1}^{n} A(i)\right)\otimes g(r) = \sum_{r=n_0}^{n-1}\left(\prod_{i=r+1}^{n} A(i)\right)\otimes g(r) + \left(\prod_{i=n+1}^{n} A(i)\right)\otimes g(n) \\ &= A(n)\sum_{r=n_0}^{n-1}\left(\prod_{i=r+1}^{n-1} A(i)\right)\otimes g(r) + g(n) = A(n)\otimes Y_2(n) + g(n), \end{aligned} \tag{48}$$

where we use the standard convention (34). Then, it follows from (47) and (48) that

$$y(n+1) = Y_1(n+1) + Y_2(n+1) = A(n)\otimes y(n) + g(n). \tag{49}$$

Furthermore, one also has

$$y(n_0) = Y_1(n_0) + Y_2(n_0) = \left(\prod_{i=n_0}^{n_0-1} A(i)\right)\otimes y_0 + \sum_{r=n_0}^{n_0-1}\left(\prod_{i=r+1}^{n_0-1} A(i)\right)\otimes g(r) = y_0 + 0_2 = y_0, \tag{50}$$

where we use the standard convention (34). Therefore, from (49) and (50), we deduce that the function y defined by (45) is a solution of Eq. (32).

+ Finally, in order to demonstrate the uniqueness of solution, we assume that z and $\widetilde{z}$ are two solutions of Eq. (32). Set $w(n) = z(n)\ominus\widetilde{z}(n)$. Then, by Proposition 2.6-(viii) one has that

$$w(n+1) = A(n)\otimes w(n), \qquad w(n_0) = 0. \tag{51}$$

By apply the formula (33), one gets the solution form of the homogeneous linear fuzzy difference system (51) as follows:

$$w(n) = \left(\prod_{i=n_0}^{n-1} A(i)\right)\otimes y_0 = 0, \quad \text{for all } n\geq n_0, \tag{52}$$

which implies $z(n)\equiv\widetilde{z}(n)$. Hence, the solution is unique.

□

Theorem 3.2. *Consider the linear fractional differential equation with a piecewise constant argument given by* (1)*. Suppose that* $\Omega_1 \neq 0$ *and* $\Omega_2 \neq 0$*. Then, the solution* $w(\mathfrak{t})$ *admits the following explicit representation:*

$$w(\mathfrak{t}) = \mathcal{M}(\mathfrak{t},\imath) \otimes w(\imath) + \mathcal{N}(\mathfrak{t},\imath) \otimes w^{(1)}(\imath) + \mathcal{I}(\mathfrak{t},\imath), \quad \imath = 0,1,2,...,N, \tag{53}$$

where $w(0) = w_0, w^{(1)}(0) = w_1$*, and, for* $\imath = 1,2,3,...,N$*,*

$$\begin{aligned}\begin{pmatrix} w(\imath) \\ w^{(1)}(\imath) \end{pmatrix} &= \prod_{\jmath=0}^{\imath-1} \begin{pmatrix} \mathcal{M}(\jmath+1,\jmath) & \mathcal{N}(\jmath+1,\jmath) \\ \mathcal{M}'(\jmath+1,\jmath) & \mathcal{N}'(\jmath+1,\jmath) \end{pmatrix} \otimes \begin{pmatrix} w(0) \\ w^{(1)}(0) \end{pmatrix} \\ &+ \sum_{k=0}^{\imath-1} \left(\prod_{\jmath=k+1}^{\imath-1} \begin{pmatrix} \mathcal{M}(\jmath+1,\jmath) & \mathcal{N}(\jmath+1,\jmath) \\ \mathcal{M}'(\jmath+1,\jmath) & \mathcal{N}'(\jmath+1,\jmath) \end{pmatrix} \right) \otimes \begin{pmatrix} \mathcal{I}(k+1,k) \\ \mathcal{I}'(k+1,k) \end{pmatrix},\end{aligned} \tag{54}$$

and the coefficient functions $\mathcal{M}(\mathfrak{t},\imath), \mathcal{N}(\mathfrak{t},\imath)$*, and* $\mathcal{I}(\mathfrak{t},\imath)$ *are defined by*

$$\mathcal{M}(\mathfrak{t},\imath) := \left(1 + \frac{\Omega_2}{\Omega_1}\right) E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}) - \frac{\Omega_2}{\Omega_1}, \tag{55}$$

$$\mathcal{N}(\mathfrak{t},\imath) := \frac{1}{\mathfrak{K}'(\imath)}(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath)) E_{\mathfrak{q}_\imath,2}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}), \tag{56}$$

$$\mathcal{I}(\mathfrak{t},\imath) := \int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath - 1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) \otimes \mathfrak{F}(s) ds. \tag{57}$$

Proof. From Eq. (1), we get, for $\mathfrak{t} \in [\imath, \imath+1), \imath = 1,2,...,N$,

$${}^C D_{\imath^+}^{\mathfrak{q}_\imath;\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t}) = \Omega_1 \otimes w(\mathfrak{t}) + \Omega_2 \otimes w(\imath) + \mathfrak{F}(\mathfrak{t}), \quad \mathfrak{t} \in [\imath, \imath+1). \tag{58}$$

By utilizing Remark 2.2, we obtain, for $\mathfrak{t} \in [\imath, \imath+1)$,

$$\begin{aligned} w(\mathfrak{t}) &= E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}) \otimes w(\imath) + \frac{1}{\mathfrak{K}'(\imath)}(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath)) E_{\mathfrak{q}_\imath,2}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}) \otimes w^{(1)}(\imath) \\ &+ \int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath - 1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) \Omega_2 \otimes w(\imath) ds \\ &+ \int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath - 1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) \otimes \mathfrak{F}(s) ds \\ &= \left(E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}) + \Omega_2 \int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath - 1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) ds \right) \otimes w(\imath) \\ &+ \frac{1}{\mathfrak{K}'(\imath)}(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath)) E_{\mathfrak{q}_\imath,2}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}) \otimes w^{(1)}(\imath) \\ &+ \int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath - 1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) \otimes \mathfrak{F}(s) ds. \end{aligned} \tag{59}$$

