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Original paper

On smooth nullnorms on bounded lattices and their extensions to smooth uninorms

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Abstract

The paper deals with smooth uninorms and smooth nullnorms on finite lattices. These operations were first introduced on the unit interval and later generalized to bounded lattices. One of the most important and most studied properties is continuity. However, when working on finite lattices (finite structures, in general), the topological continuity is not very meaningful. For this reason, Mayor and Torrens (2005) introduced several notions that replace continuity. Smoothness and divisibility are among them. We study smooth nullnorms on finite lattices looking for conditions under which, when the nullnorm is defined on a lattice L, it is possible to extend the smooth nullnorm to a smooth uninorm defined on the horizontal sum of L and a finite chain. We provide necessary and sufficient conditions for such an extension to yield a smooth uninorm. Finally, we give a necessary and sufficient condition under which a smooth uninorm is also a divisible uninorm.

Keywords: Smooth nullnorm, smooth uninorm, divisible uninorm, finite lattice, horizontal sum.

1 Introduction

Uninorms and nullnorms on the unit interval are special types of aggregation functions since, due to their associativity they can be straightforwardly extended to n-ary operations for arbitrary $n \in \mathbb{N}$. They are important in various fields of applications, e.g., neural nets, fuzzy decision making and fuzzy modelling. Some basic knowledge on aggregation functions can be found, e.g., in the monographs [5, 16]

Uninorms were introduced by Yager and Rybalov [22]. Special types of associative, commutative and monotone operations with neutral elements had already been studied in [11, 14, 15]. Deschrijver [12, 13] has shown that on the lattice $L^{[0,1]}$ of closed subintervals of the unit interval there exist uninorms which are neither conjunctive nor disjunctive (i.e., whose annihilator is different from both, $\mathbf{0}$ and $\mathbf{1}$).

Calvo et al. [4] posted a question whether there exist functions that solve the equation of Frank and Alsina where one operation is a uninorm and the about other a different one. This was solved negatively, but by solving it nullnorms were proposed. Later on, Karaçal et al. [18] introduced nullnorms on arbitrary bounded lattices and Karaçal and Mesiar [19] introduced uninorms on arbitrary bounded lattices. Afterwards, construction possibilities for uninorms and nullnorms on bounded lattices were studied, e.g., in [2, 3, 6, 7, 8, 9, 10].

The paper is organized as follows. In Section 2 some notions and results will be repeated that are important for our considerations on smooth and divisible uninorms and nullnorms. The main body of the paper is in Section 3. Finally, Section 4 contains conclusions.

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2 Preliminaries

In this section, some basic notions and known facts will be recalled to make the paper self-contained. For details on lattice theory, we recommend the monographs [1, 17].

In the whole paper, $(L, 0_L, 1_L, \leq)$ will denote a bounded lattice that will be often shortened just to L, where $L \neq \emptyset$ is a given set and $0_L, 1_L$ its two distinguished elements such that $0_L \leq x \leq 1_L$, for all $x \in L$.

For arbitrary $x_1, x_2 \in L$, $x_1 \leq x_2$,

(a) if $x_1 \neq x_2$, this will be denoted by $x_1 \lneqq x_2$,

(b) $[x_1, x_2]$ will denote the closed interval,

(c) $]x_1, x_2[$ will denote the open interval,

(d) $[x_1, x_2[$ and $]x_1, x_2]$ will denote the two kinds of semi-open intervals.

Further, if x_1 and x_2 are incomparable, this will be denoted by $x_1 \parallel x_2$. The set of all elements which are incomparable with a given one, $x \in L$, will be denoted by $\parallel_x$.

2.1 Basic definitions and known facts on commutative monoidal operations on $[0, 1]$

First, some basic notions.

Definition 2.1. *Let L be a bounded lattice and $*$ be a binary commutative operation on L. Then*

(i) *element $c \in L$ is said to be* idempotent *if $c * c = c$,*

(ii) *element $e \in L$ is said to be* neutral *if $e * x = x$ for all $x \in L$,*

(iii) *element $a \in L$ is said to be an* annihilator *if $a * x = a$ for all $x \in L$.*

The notion of *partial annihilator* (*partial neutral element*), will be used in the meaning that $a * x = a$ $(e * x = x)$ for a set $A \subset L$.

Lemma 2.2. *Let $*$ be a commutative and associative operation on L. Further, let c be an idempotent element. Assume that there exist elements $x, y \in L$ such that $x * c = y$. Then $y * c = y$.*

Schweizer and Sklar [21] introduced the notion of a triangular norm (t-norm for brevity).

