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Original paper

Colimits in the category of structural spaces and their realizations in fuzzy-type categories

H. Yavari[1] and S. N. Hosseini[2]

[1,2] *Dept of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*

hyavari@math.uk.ac.ir, nhoseini@uk.ac.ir

Abstract

In this paper, we investigate the existence of colimits in the category of $\mathbb{O}$-structural spaces, denoted by $\mathbb{OSS}$. We show that $\mathbb{OSS}$ admits all small colimits. Special types of colimits, such as coproducts, coequalizers, and pushouts, are studied in detail. Based on these results, we explicitly describe the $\mathbb{O}$-structures of these colimits in several topological structures that are $\mathbb{O}$-structural spaces, including the categories of fuzzy topological spaces in the sense of Chang, intuitionistic fuzzy topological spaces in the sense of Coker, fuzzy topological spaces in the sense of Chakraborty and Ahsanullah, soft and fuzzy soft topological spaces, neutrosophic topological spaces, hypersoft topological spaces and various types of topological molecular lattice structures. In particular, we show how these colimits can be recovered via the underlying $\mathbb{O}$-structural framework. The results shed light on the internal structure of these categories and offer categorical tools for further analysis of generalized topological structures.

Keywords: $\mathbb{O}$-structure, (fuzzy, intuitionistic fuzzy, soft, neutrosophic, structural) topology, final $\mathbb{O}$-structure, coproduct, coequalizer, pushout.

1 Introduction

The study of categorical limits and colimits has long played a central role in modern mathematics, providing unifying tools to describe constructions across various categories. In topology and its fuzzy generalizations, categorical methods have proven especially powerful, as they allow systematic comparison and transfer of structures between different frameworks. The present paper is devoted to the investigation of colimits in the category of $\mathbb{O}$-structural spaces, denoted by $\mathbb{OSS}$, a category that encompasses a broad spectrum of generalized topological structures, including both classical and fuzzy-type spaces. The notion of $\mathbb{O}$-structural spaces was recently introduced as a unifying categorical framework that extends the ordinary concept of a topological space, see [36].

Historically, several fuzzy and soft generalizations of topology had been introduced in response to the limitations of classical topology in modeling uncertainty, vagueness, and partial truth. For instance, "Chang fuzzy topological spaces (type I)" were introduced in 1968 as the first systematic attempt to combine fuzzy set theory with topological structures, see [6]. "Intuitionistic fuzzy topological spaces (type III)", introduced by Coker in 1997, extended this framework by simultaneously accounting for degrees of membership and non-membership, see [7]. Another significant contribution was made by Chakraborty and Ahsanullah through their development of "type IV fuzzy topological spaces", which further generalized the interplay between fuzziness and topological structure, see [5].

Alongside these fuzzy models, more algebraically motivated approaches emerged: "topological molecular lattices" and "topological De Morgan molecular lattices" were proposed to capture the interaction between lattice-theoretic structures and topological operators, see [21, 35]. These categories highlight the role of order-theoretic dualities and algebraic constraints in shaping generalized topologies. "Topological fuzzes" are an important special class of topological

Corresponding Author: H. Yavari

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molecular lattices, introduced by Hutton in 1980, which sought to provide a unified fuzzification of classical topologies through algebraic and logical means, see [13].

In parallel, the framework of "soft set theory", introduced by Molodtsov in 1999, opened another path of generalization by modeling uncertainty via parameterized families of subsets, see [22]. The resulting "soft topological spaces", see [4, 31], and their subsequent refinements, such as "type I fuzzy soft topological spaces", see [25], have since become a rich area of research, particularly in applications. Similarly, "neutrosophic topological spaces", introduced by Smarandache in 2002 and further developed by other researchers, generalize intuitionistic fuzzy topological spaces by allowing an explicit degree of indeterminacy, see [18, 29, 33]. More recently, the concept of "hypersoft topological spaces" has been developed as an even more flexible extension of the soft approach, see [23].

The common feature of all these structures is that they can be embedded into the general framework of $\mathbb{O}$-structural spaces. This observation motivates our study. In this paper, we first prove that $\mathbb{OSS}$ admits all small colimits and explicitly characterize some important ones, such as coproducts, coequalizers, and pushouts. Second, we show how these colimits are realized within the aforementioned fuzzy-type categories. This unified categorical treatment provides a general framework for understanding colimit constructions across diverse generalized topological settings.