By direct calculation, we get

$$\begin{aligned} &\int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}-1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) ds = \sum_{\jmath=0}^{\infty} \frac{\Omega_1^\jmath}{\Gamma(\mathfrak{q}(\jmath+1))} \left(\int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}(\jmath+1)-1} ds \right) \\ &= \sum_{\jmath=0}^{\infty} \frac{\Omega_1^\jmath (\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}(\jmath+1)}}{\Gamma(\mathfrak{q}(\jmath+1)+1)} \overset{\jmath:=\jmath+1}{=} \frac{1}{\Omega_1} \sum_{\jmath=1}^{\infty} \frac{\Omega_1^\jmath (\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\jmath\mathfrak{q}}}{\Gamma(\jmath\mathfrak{q}+1)} = \frac{1}{\Omega_1} \left(E_{\mathfrak{q},1}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}}) - 1 \right). \end{aligned} \tag{60}$$

Substituting (60) into (59) yields, for $\mathfrak{t}\in[\imath,\imath+1)$,

$$w(\mathfrak{t})=\left(\left(1+\frac{\Omega_2}{\Omega_1}\right)E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath})-\frac{\Omega_2}{\Omega_1}\right)\otimes w(\imath)+\frac{1}{\mathfrak{K}'(\imath)}(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))E_{\mathfrak{q}_\imath,2}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath})\otimes w^{(1)}(\imath)$$

$$+\int_\imath^{\mathfrak{t}}\mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath})\otimes\mathfrak{F}(s)ds. \tag{61}$$

Set

$$\begin{cases}\mathcal{M}(\mathfrak{t},\imath) & :=\left(1+\frac{\Omega_2}{\Omega_1}\right)E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath})-\dfrac{\Omega_2}{\Omega_1}, \quad \mathfrak{t}\in[\imath,\imath+1),\\ \mathcal{N}(\mathfrak{t},\imath) & :=\dfrac{1}{\mathfrak{K}'(\imath)}(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))E_{\mathfrak{q}_\imath,2}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath}), \quad \mathfrak{t}\in[\imath,\imath+1),\\ \mathcal{I}(\mathfrak{t},\imath) & :=\int_\imath^{\mathfrak{t}}\mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath})\otimes\mathfrak{F}(s)ds, \quad \mathfrak{t}\in[\imath,\imath+1).\end{cases} \tag{62}$$

By substituting (62) into (61), we obtain (53). This means that

$$w(\mathfrak{t})=\mathcal{M}(\mathfrak{t},\imath)\otimes w(\imath)+\mathcal{N}(\mathfrak{t},\imath)\otimes w^{(1)}(\imath)+\mathcal{I}(\mathfrak{t},\imath), \quad \mathfrak{t}\in[\imath,\imath+1),\ \imath=0,1,2,...,N. \tag{63}$$

We now find $w(\imath)$ and $w^{(1)}(\imath)$ for $\imath=1,2,...,N$. From (62), we can get

$$\begin{aligned}\mathcal{M}'(\mathfrak{t},\imath)&=\left(1+\frac{\Omega_2}{\Omega_1}\right)\frac{d}{d\mathfrak{t}}\left(\sum_{\jmath=0}^{\infty}\frac{\Omega_1^\jmath(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath\jmath}}{\Gamma(\mathfrak{q}_\imath\jmath+1)}\right)=\left(1+\frac{\Omega_2}{\Omega_1}\right)\sum_{\jmath=1}^{\infty}\frac{\Omega_1^\jmath\mathfrak{K}'(\mathfrak{t})(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath\jmath-1}}{\Gamma(\mathfrak{q}_\imath\jmath)}\\ &\overset{j:=\jmath-1}{=}\left(1+\frac{\Omega_2}{\Omega_1}\right)\sum_{j=0}^{\infty}\frac{\Omega_1^{j+1}\mathfrak{K}'(\mathfrak{t})(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath j+\mathfrak{q}_\imath-1}}{\Gamma(\mathfrak{q}_\imath j+\mathfrak{q}_\imath)}\\ &=(\Omega_1+\Omega_2)\,\mathfrak{K}'(\mathfrak{t})(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath-1}\sum_{j=0}^{\infty}\frac{\Omega_1^j(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath j}}{\Gamma(\mathfrak{q}_\imath j+\mathfrak{q}_\imath)}\\ &=\mathfrak{K}'(\mathfrak{t})(\Omega_1+\Omega_2)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath}),\end{aligned} \tag{64}$$

$$\begin{aligned}\mathcal{N}'(\mathfrak{t},\imath)&=\frac{1}{\mathfrak{K}'(\imath)}\frac{d}{d\mathfrak{t}}\left(\sum_{\jmath=0}^{\infty}\frac{\Omega_1^\jmath(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath\jmath+1}}{\Gamma(\mathfrak{q}_\imath\jmath+2)}\right)=\frac{\mathfrak{K}'(\mathfrak{t})}{\mathfrak{K}'(\imath)}\sum_{\jmath=0}^{\infty}\frac{\Omega_1^\jmath(\mathfrak{q}_\imath\jmath+1)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath\jmath}}{\Gamma(\mathfrak{q}_\imath\jmath+2)}\\ &=\frac{\mathfrak{K}'(\mathfrak{t})}{\mathfrak{K}'(\imath)}E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath}),\end{aligned} \tag{65}$$

$$\begin{aligned}\mathcal{I}'(\mathfrak{t},\imath)&=\frac{d}{d\mathfrak{t}}\left(\int_\imath^{\mathfrak{t}}\mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath})\otimes\mathfrak{F}(s)ds\right)\\ &=\mathfrak{K}'(\mathfrak{t})\int_\imath^{\mathfrak{t}}\mathfrak{K}'(s)\sum_{\jmath=0}^{\infty}\frac{\Omega_1^\jmath(\mathfrak{q}_\imath\jmath+\mathfrak{q}_\imath-1)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath\jmath+\mathfrak{q}_\imath-2}}{\Gamma(\mathfrak{q}_\imath\jmath+\mathfrak{q}_\imath)}\otimes\mathfrak{F}(s)ds\\ &=\mathfrak{K}'(\mathfrak{t})\int_\imath^{\mathfrak{t}}\mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath-2}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath-1}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath})\otimes\mathfrak{F}(s)ds.\end{aligned} \tag{66}$$