Definition 2.3. [21] *An operation $T\colon [0, 1]^2 \to [0, 1]$ is a t-norm if it is associative, commutative, monotone, and 1 is its neutral element.*

T-norms and t-conorms are dual to each other. If $T\colon [0, 1]^2 \to [0, 1]$ is a t-norm, then

$$S(x, y) = 1 - T(1 - x, 1 - y),$$

is the dual t-conorm to T. For details on t-norms and t-conorms see, e.g., [20].

As a generalization of both t-norms and t-conorms, Yager and Rybalov [22] proposed the notion of uninorm.

Definition 2.4. [22] *An operation $U\colon [0, 1]^2 \to [0, 1]$ is a uninorm if it is associative, commutative, monotone, and if it possesses a neutral element $e \in [0, 1]$.*

A uninorm U is *proper* if its neutral element $e \in]0, 1[$.

Every uninorm has an annihilator. A uninorm with the annihilator 0 is conjunctive, and a uninorm with annihilator 1 is disjunctive.

Lemma 2.5. [22] *Let $U\colon [0, 1]^2 \to [0, 1]$ be a uninorm whose neutral element is e. Then its dual operation*

$$U^d(x, y) = 1 - U(1 - x, 1 - y),$$

is a uninorm whose neutral element is $1 - e$. Moreover, U is conjunctive if and only if U^d is disjunctive.

Results in paper [22] imply the following assertion.

Lemma 2.6. *Let $U\colon [0,1]^2 \to [0,1]$ be a uninorm whose neutral element is e. Then there exists a t-norm $T_U\colon [0,1]^2 \to [0,1]$ and a t-conorm $S_U\colon [0,1]^2 \to [0,1]$ such that*

$$(\forall x, y \in [0,e]^2)(U(x,y) = T_U(\tfrac{x}{e}, \tfrac{y}{e})),$$
$$(\forall x, y \in [e,1]^2)(U(x,y) = S_U(\tfrac{x-e}{1-e}, \tfrac{y-e}{1-e})).$$

Lemma 2.7. [22] *Assume U is a uninorm with neutral element e. Then:*

1. *for any x and all $y > e$ we get $U(x,y) \geq x$,*
2. *for any x and all $y < e$ we get $U(x,y) \leq x$.*

Definition 2.8. [4] *An operation $V : [0,1]^2 \to [0,1]$ is said to be a nullnorm if V is associative, commutative, monotone, moreover if there exists an element $a \in [0,1]$ such that*

(1b) *$V(0,x) = x$ for all $x \in [0,a]$,*

(2b) *$V(1,x) = x$ for all $x \in [a,1]$.*

A nullnorm is *proper* if $a \in]0,1[$.

Remark 2.9. (a) *It is well-known that the element a in Definition 2.8 is the annihilator of the nullnorm V. Further, 0 is a partial neutral element of V on the interval $[0,a]$ and 1 is a partial neutral element of V on the interval $[a,1]$. Particularly, we have*

$$V(0,0) = 0, \quad V(1,1) = 1. \tag{1}$$

(b) *Setting $a = 0$ ($a = 1$) the nullnorm V becomes a t-norm (t-conorm).*

The extension of t-norms, t-conorms, uninorms and nullnorms to bounded lattices is straightforward. We skip these definitions.

An important result from [6] is formulated in the following proposition.

Proposition 2.10. [6] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $a \in L \setminus \{0_L, 1_L\}$ and V be an idempotent nullnorm on L with the annihilator a. Then the following hold:*

(a) *$V(x,y) = (x \wedge a) \vee (y \wedge a)$ if $(x,y) \in [0_L, a]^2 \cup [0_L, a] \times \parallel_a \cup \parallel_a \cup [0_L, a]$,*

(b) *$V(x,y) = (x \vee a) \wedge (y \vee a)$ if $(x,y) \in [a, 1_L]^2 \cup [a, 1_L] \times \parallel_a \cup \parallel_a \cup [a, 1_L]$,*

(c) *$V(x,y) = a$ if $(x,y) \in [0_L, a] \times [a, 1_L] \cup [a, 1_L] \times [0_L, a]$,*

(d) *$V(x,y) = (x \wedge a) \vee (y \wedge a)$ or $V(x,y) = (x \vee a) \wedge (y \vee a)$ or $v(x,y) \in \parallel_a$ if $(x,y) \in (\parallel_a)^2$.*

For more information on associative (and monotone) operations on $[0,1]$, refer to the monographs [5, 16, 20].