The notions of structural topology as well as structural continuity, on a categorical basis, are introduced in [11], where it is established that the ordinary concept of topology together with five different fuzzy topological notions appearing in the literature can all be subsumed as instances of structural topology. In [15], the category of structural topological spaces is constructed, and it is proved that under certain assumptions this category admits (finite) limits whenever the underlying category does. In [37] under a condition on the base $\mathbb{B}$ that is weaker than the one assumed in [15], it is shown that the category $\mathbb{OSS}$ has small limits whenever its underlying base category does. In the present paper, we adopt the framework developed in [36], which extends the constructions of [11, 15] by replacing the squaring functor with an arbitrary endofunctor. This rather mild modification allows for several additional examples to be included; see, for instance, Examples 6.4 and 6.5 in [38]. The categorical background for $\mathbb{O}$-structural spaces is developed in [38], where the topologicality and (co)completeness of the category were established. In the present paper, we continue that investigation by focusing on the formation and behavior of colimits and their realizations in fuzzy-type categories within the same framework. Some earlier results from [38] are restated in the preliminary section to make the exposition reasonably self-contained. Proofs are included only when needed later, and several new auxiliary results are established for subsequent use.

A natural motivation for studying colimits in this setting comes from the fact that many constructions in generalized topological structures are inherently colimit-based. Typical examples include coproduct constructions, amalgamations represented by pushouts, and identification or quotient constructions represented by coequalizers. In fuzzy-type and related generalized topological categories, such constructions often involve additional structural data and therefore require a systematic categorical treatment.

Despite the existence of several individual studies in specific frameworks, a unified categorical description of these colimit constructions within the general setting of $\mathbb{O}$-structural spaces has not been systematically developed. The present paper aims to fill this gap by developing a unified categorical treatment of colimits in $\mathbb{OSS}$, thereby complementing previous work on limits in the same framework. More precisely, we establish general existence results for colimits in $\mathbb{OSS}$ and explicitly realize important special colimits in a variety of fuzzy-type categories.

The duality between limits and colimits plays a fundamental role in category theory, often allowing constructions to be transferred from one setting to the other. In previous works [15, 37], limits were constructed based on the idea of generalizing initial topology to initial $\mathbb{O}$-structures induced by morphisms and further extended to sources. These constructions were obtained under the assumption that the base $\mathbb{B}$ is an I-base, or under the stronger condition that $\mathbb{B}$ is an IFE-base. In contrast, the main contribution of the present paper is to develop colimit constructions in the category of $\mathbb{OSS}$ based on the dual notion of final $\mathbb{O}$-structures coinduced by morphisms and extended to sinks. These results are established under the assumption that the base $\mathbb{B}$ is an M-base, and more generally under the stronger condition that $\mathbb{B}$ is an IM-base.

We briefly clarify the contribution of the present work. Section 2 is mainly expository and included for completeness, recalling results from [38] and related material from [36]. The main new contributions are contained in Sections 3 and 4, where we establish the existence of colimits in $\mathbb{OSS}$ and provide explicit constructions of coproducts, coequalizers, and pushouts, together with their realizations in various fuzzy-type categories. These results are the categorical dual of previous work on limits in $\mathbb{OSS}$ and contribute to the categorical understanding of generalized topological structures.

Throughout the paper, for simplicity of notation, we write $X \in \mathcal{C}$ to mean that X is an object of the category $\mathcal{C}$. For categorical concepts we refer the reader to [1].

2 Preliminaries

In this section, on a categorical basis, we introduce the notions of a base $\mathbb{B}$, an operant $\mathbb{O}$ relative to $\mathbb{B}$ and an $\mathbb{O}$-structure on an object X leading to the concept of $\mathbb{O}$-structural space. We also define the notion of $\mathbb{O}$-structural morphism and introduce the categories $\mathbb{OSS}$ and $[\mathbb{OSS}]$ of $\mathbb{O}$-structures and $\mathbb{O}$-structural morphisms. We prove that both $\mathbb{OSS}$ and $[\mathbb{OSS}]$ are concrete categories over $\mathcal{E}$ and that they are concretely equivalent. Finally, we show the existence of final $\mathbb{O}$-structures for both morphisms and sinks. For the convenience of the reader and to keep the paper reasonably self-contained, we recall several definitions and results from [36, 38], presented in a form adapted to the current framework. Previously established results are stated without proof unless they are used later, while proofs are provided for several new auxiliary results.

We say a category has *Mono*-pullbacks, when pullbacks of monomorphisms along every map exist. We recall from [36] that:

Definition 2.1. [36] *Let $\mathcal{E}$ and $\mathcal{C}$ be categories, with $\mathcal{C}$ well-powered and having a terminal object* 1.

a) *A triple of functors* $\mathcal{E}^{op} \xrightarrow{\mathsf{T}} \mathcal{C} \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} \mathcal{C}$ *is called a base on $(\mathcal{E}, \mathcal{C})$ and is denoted by $\mathbb{B}$. A base is said to be:*