Furthermore, from (61) and (62), one has

$$w^{(1)}(\mathfrak{t})=\mathcal{M}'(\mathfrak{t},\imath)\otimes w(\imath)+\mathcal{N}'(\mathfrak{t},\imath)\otimes w^{(1)}(\imath)+\mathcal{I}'(\mathfrak{t},\imath), \tag{67}$$

where $\mathcal{M}'(\mathfrak{t},\imath),\mathcal{N}'(\mathfrak{t},\imath)$ and $\mathcal{I}'(\mathfrak{t},\imath)$ are given by (64), (65) and (66), respectively. This means

$$\begin{cases}\mathcal{M}'(\mathfrak{t},\imath) & =\mathfrak{K}'(\mathfrak{t})(\Omega_1+\Omega_2)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath}), \quad \mathfrak{t}\in[\imath,\imath+1),\\ \mathcal{N}'(\mathfrak{t},\imath) & =\dfrac{\mathfrak{K}'(\mathfrak{t})}{\mathfrak{K}'(\imath)}E_{\mathfrak{q}_\imath,1}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(\imath))^{\mathfrak{q}_\imath}), \quad \mathfrak{t}\in[\imath,\imath+1),\\ \mathcal{I}'(\mathfrak{t},\imath) & =\mathfrak{K}'(\mathfrak{t})\int_\imath^{\mathfrak{t}}\mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath-2}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath-1}(\Omega_1(\mathfrak{K}(\mathfrak{t})-\mathfrak{K}(s))^{\mathfrak{q}_\imath})\otimes\mathfrak{F}(s)ds, \quad \mathfrak{t}\in[\imath,\imath+1).\end{cases} \tag{68}$$

Next, we determine $w(\imath)$ and $w^{(1)}(\imath)$, and show that they satisfy the formula (54) for $\imath = 0, 1, 2, \ldots, N$. It is easy to see that $w(0) = w_0$ and $w^{(1)}(0) = w_1$. Taking the limit as $\mathfrak{t} \to \imath + 1$ in (61) and (67) yields the following linear recurrence system:

$$\begin{pmatrix} w(\imath+1) \\ w^{(1)}(\imath+1) \end{pmatrix} = \begin{pmatrix} \mathcal{M}(\imath+1,\imath) & \mathcal{N}(\imath+1,\imath) \\ \mathcal{M}'(\imath+1,\imath) & \mathcal{N}'(\imath+1,\imath) \end{pmatrix} \otimes \begin{pmatrix} w(\imath) \\ w^{(1)}(\imath) \end{pmatrix} + \begin{pmatrix} \mathcal{I}(\imath+1,\imath) \\ \mathcal{I}'(\imath+1,\imath) \end{pmatrix}. \tag{69}$$

By applying Lemma 3.1 to Eq. (69), we obtain (54). This means that, for $\imath = 1, 2, 3, ..., N$,

$$\begin{aligned} \begin{pmatrix} w(\imath) \\ w^{(1)}(\imath) \end{pmatrix} &= \prod_{\jmath=0}^{\imath-1} \begin{pmatrix} \mathcal{M}(\jmath+1,\jmath) & \mathcal{N}(\jmath+1,\jmath) \\ \mathcal{M}'(\jmath+1,\jmath) & \mathcal{N}'(\jmath+1,\jmath) \end{pmatrix} \otimes \begin{pmatrix} w(0) \\ w^{(1)}(0) \end{pmatrix} \\ &\quad + \sum_{k=0}^{\imath-1} \left(\prod_{\jmath=k+1}^{\imath-1} \begin{pmatrix} \mathcal{M}(\jmath+1,\jmath) & \mathcal{N}(\jmath+1,\jmath) \\ \mathcal{M}'(\jmath+1,\jmath) & \mathcal{N}'(\jmath+1,\jmath) \end{pmatrix} \right) \otimes \begin{pmatrix} \mathcal{I}(k+1,k) \\ \mathcal{I}'(k+1,k) \end{pmatrix}. \end{aligned} \tag{70}$$

□

4 Application to a fuzzy vehicle-following model

In this subsection, we present an application of the proposed fuzzy fractional differential equation with piecewise constant argument (1) to a vehicle-following system under uncertain observations and driver-memory effects. The purpose of this application is to illustrate how the proposed analytical representation can be used to describe a realistic hybrid dynamical system involving continuous motion, discrete observation, memory effect, inertia, and fuzzy uncertainty.

Consider a following vehicle moving behind a leading vehicle on a single lane. The objective of the following vehicle is to maintain a desired safety distance from the leader. *Let $w(\mathfrak{t}) \in \mathbb{E}$ denote the fuzzy spacing error at time $\mathfrak{t}$, defined as the difference between the actual inter-vehicle distance and the desired safe distance.* A positive value of $w(\mathfrak{t})$ means that the following vehicle is farther than desired, while a negative value indicates that it is too close to the preceding vehicle. Due to sensor errors, uncertain road conditions, driver reaction variability, and imperfect estimation of relative velocity, the spacing error is naturally described as a fuzzy-valued function. The vehicle-following dynamics are modeled by

$$\begin{cases} {}^{C}D_{\imath^+}^{\mathfrak{q}_\imath;\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t}) = \Omega_1 \otimes w(\mathfrak{t}) + \Omega_2 \otimes w([\mathfrak{t}]) + \mathfrak{F}(\mathfrak{t}), \quad \mathfrak{t} \in [\imath, \imath+1), \quad \imath = 0, 1, 2, \ldots, N, \\ w(0) = w_0, \quad w^{(1)}(0) = w_1. \end{cases} \tag{1}$$