2.2 Smoothness and divisibility

The following notion will be important for studying smoothness.

Definition 2.11. [17] *Let $(L, \leq)$ be a lattice. For $x, z \in L$, z* is said to cover *x, denoted by $x \prec z$, if $x \lneqq z$ and there is no element $w \in L$ such that $x \lneqq w \lneqq z$.*

Definition 2.12. *Let $(L, \leq)$ be a bounded lattice. A non-decreasing binary commutative operation $F : L^2 \to L$ is said to be* smooth *if for all $x, y, z \in L$*

$$x \prec z \quad \Rightarrow \quad F(x,y) = F(z,y) \text{ or } F(x,y) \prec F(z,y).$$

Definition 2.13. *Let $(L, 0_L, 1_L, \leq)$ be a bounded lattice. A non-decreasing binary commutative operation $F : L^2 \to L$ is said to be* divisible *if for all $x \in L$*

$$z \in [F(x, 0_L), F(x, 1_L)] \quad \exists y \in L \quad \text{such that} \quad F(x,y) = z.$$

Remark 2.14. *Smoothness and divisibility of a binary function F represent a kind of its continuity. On a general bounded lattice L, the operations meet and join are divisible.*
Smoothness means that going one step in one variable, the value of the function either remains the same or moves also just by one step.

3 Extension of smooth nullnorms to smooth uninorms

At first, we show an example of a smooth nullnorm that cannot be extended to a uninorm – regardless of whether smooth or not.

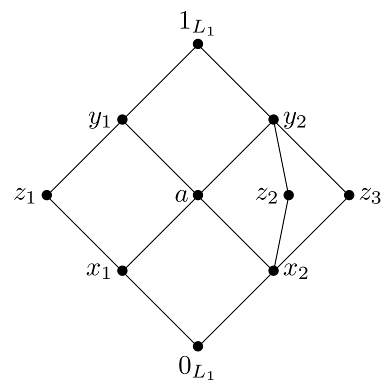

Figure 1: Hasse diagram of the lattice L_1

N_1	0_{L_1}	x_1	x_2	a	z_1	z_2	z_3	y_1	y_2	1_{L_1}
0_{L_1}	0_{L_1}	x_1	x_2	a	a	x_2	x_2	a	a	a
x_1	x_1	x_1	a	a	a	a	a	a	a	a
x_2	x_2	a	x_2	a	a	x_2	x_2	a	a	a
a	a	a	a	a	a	a	a	a	a	a
z_1	a	a	a	a	a	a	a	a	a	a
z_2	x_2	a	x_2	a	a	z_2	z_3	a	y_2	y_2
z_3	x_2	a	x_2	a	a	z_3	z_3	a	y_2	y_2
y_1	a	a	a	a	a	a	a	y_1	a	y_1
y_2	a	a	a	a	a	y_2	y_2	a	y_2	y_2
1_{L_1}	a	a	a	a	a	y_2	y_2	y_1	y_2	1_{L_1}

Table 1: The smooth nullnorm $N_1 : L_1 \times L_1 \to L_1$

Example 3.1. *Let L_1 be a finite lattice whose Hasse diagram is laid out on Figure 1. Table 1 presents the nullnorm N_1 defined on, L_1 Looking at Table 1, we can easily check that for all $s \in L_1$ and all $t_1 \neq 1_{L_1}$, $t_2 \in L_1$ such that $t_1 \prec t_2$, $N_1(s,t_1) \preceq N_1(s,t_2)$.*

Let $L = \{0_L, e, 1_L\}$ and denote $\tilde{L}$ the horizontal sum of L_1 and L. On $\tilde{L}$ we define the following function $\tilde{N}$

$$\tilde{N}(s,t) = \begin{cases} N_1(s,t) & \text{for } (s,t) \in L_1 \times L_1, \\ s & \text{if } t = e, \\ t & \text{if } s = e. \end{cases}$$

Check: $\tilde{N}(e,z_1) = z_1$, $\tilde{N}(z_1, 0_{L_1}) = a$. This gives that $\tilde{N}$ is not monotone. This means, $\tilde{N}$ is the extension of the nullnorm N_1 however, $\tilde{N}$ is not a nullnorm and neither a uninorm.

We are now going to study conditions under which it is possible to extend a smooth nullnorm to a uninorm.