- *an* I-*base if $\mathcal{C}$ has intersections,*
- *an* M-*base if $\mathcal{C}$ has Mono-pullbacks,*
- *an* IM-*base if it is both an* I-*base and an* M-*base.*

b) *An operant relative to a base $\mathbb{B} = (\mathsf{T}, \mathsf{P}, \mathsf{Q})$ or a $\mathbb{B}$-operant is a quadruple $\mathbb{O} = (\beta, \tau, \mu, \nu)$, where*

$$1 \xrightarrow{\beta} \mathsf{T}\,, \qquad 1 \xrightarrow{\tau} \mathsf{T}\,, \qquad \mathsf{P} \circ \mathsf{T} \xrightarrow{\mu} \mathsf{T} \;\; \text{and} \;\; \mathsf{Q} \circ \mathsf{T} \xrightarrow{\nu} \mathsf{T}\,,$$

are natural transformations, in which $\mathcal{E}^{op} \xrightarrow{1} \mathcal{C}$ *denotes the constant functor with terminal object 1 as its value.*

c) *A structure with respect to an operant $\mathbb{O}$ or an $\mathbb{O}$-structure, on an object X of $\mathcal{E}$, is a $\mathcal{C}$-monomorphism* $T_X \overset{t_X}{\rightarrowtail} \mathsf{T}(X)$ *, such that the morphisms β_X, τ_X, μ_X and ν_X restrict to T_X in the sense that there exist morphisms* $1 \xrightarrow{\beta'_X} T_X$, $1 \xrightarrow{\tau'_X} T_X$, $\mathsf{P}(T_X) \xrightarrow{\mu'_X} T_X$ *and* $\mathsf{Q}(T_X) \xrightarrow{\nu'_X} T_X$ *rendering commutative the following diagrams.*

$$\begin{tikzcd} & T_X \arrow[d, tail, "t_X"] \\ 1 \arrow[ur, "\beta'_X"] \arrow[r, "\beta_X"'] & \mathsf{T}(X) \end{tikzcd} \qquad \begin{tikzcd} & T_X \arrow[d, tail, "t_X"] \\ 1 \arrow[ur, "\tau'_X"] \arrow[r, "\tau_X"'] & \mathsf{T}(X) \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{P}(T_X) \arrow[r, "\mu'_X"] \arrow[d, "\mathsf{P}(t_X)"'] & T_X \arrow[d, tail, "t_X"] \\ \mathsf{P}(\mathsf{T}(X)) \arrow[r, "\mu_X"'] & \mathsf{T}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Q}(T_X) \arrow[r, "\nu'_X"] \arrow[d, "\mathsf{Q}(t_X)"'] & T_X \arrow[d, tail, "t_X"] \\ \mathsf{Q}(\mathsf{T}(X)) \arrow[r, "\nu_X"'] & \mathsf{T}(X) \end{tikzcd}$$

When t_X is an $\mathbb{O}$-structure on X, we call the pair (X, t_X) an $\mathbb{O}$-structural space. When it causes no ambiguity, we also write β_X for β'_X, etc.

Definition 2.2. [36] *Given $\mathbb{O}$-structural spaces (X, t_X) and (Y, t_Y), an $\mathcal{E}$-morphism $f : X \longrightarrow Y$ is said to be an $\mathbb{O}$-structural morphism, if there exists a (necessarily unique) $\mathcal{C}$-morphism $T_{(f,t_X,t_Y)} : T_Y \longrightarrow T_X$ induced by f and $\mathbb{O}$-structures t_X and t_Y such that the following diagram commutes.*

$$\begin{tikzcd} T_Y \arrow[r, tail, "t_Y"] \arrow[d, "T_{(f,t_X,t_Y)}"'] & \mathsf{T}(Y) \arrow[d, "\mathsf{T}(f)"] \\ T_X \arrow[r, tail, "t_X"'] & \mathsf{T}(X) \end{tikzcd}$$

In this case, we write $f : (X, t_X) \longrightarrow (Y, t_Y)$ *. When it causes no ambiguity, we also write* T_f *for* $T_{(f,t_X,t_Y)}$*.*

We recall that for morphisms $A \xrightarrow{a} C$ and $B \xrightarrow{b} C$, we write $a \leq b$, if there is a morphism $A \xrightarrow{h} B$ such that $a = bh$. When restricted to monomorphisms, this preorder induces the standard partial order on $\mathrm{Sub}(A)$, where $\mathrm{Sub}(A)$ is the collection of equivalence classes of monomorphisms with codomain A that is a partially ordered set, because $\mathcal{C}$ is well-powered. In particular, the identity morphism 1_A represents the greatest element of $\mathrm{Sub}(A)$. By a similar argument that is given in [19] on page 145, since for an I-base the category $\mathcal{C}$ has intersections, the collection $\mathrm{Sub}(A)$ of subobjects of $A \in \mathcal{C}$ is a complete lattice, in which meet (denoted by $\wedge$) is the same as intersection and the existence of join (denoted by $\vee$) is implied by that of meet.