Here, the term $\Omega_1 \otimes w(\mathfrak{t})$ describes the instantaneous correction based on the current spacing error. If $\Omega_1 < 0$ the driver or controller tends to reduce the spacing error. A larger absolute value of Ω_1 corresponds to a stronger continuous correction. The term $\Omega_2 \otimes w([\mathfrak{t}])$ represents the sampled-data effect. In a real vehicle-following process, the driver or onboard controller does not update the spacing information continuously. Instead, the perceived spacing error is updated at discrete instants. Hence, during the interval $[\imath, \imath+1)$, the latest available information is $w([\mathfrak{t}]) = w(\imath)$. If $\Omega_2 < 0$, the sampled information acts in a stabilizing direction. If $\Omega_2 > 0$, delayed or outdated information may reinforce the previous error and produce a destabilizing effect. The fuzzy forcing term $\mathfrak{F}(\mathfrak{t})$ describes uncertain external perturbations, such as road slope, wind, temporary braking of the leading vehicle, actuator fluctuations, measurement noise, or reaction-delay errors. The choice $\mathfrak{q}_\imath \in (1, 2)$ is physically meaningful for a vehicle-following system. The spacing error $w(\mathfrak{t})$ is a position-type variable. Its evolution is not governed only by the present spacing error, but also by the relative velocity and acceleration history between the two vehicles. A fractional order $\mathfrak{q}_\imath \in (1, 2)$ describes dynamics between first-order damping and second-order acceleration. Therefore, two fuzzy initial conditions, $w(0)$ and $w^{(1)}(0)$, are physically meaningful: they represent the initial spacing error and the initial rate of change of the spacing error. The multi-order sequence $\{\mathfrak{q}_\imath\}_{\imath=0}^{N}$ allows the driver-memory and inertial effects to vary across different driving regimes. When $\mathfrak{q}_\imath$ is close to 1, the system behaves closer to a damped first-order response with weak inertia. When $\mathfrak{q}_\imath$ is close to 2, the dynamics exhibit stronger persistence of velocity and acceleration history. Hence, different values of $\mathfrak{q}_\imath \in (1, 2)$ can model different levels of driver memory, traffic density, and inertial response. In the model (1), we choose the fuzzy disturbance as

$$\mathfrak{F}(\mathfrak{t}) = \widetilde{c_\imath} \otimes (\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\beta_\imath}, \quad \mathfrak{t} \in [\imath, \imath+1), \tag{2}$$

where $\widetilde{c_\imath} \in \mathbb{E}$ is the fuzzy intensity of the disturbance on the interval $[\imath, \imath+1)$ and, $\beta_\imath \geq 0$ controls the growth rate of the disturbance within each observation interval. If $\beta_\imath \in (0, 1)$, the disturbance grows slowly. This is suitable for smooth driving or weak external perturbations. If $\beta_\imath = 1$ the disturbance grows approximately linearly. If $\beta_\imath > 1$, the disturbance becomes stronger near the end of the observation interval, which may represent sudden braking, stronger wind, road slope effects, or increasing traffic density.

4.1 General solution representation

By utilizing the formula (53) in Theorem 3.2, we obtain the solution of problem (1) as follows:

$$w(\mathfrak{t}) = \mathcal{M}(\mathfrak{t},\imath) \otimes w(\imath) + \mathcal{N}(\mathfrak{t},\imath) \otimes w^{(1)}(\imath) + \mathcal{I}(\mathfrak{t},\imath), \quad \mathfrak{t} \in [\imath, \imath+1), \imath = 0, 1, \ldots, N. \tag{3}$$

Here, $\mathcal{M}(\mathfrak{t},\imath)$ and $\mathcal{N}(\mathfrak{t},\imath)$ are determined by (55)-(56), respectively, and

$$\begin{aligned}\mathcal{I}(\mathfrak{t},\imath) &= \int_\imath^{\mathfrak{t}} \mathfrak{K}'(s)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath - 1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(s))^{\mathfrak{q}_\imath}) \otimes \mathfrak{F}(s) ds \\ &= \widetilde{c_\imath} \otimes \Gamma(\beta_\imath + 1)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath + \beta_\imath} E_{\mathfrak{q}_\imath, \mathfrak{q}_\imath + \beta_\imath + 1}\left(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}\right).\end{aligned} \tag{4}$$

Furthermore, we also have

$$w^{(1)}(\mathfrak{t}) = \mathcal{M}'(\mathfrak{t},\imath) \otimes w(\imath) + \mathcal{N}'(\mathfrak{t},\imath) \otimes w^{(1)}(\imath) + \mathcal{I}'(\mathfrak{t},\imath), \tag{5}$$

where $\mathcal{M}'(\mathfrak{t},\imath), \mathcal{N}'(\mathfrak{t},\imath)$ are given by (64) and (65), respectively, and

$$\mathcal{I}'(\mathfrak{t},\imath) = \widetilde{c_\imath} \otimes \mathfrak{K}'(\mathfrak{t})\Gamma(\beta_\imath + 1)(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath + \beta_\imath - 1} E_{\mathfrak{q}_\imath, \mathfrak{q}_\imath + \beta_\imath}\left(\Omega_1(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))^{\mathfrak{q}_\imath}\right). \tag{6}$$

Define

$$Y(\imath) = \begin{pmatrix} w(\imath) \\ w^{(1)}(\imath) \end{pmatrix}, \quad A_\imath = \begin{pmatrix} \mathcal{M}(\imath+1,\imath) & \mathcal{N}(\imath+1,\imath) \\ \mathcal{M}'(\imath+1,\imath) & \mathcal{N}'(\imath+1,\imath) \end{pmatrix}, \quad \text{and} \quad G_\imath = \begin{pmatrix} \mathcal{I}(\imath+1,\imath) \\ \mathcal{I}'(\imath+1,\imath) \end{pmatrix}. \tag{7}$$

Then, from (54) it follows that the sampled fuzzy state satisfies

$$Y(\imath) = \left(\prod_{\jmath=0}^{\imath-1} A_\jmath\right) \otimes Y(0) + \sum_{k=0}^{\imath-1} \left(\prod_{\jmath=k+1}^{\imath-1} A_\jmath\right) \otimes G_k, \quad \imath = 1, \ldots, N+1. \tag{8}$$

4.2 Numerical scenarios

To demonstrate the practical meaning of the parameters, we consider two representative driving regimes.

4.2.1 Situation 1: Smooth driving with stabilizing response

This situation corresponds to normal driving on a relatively smooth road. The driver has reliable observations, the spacing error is corrected effectively, and the memory effect is moderate.