Lemma 3.2. *Let L be a finite lattice and N a nullnorm with its annihilator $a \in L \setminus \{0_L, 1_L\}$. Assume there exists $z \parallel a$. Denote $x_1 = a \wedge z$ and $x_2 = a \vee z$. Let $\tilde{L}$ be another finite lattice and $e \in \tilde{L} \setminus \{1_{\tilde{L}}, 0_{\tilde{L}}\}$ and let $\tilde{U} : \tilde{L} \times \tilde{L}$ be a uninorm without $0_{\tilde{L}}$ and $1_{\tilde{L}}$ divisors. Denote $L_\oplus = L \oplus \tilde{L}$ the horizontal sum. Define a function U by the following*

$$U(x,y) = \begin{cases} N(x,y) & \text{if } (x,y) \in L \times L, \text{ and } x \in L, \\ x & \text{if } y \in \tilde{L} \setminus \{1_{L_\oplus}, 0_{L_\oplus}\}, \text{ and } y \in L, \\ y & \text{if } x \in \tilde{L} \setminus \{1_{L_\oplus}, 0_{L_\oplus}\} \text{ and } x \in L, \\ \tilde{U}(x,y) & \text{if } (x,y) \in (\tilde{L} \setminus \{1_{L_\oplus}, 0_{L_\oplus}\}) \times (\tilde{L} \setminus \{1_{L_\oplus}, 0_{L_\oplus}\}). \end{cases} \tag{2}$$

Then U is associative and monotone if

$$U(z, 0_{L_\oplus}) = N(z, 0_{L_\oplus}) = x_1 \quad \textit{and} \quad U(z, 1_{L_\oplus}) = N(z, 1_{L_\oplus}) = x_2. \tag{3}$$

Proof. Take a $z \parallel a$ such that $x_1 = a \wedge z$ and $x_2 = a \vee z$. We have $U(z, e) = z$ and $0_{L_\oplus} \leq e$. This implies $U(z, 0_{L_\oplus}) = N(z, 0_{L_\oplus}) \leq z$. Further, we know $N(z, 0_{L_\oplus}) \in [x_1, a]$, hence $N(z, 0_{L_\oplus}) = x_1$, what was to be proved. The equality $U(z, 1_{L_\oplus}) = N(z, 1_{L_\oplus}) = x_2$ we get by duality. Now, the associativity and monotonicity is obvious. □

Proposition 3.3. *Let $L_\oplus$ be the horizontal sum of finite lattices from Lemma 3.2. Assume U, defined by (2), is a smooth uninorm. Then*

1. $0_{L_\oplus} \prec a \prec 1_{L_\oplus}$,
2. $0_{L_\oplus} \prec e \prec 1_{L_\oplus}$.

Proof.
1. We are to prove item 1. For arbitrary $x \in [0_{L_\oplus}, e] \cup [e, 1_{L_\oplus}]$, $x \neq 1_{L_\oplus}$, we have $U(x, 0_{L_\oplus}) = 0_{L_\oplus}$. There is a $z \in [e, 1_{L_\oplus}]$, $z \prec 1_{L_\oplus}$ and
$$U(0_{L_\oplus}, z) = 0_{L_\oplus}, \quad U(0_{L_\oplus}, 1_{L_\oplus}) = a.$$
In order for U to be smooth, $0_{L_\oplus} \prec a$. Dually, $a \prec 1_{L_\oplus}$.
2. To prove item 2, take $0_{L_\oplus} \leq z \leq e \leq x \leq 1_{L_\oplus}$ such that $0_{L_\oplus} \prec z \leq e \leq x \prec 1_{L_\oplus}$. Then we get $0_{L_\oplus} \prec z \leq U(z, x) \leq x \prec 1_{L_\oplus}$ and $U(0_{L_\oplus}, x) = 0_{L_\oplus}, U(z, 1_{L_\oplus}) = 1_{L_\oplus}$, and hence, on the one hand $U(z, x) = z$ and on the other hand $U(z, x) = x$. This is possible only if $z = e = x$, proving item 2.

□

Proposition 3.4. *Let $L, \tilde{L}$ be finite lattices such that $\tilde{L} = \{0_{\tilde{L}}, e, 1_{\tilde{L}}\}$ and assume constraints 1. and 2. of Proposition 3.3 are fulfilled. If N is a smooth nullnorm on L then U defined on the horizontal sum $L_\oplus = L \oplus \tilde{L}$ by formula (2) is a smooth uninorm if and only if all maximal chains in $L_\oplus$ are of length 2.*

Proof. Let there exist a maximal chain of length greater than 2. Then there are x_1, x_2 such that $0_{L_\oplus} \lneqq x_1 \lneqq x_2 \lneqq 1_{L_\oplus}$. We have $e \prec 1_{L_\oplus}$ and
$$x_1 = U(x_1, e) \not\prec 1_{L_\oplus} = U(x_1, 1_{L_\oplus}),$$
meaning that smoothness is violated.
On the other hand, if all maximal chains are of length 2, for all $x \parallel a$, $x \neq e$, we have

$$0_{L_\oplus} = U(x, 0_{L_\oplus}) \prec x = U(x, e) \prec 1_{L_\oplus} = U(x, 1_{L_\oplus}).$$

□

Gathering the information from Lemma 3.2 and Propositions 3.3 and 3.4, we get the following assertion that we leave without proof.