Proposition 2.3. [38] *Let* $\mathbb{O}$ *be a* $\mathbb{B}$*-operant on an* I*-base* $\mathbb{B}$*. For an object* X *of* $\mathcal{E}$*, the intersection of any collection* $\{t_i\}_I$ *of* $\mathbb{O}$*-structures on* X *is an* $\mathbb{O}$*-structure on* X *and is denoted by* $\bigwedge_{i\in I} t_i$*.*

Proposition 2.4. [38] *Let* $\mathbb{O}$ *be a* $\mathbb{B}$*-operant on an* I*-base* $\mathbb{B}$*. For* $a \in \mathrm{Sub}(\mathsf{T}(X))$*, there exists a smallest* $\mathbb{O}$*-structure* t *on* X *containing* a*, i.e.,*

a) $a \leq t$ *and*

b) *if* t' *is an* $\mathbb{O}$*-structure on* X *with* $a \leq t'$*, then* $t \leq t'$*.*

The $\mathbb{O}$-structure in the above proposition is called the $\mathbb{O}$-structure induced by a and is denoted by $\langle a \rangle$.

Proposition 2.5. [38] *Let* $\mathbb{O}$ *be a* $\mathbb{B}$*-operant on an* I*-base* $\mathbb{B}$*. For* $\mathbb{O}$*-structures* t_i *on* X*,* $i \in I$*, there is a smallest* $\mathbb{O}$*-structure* t *containing* t_i *for all* $i \in I$*.*

The $\mathbb{O}$-structure in the above proposition is called the $\mathbb{O}$-structure induced by the $\mathbb{O}$-structures t_i and is denoted by $\langle \bigvee_{i\in I} t_i \rangle$.

Definition 2.6. [38] $\mathbb{O}$*-structures* $S_X \xrightarrow{s_X} \mathsf{T}(X)$ *and* $T_X \xrightarrow{t_X} \mathsf{T}(X)$ *are said to be isomorphic if there is a* $\mathcal{C}$*-isomorphism* $\phi : S_X \longrightarrow T_X$ *such that* $s_X = t_X\phi$*. In this case we write* $s_X \cong t_X$*. The isomorphism class of* t_X *is denoted by* $[t_X]$*.*

Theorem 2.7. [38] *Let* $\mathbb{O}$ *be a* $\mathbb{B}$*-operant on a base* $\mathbb{B}$*. Given* $\mathbb{O}$*-structural spaces* (X, t_X)*,* (Y, t_Y) *and* (Z, t_Z)*,*

1. *the identity morphism* $(X, t_X) \xrightarrow{1_X} (X, t_X)$ *is an* $\mathbb{O}$*-structural morphism.*
2. *the composition of two* $\mathbb{O}$*-structural morphisms*

$$(X, t_X) \xrightarrow{f} (Y, t_Y) \xrightarrow{g} (Z, t_Z)$$

is an $\mathbb{O}$*-structural morphism.*

Theorem 2.8. [38] *Let* $\mathbb{O}$ *be a* $\mathbb{B}$*-operant on a base* $\mathbb{B}$ *and* $(X, t_X) \xrightarrow{f} (Y, t_Y)$ *be an* $\mathbb{O}$*-structural morphism. If* t'_X *is an* $\mathbb{O}$*-structure on* X *such that* $t_X \leq t'_X$ *and* t'_Y *is an* $\mathbb{O}$*-structure on* Y *such that* $t'_Y \leq t_Y$*, then*

a) $(X, t'_X) \xrightarrow{f} (Y, t_Y)$ *and*

b) $(X, t_X) \xrightarrow{f} (Y, t'_Y)$

are $\mathbb{O}$*-structural morphisms.*

Corollary 2.9. [38] *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. Consider the $\mathbb{O}$-structures* $S_X \overset{s_X}{\rightarrowtail} \mathsf{T}(X)$, $T_X \overset{t_X}{\rightarrowtail} \mathsf{T}(X)$, $S_Y \overset{s_Y}{\rightarrowtail} \mathsf{T}(Y)$, *and* $T_Y \overset{t_Y}{\rightarrowtail} \mathsf{T}(Y)$. *Assume that* $s_X \cong t_X$ *and* $s_Y \cong t_Y$. *If a morphism* $f : (X, t_X) \longrightarrow (Y, t_Y)$ *is $\mathbb{O}$-structural, so is* $f : (X, s_X) \longrightarrow (Y, s_Y)$.

By an abuse of language, we also refer to the isomorphism class $[t_X]$ as an $\mathbb{O}$-structure on X, and we say that a morphism $f : (X, [t_X]) \longrightarrow (Y, [t_Y])$ is $\mathbb{O}$-structural if the morphism $f : (X, t_X) \longrightarrow (Y, t_Y)$ is, we have:

Theorem 2.10. [38] *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. The collections of $\mathbb{O}$-structural spaces and $\mathbb{O}$-structural morphisms (in both senses) form a category.*

The categories in the above theorem are denoted by $\mathbb{OSS}$ and $[\mathbb{OSS}]$, respectively.