We chose $\Omega_1 = -0.55$ and $\Omega_2 = -0.30$. The choice $\Omega_1 < 0$ means that the instantaneous response reduces the current spacing error. The choice $\Omega_2 < 0$ means that the last observed spacing error also acts in the stabilizing direction. This corresponds to a cautious driver or an effective adaptive cruise controller.

We choose $\mathfrak{q}_\imath = 1.12 + 0.18(1 - e^{-0.25\imath}), \imath = 0, 1, \ldots, N$. Thus, $1.12 \leq \mathfrak{q}_\imath < 1.30$. These values are slightly larger than 1, which means that the dynamics are close to a first-order correction process but still include weak inertial memory. This is appropriate for smooth driving, where the driver reacts mainly to the current spacing error and only mildly to velocity-acceleration history.

For the memory kernel, we take $\mathfrak{K}(\mathfrak{t}) = \ln(0.35\mathfrak{t}+1)$. This kernel grows slowly and therefore weakens the accumulation of remote memory. It is suitable for stable driving because older spacing information becomes less influential.

The disturbance $\mathfrak{F}(\mathfrak{t})$ is chosen as (2) with $\widetilde{c_\imath} = (0, 0.03, 0.06)$, and $\beta_\imath = 0.40 + 0.60\dfrac{\imath}{N+1}$. Here the disturbance is small and grows moderately. This models mild road irregularities, small radar errors, or weak variations in the leading vehicle speed.

The fuzzy initial data are $w(0) = (0.8, 1.0, 1.2)$ and $w^{(1)} = (-0.08, 0, 0.08)$. The initial spacing error is approximately one meter with small uncertainty, and the initial rate of change is close to zero. This is consistent with a vehicle already operating near the desired safe distance.

For numerical illustration, we consider the finite time horizon corresponding to $N = 9$, namely $\mathbb{T} = \bigcup_{\imath=0}^{9} [\imath, \imath + 1)$. By applying the formulas derived in Subsection 4.1 to the prescribed data set, we obtain the following quantities:

$$\mathcal{M}(\mathfrak{t}, \imath) = (17/11)E_{\mathfrak{q}_\imath,1}(-0.55[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}) - (6/11), \quad \mathcal{N}(\mathfrak{t}, \imath) = \frac{\Delta_\imath(\mathfrak{t})}{\mathfrak{K}'(\imath)} E_{\mathfrak{q}_\imath,2}(-0.55[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}), \tag{9}$$

$$\mathcal{I}(\mathfrak{t}, \imath) := \widetilde{c}_\imath \otimes \mathcal{J}_\imath(\mathfrak{t}) = \widetilde{c}_\imath \otimes \Gamma(\beta_\imath + 1)[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath+\beta_\imath} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath+\beta_\imath+1}\left(-0.55[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}\right), \tag{10}$$

$$\mathcal{M}'(\mathfrak{t}, \imath) = -0.85\mathfrak{K}'(\mathfrak{t})\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath-1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(-0.55\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}), \quad \mathcal{N}'(\mathfrak{t}, \imath) = \frac{\mathfrak{K}'(\mathfrak{t})}{\mathfrak{K}'(\imath)} E_{\mathfrak{q}_\imath,1}(-0.55[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}), \tag{11}$$

$$\mathcal{I}'(\mathfrak{t}, \imath) := \widetilde{c}_\imath \otimes \mathcal{J}'_\imath(\mathfrak{t}) = \widetilde{c}_\imath \otimes \mathfrak{K}'(\mathfrak{t})\Gamma(\beta_\imath + 1)[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath+\beta_\imath-1} E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath+\beta_\imath}\left(-0.55[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}\right), \tag{12}$$

where $\Delta_\imath(\mathfrak{t}) := \mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath) = \ln\left((0.35t + 1)/(0.35i + 1)\right)$ and $\mathfrak{K}'(\mathfrak{t}) = 0.35/(0.35\mathfrak{t}+1)$. These are the six scalar coefficient functions needed to reconstruct both $w(\mathfrak{t})$ and $w^{(1)}(\mathfrak{t})$ on every interval $\mathbb{T}_\imath := [\imath, \imath + 1), \imath = 0, 1, \ldots, 9$. At the nodes, we define $\Delta_\imath = \mathfrak{K}(\imath + 1) - \mathfrak{K}(\imath)$. For notational convenience, we further define

$$\mathcal{M}_\imath := \mathcal{M}(\imath + 1, \imath),\ \mathcal{N}_\imath := \mathcal{N}(\imath + 1, \imath),\ \mathbb{J}_\imath := \mathcal{J}_\imath(\imath + 1), \mathcal{M}'_\imath := \mathcal{M}'(\imath + 1, \imath),\ \mathcal{N}'_\imath := \mathcal{N}'(\imath + 1, \imath),\ \mathbb{J}'_\imath := \mathcal{J}'_\imath(\imath + 1).$$

Substituting the prescribed parameter values into the expressions (9)-(12) yields the numerical coefficients listed in Table 1.

Table 1: Numerical values of the coefficient functions at the nodal points for Situation 1 with $N = 9$.

$\imath$	$\mathfrak{q}_\imath$	β_i	Δ_i	$\mathcal{M}_i$	$\mathcal{N}_i$	$\mathcal{M}'_i$	$\mathcal{N}'_i$	$\mathbb{J}_\imath$	$\mathbb{J}'_\imath$
0	1.120000	0.400000	0.300105	0.776882	0.631543	-0.179263	0.646380	0.100554	0.127303
1	1.159816	0.460000	0.230524	0.834748	0.726230	-0.137520	0.723121	0.054892	0.077622
2	1.190824	0.520000	0.187212	0.872115	0.805300	-0.108232	0.774258	0.031691	0.048695
3	1.214974	0.580000	0.157629	0.897377	0.873867	-0.087373	0.810286	0.019070	0.031327
4	1.233782	0.640000	0.136132	0.915274	0.934800	-0.072176	0.836804	0.011849	0.020592
5	1.248429	0.700000	0.119801	0.928508	0.989790	-0.060834	0.857019	0.007552	0.013783
6	1.259837	0.760000	0.106972	0.938654	1.040065	-0.052174	0.872878	0.004913	0.009365
7	1.268721	0.820000	0.096627	0.946671	1.086556	-0.045422	0.885622	0.003249	0.006444
8	1.275640	0.880000	0.088107	0.953167	1.129997	-0.040053	0.896069	0.002178	0.004480
9	1.281028	0.940000	0.080969	0.958547	1.170980	-0.035710	0.904781	0.001476	0.003141

Applying the recurrence relation (8), we obtain the nodal values of $w(\imath)$ and $w^{(1)}(\imath)$ listed in Table 2.