Proposition 3.5. *Let L be a finite lattice such that all maximal chains are of length 2. Further, there are distinct elements $a, e \in L \setminus \{0_L, 1_L\}$. Then a two-place function $U : L \times L \to L$ is a smooth uninorm with annihilator a and neutral element e, if and only if U is associative and*

1. *$a, e, 0_L, 1_L$ are idempotent elements,*
2. *$0_L, 1_L$ are partial annihilators for elements in $\|_a$,*
3. *$(\|_a, U)$ is a commutative monoid with the neutral element e,*
4. *$U(0_L, 1_L) = U(1_L, 0_L) = a$.*

Proposition 3.6. *Let L be a finite lattice fulfilling the assumptions of Proposition 3.5 and U be a smooth uninorm on L having annihilator a and neutral element e with $a \parallel e$. Then U is a divisible uninorm if and only if $(\|_a, U)$ is a commutative group.*

Proof. Let $(\|_a, U)$ be a commutative group. We have to prove that for all $z_1 \in L$ and $z_2 \in [U(z, 0_L), U(z, 1_L)]$ there exists $x \in L$ fulfilling $U(z_1, x) = z_2$. For $z_1 \in \{0_L, 1_L, a\}$ this property is due to items 1., 4, of Proposition 3.5.
If $z_1 \in \|_a$ and $z_2 \in \|_a$, since $(\|_a, U)$ is a commutative group, there exists $x \in \|_a$ such that $U(z_1, x) = z_2$. The remaining part is due to item 2. of Proposition 3.5, and due to the fact that a is annihilator.

Now, assume that U is divisible. For $z_1 \in \|_a$ we have $U(z_1, 0_L) = 0_L$ and $U(z_1, 1_L) = 1_L$. Hence, for all $z_2 \in \|_a$ there exists $x \in L$ such that $U(z_1, x) = z_2$ only if $(\|_a, U)$ is a commutative group, since particularly for $z_2 = e$ we have $x = z_1^{-1}$, and this means that $(\|_a, U)$ is a commutative monoid possessing inverse elements to all $z_1 \in \|_a$ meaning that $(\|_a, U)$ is a commutative group. □

Example 3.7. *Consider the lattice L_2 depicted in Figure 2. On the lattice L_2, we can define a smooth uninorm U_1. By Proposition 3.5, if we define a commutative monoidal operation on $\|_a$, the uninorm will be given uniquely. Denote $e = x_0$. Then $U_1(x_i, x_j) = \max(i, j)$ for $i, j \in \{0, 1, 2, 3\}$, is a commutative monoidal operation. By proposition 3.5, we know that U_1 is smooth. However, this uninorm is not divisible since, e.g., there is no x such that $U_1(z_1, x) = x_0 = e$.*

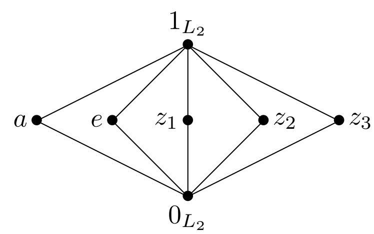

Figure 2: Hasse diagram of the lattice L_2

Now, we define a uninorm U_2 such that for $i, j \in \{0, 1, 2, 3\}$ the following holds

$$U_2(x_i, x_j) = \begin{cases} x_{i+1} & \text{if } i + j \le 3, \\ x_{i+j-4} & \text{otherwise.} \end{cases}$$

In this case, $(\|_a, U_2)$ is a cyclic commutative group. This gives that U_2 is both, smooth and divisible.

4 Conclusions

In the paper, we have studied smooth nullnorms on finite lattices, and a possibility to extend them to smooth uninorms, extending the domain by the horizontal sum of lattices. We have found necessary and sufficient conditions when this is possible. Moreover, on finite lattices whose maximal chains are of length 2, we have shown how smooth uninorms may look like. We have also found conditions under which such a smooth uninorm is also divisible.

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