Theorem 2.11. [38] *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. The categories $\mathbb{OSS}$ and $[\mathbb{OSS}]$ are concrete categories over $\mathcal{E}$ and they are concretely equivalent.*

Proof. The mapping $U : \mathbb{OSS} \longrightarrow \mathcal{E}$ that takes $f : (X, t_X) \longrightarrow (Y, t_Y)$ in $\mathbb{OSS}$ to $f : X \longrightarrow Y$ in $\mathcal{E}$ is easily seen to be a faithful functor. There also exists a similar faithful functor $V : [\mathbb{OSS}] \longrightarrow \mathcal{E}$. The last assertion holds by a straight verification that the mapping $G : \mathbb{OSS} \longrightarrow [\mathbb{OSS}]$ sending $f : (X, t_X) \longrightarrow (Y, t_Y)$ to $f : (X, [t_X]) \longrightarrow (Y, [t_Y])$ is the desired concrete equivalence. □

Lemma 2.12. *Let* $(\mathcal{A}, U) \xrightarrow{G} (\mathcal{B}, V)$ *be a concrete equivalence between concrete categories over the base category $\mathcal{X}$. Then a source* $\mathcal{S} = \big(A \xrightarrow{\alpha_i} A_i\big)_I$ *in $\mathcal{A}$ is U-initial if and only if the source* $G(\mathcal{S}) = \big(G(A) \xrightarrow{G(\alpha_i)} G(A_i)\big)_I$ *in $\mathcal{B}$ is V-initial. Equivalently, concrete equivalences preserve and reflect initial sources.*

Proof. Assume first that $\mathcal{S} = \big(A \xrightarrow{\alpha_i} A_i\big)_I$ is a U-initial source in $\mathcal{A}$. We show that $G(\mathcal{S}) = \big(G(A) \xrightarrow{G(\alpha_i)} G(A_i)\big)_I$ is a V-initial source in $\mathcal{B}$. Let B be an object of $\mathcal{B}$ and $h : V(B) \longrightarrow V(G(A))$ be a morphism in $\mathcal{X}$ such that, for each $i \in I$, $V(G(\alpha_i))h$ is a $\mathcal{B}$-morphism. Hence, for each $i \in I$, there exists a morphism $h_i : B \longrightarrow G(A_i)$ such that $V(h_i) = V(G(\alpha_i))h$. Since G is an equivalence, it is isomorphism-dense. Thus, for the object B of $\mathcal{B}$, there exists an object A' of $\mathcal{A}$ and an isomorphism $G(A') \xrightarrow{\varepsilon_B} B$. Using the fullness and faithfulness of G, for each $i \in I$ there exists a unique morphism $k_i : A' \longrightarrow A_i$ such that $G(k_i) = h_i\varepsilon_B$. Putting $g = hV(\varepsilon_B)$, for each $i \in I$, one obtains an $\mathcal{A}$-morphism $U(\alpha_i)g : U(A') \longrightarrow U(A_i)$ such that $U(k_i) = U(\alpha_i)g$. Since $\mathcal{S}$ is U-initial, there exists a morphism $u : A' \longrightarrow A$ such that $U(u) = g$ and $\alpha_i u = k_i$ for all $i \in I$. Define $w := G(u)\varepsilon_B^{-1} : B \longrightarrow G(A)$. Then $V(w) = h$, and from the faithfulness of V it follows that $G(\alpha_i)w = h_i$ for all $i \in I$. Hence $G(\mathcal{S})$ is V-initial.

Conversely, assume that $G(\mathcal{S})$ is V-initial. Let $h : U(A') \longrightarrow U(A)$ be a morphism in $\mathcal{X}$ such that, for each $i \in I$, $U(\alpha_i)h$ is an $\mathcal{A}$-morphism. For each $i \in I$, let $k_i : A' \longrightarrow A_i$ be the morphism satisfying $U(k_i) = U(\alpha_i)h$. By applying G and using $U = VG$, we obtain a family of $\mathcal{B}$-morphisms $G(k_i)$ satisfying $V(G(k_i)) = V(G(\alpha_i))h$. Since $G(\mathcal{S})$ is V-initial, there exists a morphism $\bar{h} : G(A') \longrightarrow G(A)$ lifting h. By fullness and faithfulness of G, there exists a unique morphism $g : A' \longrightarrow A$ such that $G(g) = \bar{h}$. Then $U(g) = VG(g) = V(\bar{h}) = h$, and the faithfulness of U yields $\alpha_i g = k_i$ for all $i \in I$. Therefore $\mathcal{S}$ is U-initial. □

Proposition 2.13. *Let* $(\mathcal{A}, U) \xrightarrow{G} (\mathcal{B}, V)$ *be a surjective concrete equivalence between concrete categories over the base $\mathcal{X}$. Then $(\mathcal{A}, U)$ is topological over $\mathcal{X}$ if and only if $(\mathcal{B}, V)$ is topological over $\mathcal{X}$.*