Table 2: Fuzzy nodal values of $w(i)$ and $w^{(1)}(i)$ represented as triangular fuzzy numbers.

$\imath$	$w(\imath)$	$w^{(1)}(\imath)$
0	(0.800000, 1.000000, 1.200000)	(−0.080000, 0.000000, 0.080000)
1	(0.578116, 0.806147, 1.034177)	(−0.195121, −0.175444, −0.155767)
2	(0.332026, 0.546958, 0.761891)	(−0.250201, −0.235400, −0.220598)
3	(0.103528, 0.284329, 0.465130)	(−0.273259, −0.239997, −0.206736)
4	(−0.091102, 0.045926, 0.182953)	(−0.260178, −0.218370, −0.176561)
5	(−0.247067, −0.156705, −0.066343)	(−0.229688, −0.185429, −0.141171)
6	(−0.364991, −0.320167, −0.275342)	(−0.191984, −0.148970, −0.105956)
7	(−0.447991, −0.445235, −0.442479)	(−0.152651, −0.113047, −0.073444)
8	(−0.569325, −0.534762, −0.500200)	(−0.114706, −0.079701, −0.044695)
9	(−0.659059, −0.592535, −0.526010)	(−0.079713, −0.049864, −0.020015)
10	(−0.715601, −0.622647, −0.529693)	(−0.048399, −0.023862, 0.000674)

Then, it follows from (3) that the continuous fuzzy solution in Situation 1 is then reconstructed by

$$w(\mathfrak{t}) = \begin{cases} w_0(\mathfrak{t}), & \mathfrak{t} \in [0,1) \\ w_1(\mathfrak{t}), & \mathfrak{t} \in [1,2) \\ \vdots & \\ w_9(\mathfrak{t}), & \mathfrak{t} \in [9,10) \end{cases} \tag{13}$$

where $w_\imath(\mathfrak{t}) = \mathcal{M}(\mathfrak{t},\imath) \otimes w(\imath) + \mathcal{N}(\mathfrak{t},\imath) \otimes w^{(1)}(\imath) + \widetilde{c}_\imath \otimes \mathcal{J}_\imath(\mathfrak{t}),\ \mathfrak{t} \in [\imath, \imath+1), \imath = 0, 1, \ldots, 9$. The resulting trajectory is shown in Figure 1.

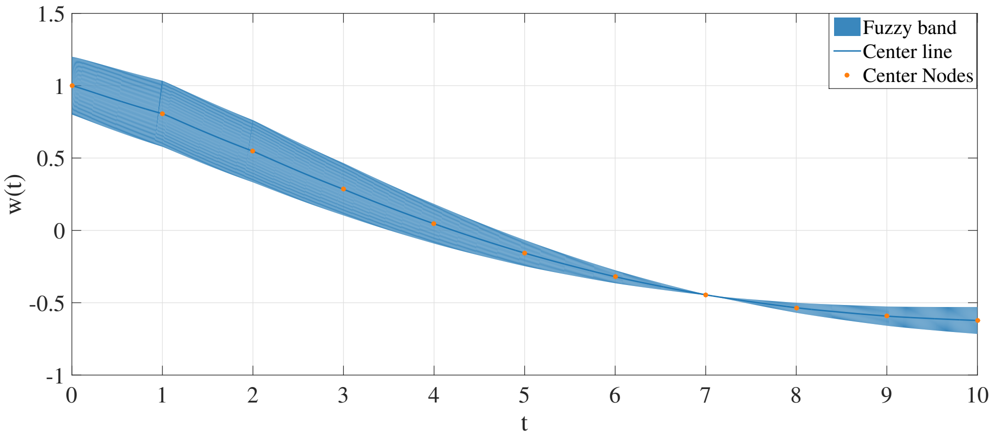

Figure 1: Trajectory of the fuzzy spacing error $w(t)$ for Situation 1. The shaded region represents the fuzzy band, while the solid curve denotes the center trajectory.

4.2.2 Situation 2. Dense traffic with stronger driver-memory effect

The second situation represents denser traffic, where the driver or controller still attempts to correct the spacing error, but the available sampled information may be delayed or slightly outdated. In this case, the sampled feedback may weaken the stabilizing effect, but it should not cause an unrealistic blow-up under normal dense-traffic conditions.

We chose $\Omega_1 = -0.18$ and $\Omega_2 = 0.08$. The choice $\Omega_1 < 0$ indicates that the instantaneous response still acts in the stabilizing direction. However, $\Omega_2 > 0$ means that the sampled term has a destabilizing influence. This may occur when the driver reacts to outdated spacing information, or when the controller receives delayed measurements.

We choose $\mathfrak{q}_\imath = 1.25 + 0.35(1 - e^{-0.25\imath}), \imath = 0, 1, \ldots, N$. Thus, $1.25 \leq \mathfrak{q}_\imath < 1.60$. This choice still reflects stronger memory and inertial persistence than Situation 1, but avoids a near-second-order regime over a long horizon. This is more suitable for dense but non-failure traffic.

The kernel is chosen as $\mathfrak{K}(\mathfrak{t}) = 0.08\mathfrak{t}^2 + 0.45\mathfrak{t}$. Compared with the logarithmic kernel in Situation 1, this kernel produces a faster growth of the effective memory variable $(\mathfrak{K}(\mathfrak{t}) - \mathfrak{K}(\imath))$. As a result, historical information has a stronger impact on the system evolution. The coefficients are chosen moderately so that the solution remains physically meaningful while still exhibiting memory effects.