Proof. By Lemma 2.12, a source in $\mathcal{A}$ is U-initial if and only if its image under G is a V-initial source in $\mathcal{B}$. Moreover, since G is surjective on objects and $U = VG$, every V-structured source can be regarded as a U-structured source, and vice versa. Hence, the existence and uniqueness of U-initial lifts of U-structured sources are equivalent to the existence and uniqueness of V-initial lifts of V-structured sources. □

Corollary 2.14. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. Then the concrete category $(\mathbb{OSS}, U)$ is topological over $\mathcal{E}$.*

Proof. By Theorem 2.11, the concrete categories $(\mathbb{OSS}, U)$ and $([\mathbb{OSS}], V)$ are concretely equivalent via a surjective functor G. Moreover, it was shown in Theorem 5.5 of [38] that $([\mathbb{OSS}], V)$ is topological over $\mathcal{E}$. Therefore, Proposition 2.13 implies that $(\mathbb{OSS}, U)$ is topological over $\mathcal{E}$. $\square$

For simplicity, we prove many of our results for the category $\mathbb{OSS}$. One can easily verify that results similar to those proved for $\mathbb{OSS}$ hold for $[\mathbb{OSS}]$.

Now we prove the existence of final $\mathbb{O}$-structures coinduced by morphisms and we use that to prove the same for sinks.

Proposition 2.15. [38] *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. Let $f : X \longrightarrow Y$ be an $\mathcal{E}$-morphism and $T_X \overset{t_X}{\rightarrowtail} \mathsf{T}(X)$ be an $\mathbb{O}$-structure on X. There is a largest $\mathbb{O}$-structure t_Y on Y making f an $\mathbb{O}$-structural morphism, obtained as:*

- **a)** *the pullback of t_X along $\mathsf{T}(f)$, provided that $\mathbb{B}$ is an* M*-base, and*
- **b)** *the join $\langle \bigvee_{t \in \mathfrak{F}} t \rangle$, where*

$$\mathfrak{F} = \{t : t \textit{ is an } \mathbb{O}\textit{-structure on } Y \textit{ making } f\ \mathbb{O}\textit{-structural}\},$$

provided that $\mathbb{B}$ is an IM*-base.*

Proof. **a)** We take the pullback of $T_X \overset{t_X}{\rightarrowtail} \mathsf{T}(X)$ along $\mathsf{T}(f) : \mathsf{T}(Y) \longrightarrow \mathsf{T}(X)$ to get $T_Y \overset{t_Y}{\rightarrowtail} \mathsf{T}(Y)$ as the following diagram shows.

$$\begin{tikzcd}
T_Y \arrow[r, tail, "t_Y"] \arrow[d, "T_f"'] \arrow[dr, phantom, "p.b."] & \mathsf{T}(Y) \arrow[d, "\mathsf{T}(f)"] \\
T_X \arrow[r, tail, "t_X"'] & \mathsf{T}(X)
\end{tikzcd}$$

First we show t_Y is an $\mathbb{O}$-structure on Y. In the following diagram since the outer square commutes, there is a unique morphism $1 \xrightarrow{\beta_Y} T_Y$ making the triangles commute.

$$\begin{tikzcd}
1 \arrow[dr, "\beta_Y"] \arrow[drr, bend left, "\beta_Y"] \arrow[ddr, bend right, "\beta_X"'] & & \\
& T_Y \arrow[r, tail, "t_Y"] \arrow[d, "T_f"'] & \mathsf{T}(Y) \arrow[d, "\mathsf{T}(f)"] \\
& T_X \arrow[r, tail, "t_X"'] & \mathsf{T}(X)
\end{tikzcd}$$

The commutativity of the upper triangle gives $t_Y \beta_Y = \beta_Y$ as desired. Similarly we get a morphism $1 \xrightarrow{\tau_Y} T_Y$ rendering commutative the triangles in the following diagram.

$$\begin{tikzcd}
1 \arrow[dr, "\tau_Y"] \arrow[drr, bend left, "\tau_Y"] \arrow[ddr, bend right, "\tau_X"'] & & \\
& T_Y \arrow[r, tail, "t_Y"] \arrow[d, "T_f"'] & \mathsf{T}(Y) \arrow[d, "\mathsf{T}(f)"] \\
& T_X \arrow[r, tail, "t_X"'] & \mathsf{T}(X)
\end{tikzcd} .$$

Thus we have $t_Y \tau_Y = \tau_Y$. To show the existence of $\mu_Y : \mathsf{P}(T_Y) \longrightarrow T_Y$, we have:

$$\mathsf{T}(f)\mu_Y \mathsf{P}(t_Y) = \mu_X \mathsf{P}(\mathsf{T}(f))\mathsf{P}(t_Y) = \mu_X \mathsf{P}(\mathsf{T}(f) t_Y) = \mu_X \mathsf{P}(t_X T_f) = \mu_X \mathsf{P}(t_X)\mathsf{P}(T_f) = t_X \mu_X \mathsf{P}(T_f).$$

This shows that in the following diagram, the outer square commutes, and hence there exists a unique morphism $\mu_Y : \mathsf{P}(T_Y) \longrightarrow T_Y$ making the triangles commute.

$$\begin{tikzcd}
\mathsf{P}(T_Y) \arrow[dr, "\mu_Y"] \arrow[drr, bend left, "\mu_Y \mathsf{P}(t_Y)"] \arrow[ddr, bend right, "\mu_X \mathsf{P}(T_f)"'] & & \\
& T_Y \arrow[r, tail, "t_Y"] \arrow[d, "T_f"'] & \mathsf{T}(Y) \arrow[d, "\mathsf{T}(f)"] \\
& T_X \arrow[r, tail, "t_X"'] & \mathsf{T}(X)
\end{tikzcd}$$

The commutativity of the upper triangle results in the commutativity of the desired square.

Replacing P by Q, and μ by ν, similar arguments show the existence of a morphism $\nu_Y : \mathsf{Q}(T_Y) \longrightarrow T_Y$ making the corresponding square commutative. This proves t_Y is an $\mathbb{O}$-structure on Y.

The pullback square that defines t_Y shows that $f : (X, t_X) \longrightarrow (Y, t_Y)$ is an $\mathbb{O}$-structural morphism and one can verify that t_Y is the largest $\mathbb{O}$-structure on Y making f an $\mathbb{O}$-structural morphism.

b) With t_Y as in part (a), we have for all $t \in \mathfrak{F}$, $t \leq t_Y$. So $\bigvee_{t\in\mathfrak{F}} t \leq t_Y$. By Proposition 2.4, $\langle \bigvee_{t\in\mathfrak{F}} t \rangle$ is the smallest $\mathbb{O}$-structure on Y containing $\bigvee_{t\in\mathfrak{F}} t$, implying $\langle \bigvee_{t\in\mathfrak{F}} t \rangle \leq t_Y$. On the other hand, by part (a), $t_Y \in \mathfrak{F}$ and so $t_Y \leq \bigvee_{t\in\mathfrak{F}} t$. Thus $t_Y \leq \langle \bigvee_{t\in\mathfrak{F}} t \rangle$. Hence $\langle \bigvee_{t\in\mathfrak{F}} t \rangle \cong t_Y$ and therefore t_Y is the largest $\mathbb{O}$-structure on Y making f $\mathbb{O}$-structural. □

The $\mathbb{O}$-structure in part (a) of the above theorem is called the final $\mathbb{O}$-structure on Y coinduced by the morphism f and the $\mathbb{O}$-structure t_X and is denoted by $t_Y^{(\!(f)\!)}$.

Let's remark that under the stronger condition that $\mathbb{B}$ is an IM-base, the $\mathbb{O}$-structures obtained in parts (a) and (b) of the above proposition are obviously isomorphic, in which case $\langle \bigvee_{t\in\mathfrak{F}} t \rangle$ is the pullback of t_X along $\mathsf{T}(f)$.

Theorem 2.16. [38] *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. Let for each $i \in I$, $f_i : X_i \longrightarrow Y$ be an $\mathcal{E}$-morphism and t_i be an $\mathbb{O}$-structure on X_i. There is a largest $\mathbb{O}$-structure on Y making f_i an $\mathbb{O}$-structural morphism, for all i.*

Proof. By Proposition 2.15, for each i there is a largest $\mathbb{O}$-structure s_i on Y, making f_i an $\mathbb{O}$-structural morphism. Set $t_Y = \bigwedge_{i\in I} s_i$. By Proposition 2.3, t_Y is an $\mathbb{O}$-structure on Y. Since for each i $f_i : (X_i, t_i) \longrightarrow (Y, s_i)$ is an $\mathbb{O}$-structural morphism and $t_Y \leq s_i$, by Theorem 2.8 $f_i : (X_i, t_i) \longrightarrow (Y, t_Y)$ is an $\mathbb{O}$-structural morphism. To show t_Y is the largest such, suppose t'_Y is another $\mathbb{O}$-structure on Y making $f_i : (X_i, t_i) \longrightarrow (Y, t'_Y)$ $\mathbb{O}$-structural for all $i \in I$. Since for each i, $f_i : (X_i, t_i) \longrightarrow (Y, s_i)$ is $\mathbb{O}$-structural, by Proposition 2.15 we get $t'_Y \leq s_i$. This is the case for all i, therefore $t'_Y \leq \bigwedge_{i\in I} s_i = t_Y$. □

The $\mathbb{O}$-structure in the above theorem is called the final $\mathbb{O}$-structure on Y coinduced by the morphisms f_i (or by the sink $(\!(f_i)\!)_I$) and the $\mathbb{O}$-structures t_i, $i \in I$ and is denoted by $t_Y^{(\!(f_i)\!)_I}$.