The disturbance $\mathfrak{F}(\mathfrak{t})$ is chosen as (2) with $\widetilde{c}_\imath = (0.025, 0.06, 0.095)$, and $\beta_\imath = 0.5 + 0.4\dfrac{\imath}{N+1}$. The larger support of $\widetilde{c}_\imath$, compared with Situation 1, represents stronger uncertainty caused by traffic density, braking fluctuations, and incomplete observations. The exponents $\beta_\imath$ increase slowly, meaning that disturbances accumulate moderately within each interval.

The fuzzy initial data are $w(0) = (0.4, 0.6, 0.8)$ and $w^{(1)} = (-0.03, 0.05, 0.13)$. The initial spacing error is smaller than in Situation 1, but the positive center of $w^{(1)}$ indicates that the spacing error is already increasing. This is typical in dense traffic, where the following vehicle may not immediately stabilize the desired distance.

For numerical illustration, we again take $N = 9$. Substituting the chosen parameters into the formulas of Subsection 4.1

yields:

$$\mathcal{M}(\mathfrak{t},\imath)=(5/9)E_{\mathfrak{q}_\imath,1}(-0.18[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath})+(4/9),\quad \mathcal{N}(\mathfrak{t},\imath)=\frac{\Delta_\imath(\mathfrak{t})}{\mathfrak{K}'(\imath)}E_{\mathfrak{q}_\imath,2}(-0.18[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}),\tag{14}$$

$$\mathcal{I}(\mathfrak{t},\imath):=\widetilde{c_\imath}\otimes\mathcal{J}_\imath(\mathfrak{t})=\widetilde{c_\imath}\otimes\Gamma(\beta_\imath+1)[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath+\beta_\imath}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath+\beta_\imath+1}\left(-0.18[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}\right),\tag{15}$$

$$\mathcal{M}'(\mathfrak{t},\imath)=-0.1\mathfrak{K}'(\mathfrak{t})\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath}(-0.18\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}),\quad \mathcal{N}'(\mathfrak{t},\imath)=\frac{\mathfrak{K}'(\mathfrak{t})}{\mathfrak{K}'(\imath)}E_{\mathfrak{q}_\imath,1}(-0.18[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}),\tag{16}$$

$$\mathcal{I}'(\mathfrak{t},\imath):=\widetilde{c_\imath}\otimes\mathcal{J}'_\imath(\mathfrak{t})=\widetilde{c_\imath}\otimes\mathfrak{K}'(\mathfrak{t})\Gamma(\beta_\imath+1)[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath+\beta_\imath-1}E_{\mathfrak{q}_\imath,\mathfrak{q}_\imath+\beta_\imath}\left(-0.18[\Delta_\imath(\mathfrak{t})]^{\mathfrak{q}_\imath}\right),\tag{17}$$

where $\Delta_\imath(\mathfrak{t}):=(\mathfrak{t}-\imath)(0.08(\mathfrak{t}+\imath)+0.45)$ and $\mathfrak{K}'(\mathfrak{t})=0.16\mathfrak{t}+0.45$. The numerical coefficients obtained from (14)–(17) are listed in Table 3. Consequently, applying the recurrence relation (8), we obtain the values of $w(\imath)$ and $w^{(1)}(\imath)$, as reported in Table 4.

Table 3: Coefficient values at the nodal points for Situation 2.

$\imath$	$\mathfrak{q}_\imath$	$\beta_\imath$	$\Delta_\imath$	$\mathcal{M}_\imath$	$\mathcal{N}_\imath$	$\mathcal{M}'_\imath$	$\mathcal{N}'_\imath$	$\mathbb{J}_\imath$	$\mathbb{J}'_\imath$
0	1.250000	0.500000	0.530000	0.961177	1.140834	-0.054312	1.260827	0.177499	0.351971
1	1.327420	0.540000	0.690000	0.950157	1.087019	-0.071409	1.149046	0.243867	0.499098
2	1.387714	0.580000	0.850000	0.937697	1.051303	-0.090936	1.072345	0.323861	0.682954
3	1.434672	0.620000	1.010000	0.923732	1.023742	-0.112735	1.011143	0.420183	0.910042
4	1.471242	0.660000	1.170000	0.908288	1.000257	-0.136551	0.957475	0.535464	1.187251
5	1.499723	0.700000	1.330000	0.891446	0.978892	-0.162055	0.907593	0.672431	1.522075
6	1.521904	0.740000	1.490000	0.873326	0.958621	-0.188878	0.859589	0.834013	1.922825
7	1.539179	0.780000	1.650000	0.854068	0.938871	-0.216630	0.812464	1.023437	2.398861
8	1.552633	0.820000	1.810000	0.833824	0.919319	-0.244918	0.765704	1.244314	2.960837
9	1.563110	0.860000	1.970000	0.812746	0.899783	-0.273370	0.719065	1.500730	3.620970

Table 4: Fuzzy nodal values of $w(i)$ and $w^{(1)}(i)$ represented as triangular fuzzy numbers.

i	$w(i)$	$w^{(1)}(i)$
0	(0.400000, 0.600000, 0.800000)	(−0.030000, 0.050000, 0.130000)
1	(0.354614, 0.644382, 0.934150)	(−0.048829, 0.051860, 0.152549)
2	(0.297125, 0.681151, 1.065177)	(−0.113019, 0.041822, 0.196662)
3	(0.215312, 0.705589, 1.195865)	(−0.180008, 0.012790, 0.205588)
4	(0.112172, 0.713983, 1.315793)	(−0.239440, −0.032837, 0.173766)
5	(−0.006180, 0.699067, 1.404315)	(−0.282444, −0.088795, 0.104854)
6	(−0.125716, 0.661931, 1.449578)	(−0.305294, −0.146773, 0.011748)
7	(−0.227733, 0.607158, 1.442048)	(−0.309006, −0.197926, −0.086846)
8	(−0.292309, 0.542657, 1.377622)	(−0.298989, −0.235376, −0.171764)
9	(−0.307086, 0.482537, 1.272160)	(−0.283677, −0.253630, −0.223582)
10	(0.270416, 0.533941, 0.797466)	(0.094154, 0.122446, 0.150739)

Then, it follows from (3) that the continuous fuzzy solution in Situation 2 is then reconstructed by

$$w(\mathfrak{t})=\begin{cases}w_0(\mathfrak{t}), & \mathfrak{t}\in[0,1)\\ w_1(\mathfrak{t}), & \mathfrak{t}\in[1,2)\\ \vdots & \\ w_9(\mathfrak{t}), & \mathfrak{t}\in[9,10)\end{cases}\tag{18}$$

where $w_\imath(\mathfrak{t})=\mathcal{M}(\mathfrak{t},\imath)\otimes w(\imath)+\mathcal{N}(\mathfrak{t},\imath)\otimes w^{(1)}(\imath)+\widetilde{c_\imath}\otimes\mathcal{J}_\imath(\mathfrak{t}),\ \mathfrak{t}\in[\imath,\imath+1),\imath=0,1,\dots,9$. The resulting trajectory is shown in Figure 2.

Discussion of Situation 1. Figure 1 shows that the center trajectory decreases from its initial positive value and eventually enters the negative region, while the fuzzy band remains bounded throughout the simulation interval. This

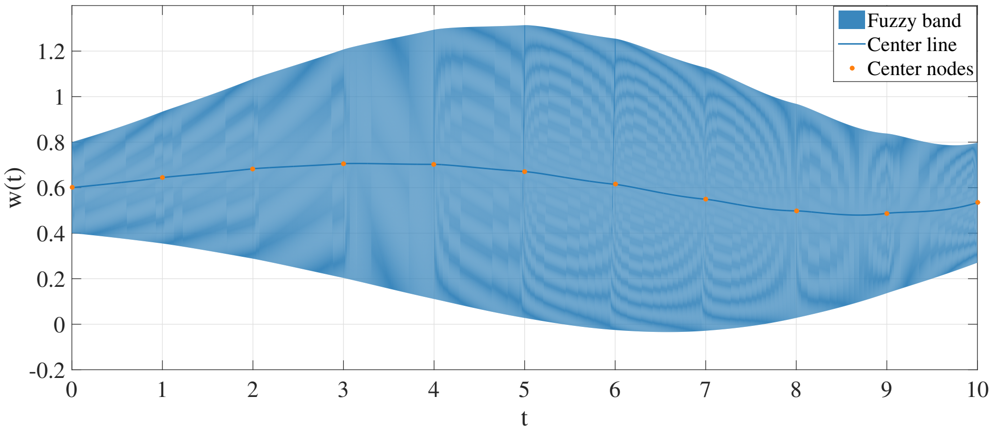

Figure 2: Trajectory of the fuzzy spacing error $w(t)$ for the revised Situation 2. The stronger memory effect and denser traffic conditions lead to a wider uncertainty band compared with Situation 1.

indicates that fuzzy uncertainty, disturbances, and memory effects do not cause excessive growth of the spacing error. Since $w(\mathfrak{t})$ denotes the difference between the actual and desired spacings, the sign change of $w(\mathfrak{t})$ simply reflects a transition from a larger-than-desired spacing to a smaller-than-desired one. Therefore, the negative values remain physically meaningful and can be interpreted as a moderate overshoot of the reference spacing level.
Discussion of Situation 2. Figure 2 exhibits a different pattern. Although the fuzzy band becomes wider than in Situation 1, the center trajectory remains positive throughout the simulation interval. This suggests that stronger memory effects and larger disturbances mainly increase the uncertainty of the spacing dynamics without significantly altering the representative system behavior. The partial overlap of the fuzzy band with the negative region indicates that some admissible fuzzy realizations may temporarily experience a spacing smaller than the desired one, providing a potential indicator of safety risk under denser traffic conditions.

5 Conclusion

This paper establishes, for the first time, an explicit solution representation for fuzzy non-homogeneous linear differential equations with piecewise constant arguments involving generalized multi-order cf-FD in the case where $\mathfrak{q}_\imath \in (1, 2)$, for all $\imath = 0, 1, 2, \ldots, N$. It should be emphasized that the proposed explicit solution formula is not a straightforward extension of the existing results for fuzzy fractional differential equations of order $\mathfrak{q}_\imath \in (0, 1)$. Instead, it constitutes a new analytical representation specifically developed for fuzzy fractional systems with piecewise constant arguments involving generalized multi-order Caputo-type derivatives of order $\mathfrak{q}_\imath \in (1, 2)$. In addition, the obtained formula is valid for a broad class of kernel functions $\mathfrak{K}(\mathfrak{t})$ and allows the fractional order to vary from one subinterval to another. As special cases, the proposed framework includes several well-known fractional derivatives, such as the Caputo, Caputo-Katugampola, and Caputo-Hadamard derivatives.

The derived solution representation provides a unified framework for studying fuzzy fractional differential equations with piecewise constant arguments and multi-order memory effects. Moreover, it enables the explicit characterization of solution trajectories on successive time subintervals, thereby facilitating the analysis of their qualitative behavior. The vehicle-following examples illustrate how different kernel functions, fractional orders, and uncertainty levels influence the evolution of the fuzzy spacing error. In particular, the numerical results show that stronger memory effects and larger disturbances increase the uncertainty of the system response, whereas moderate memory and disturbances lead to bounded trajectories with a narrower fuzzy band.

Moreover, the results of this paper can be employed to derive a solution representation for problem (1) in the following semi-linear form:

$$\begin{cases} {}^{C}D_{\imath^+}^{\mathfrak{q}_\imath;\mathfrak{K}(\mathfrak{t})} w(\mathfrak{t}) = \Omega_1 \otimes w(\mathfrak{t}) + \Omega_2 \otimes w([\mathfrak{t}]) + \mathfrak{F}(\mathfrak{t}, w(\mathfrak{t})), \quad \mathfrak{t} \in [\imath, \imath+1), \quad \imath = 0, 1, 2, \ldots, N, \\ w(0) = w_0, \quad w^{(1)}(0) = w_1. \end{cases} \tag{19}$$

where Ω_1, Ω_2, w_0, and w_1 are fuzzy numbers, and $\mathfrak{F}(\mathfrak{t}, w(\mathfrak{t}))$ is a nonlinear fuzzy function. This extension will be the subject of future work, where the explicit representation obtained here may serve as a basis for studying finite-time stability, Mittag-Leffler stability, asymptotic stability, and related qualitative properties.

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