3 Colimits in $\mathbb{OSS}$

In this section, we prove that the category $\mathbb{OSS}$ admits colimits for every diagram $D : \mathcal{I} \longrightarrow \mathbb{OSS}$, where $\mathcal{I}$ is any category, provided that the corresponding diagram has colimits in the underlying base category. We then give explicit constructions of certain colimits such as coproducts, coequalizers, and pushouts in $\mathbb{OSS}$.

Theorem 3.1. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. If $\mathcal{E}$ has colimits, then so does the category $\mathbb{OSS}$.*

Proof. Suppose $\mathcal{I} \xrightarrow{D} \mathbb{OSS}$ is a diagram in $\mathbb{OSS}$ consisting of a family $\{(D_i, t_i)\}_{\mathcal{I}}$ of $\mathbb{O}$-structural spaces, where $T_i \overset{t_i}{\rightarrowtail} \mathsf{T}(D_i)$ is an $\mathbb{O}$-structure on D_i, for each $i \in \mathcal{I}$. Also let $\mathcal{L} = (\!(D_i \xrightarrow{\theta_i} C)\!)_{\mathcal{I}}$ be the colimit of $\mathcal{I} \xrightarrow{U\circ D} \mathcal{E}$ in $\mathcal{E}$, by Theorem 2.16, we equip C with the final $\mathbb{O}$-structure $T_C^{(\!(\theta_i)\!)_{\mathcal{I}}} \overset{t_C^{(\!(\theta_i)\!)_{\mathcal{I}}}}{\rightarrowtail} \mathsf{T}(C)$ coinduced by the cocone $\mathcal{L}$ (a natural sink for $U \circ D$), in which $t_C^{(\!(\theta_i)\!)_{\mathcal{I}}} = \bigwedge_{i\in\mathcal{I}} t_C^{(\!(\theta_i)\!)}$ where by Proposition 2.15, $t_C^{(\!(\theta_i)\!)}$ is the final $\mathbb{O}$-structure coinduced by the morphism θ_i, for each $i \in \mathcal{I}$. Now let $\llparenthesis (D_i, t_i) \xrightarrow{\alpha_i} (A, t_A) \rrparenthesis_{\mathcal{I}}$ be a given cocone on D, where $T_A \rightarrowtail^{t_A} \mathsf{T}(A)$ is an $\mathbb{O}$-structure on A. In the following diagram, for each $i \xrightarrow{f} j$ in $\mathcal{I}$, by the assumption and the faithfulness of $\mathbb{OSS} \xrightarrow{U} \mathcal{E}$, right and left triangles commute. So by the universal property of colimits in $\mathcal{E}$, there exists a unique morphism $C \xrightarrow{\alpha} A$ making the lower triangle commutative.

$$\begin{tikzcd}
& D_i \arrow[d, "D(f)"] \arrow[ddl, bend right, "\theta_i"'] \arrow[ddr, bend left, "\alpha_i"] & \\
& D_j \arrow[dl, "\theta_j"'] \arrow[dr, "\alpha_j"] & \\
C \arrow[rr, dotted, "\alpha"] & & A
\end{tikzcd}$$

We need to show that α is an $\mathbb{O}$-structural morphism. In the following diagram, the inner square is a pullback by Proposition 2.15. Moreover, for each $i \in \mathcal{I}$, the outer square commutes by the $\mathbb{O}$-structurality of α_i, and by the commutativity of outer triangle, as seen in the above diagram. Hence, for each $i \in \mathcal{I}$, there exists a morphism $T_A \xrightarrow{\lambda_i} T_C^{\llparenthesis \theta_i \rrparenthesis}$ making both triangles commute.

$$\begin{tikzcd}
T_A \arrow[dr, dotted, "\lambda_i"] \arrow[drr, bend left, "\mathsf{T}(\alpha)t_A"] \arrow[ddr, bend right, "T_{\alpha_i}"'] & & \\
& T_C^{\llparenthesis \theta_i \rrparenthesis} \arrow[r, tail, "t_C^{\llparenthesis \theta_i \rrparenthesis}"] \arrow[d, "T_{\theta_i}"'] \arrow[dr, phantom, "p.b."] & \mathsf{T}(C) \arrow[d, "\mathsf{T}(\theta_i)"] \\
& T_i \arrow[r, tail, "t_i"'] & \mathsf{T}(D_i)
\end{tikzcd}$$

The commutativity of upper triangle in the above diagram implies that for each $i \in \mathcal{I}$, $\mathsf{T}(\alpha)t_A \leq t_C^{\llparenthesis \theta_i \rrparenthesis}$. Now by the proof of Proposition 2.15 since $t_C^{\llparenthesis \theta_i \rrparenthesis_{\mathcal{I}}} = \bigwedge_{i \in \mathcal{I}} t_C^{\llparenthesis \theta_i \rrparenthesis}$, then $\mathsf{T}(\alpha)t_A \leq t_C^{\llparenthesis \theta_i \rrparenthesis_{\mathcal{I}}}$. So there exists a unique morphism $T_A \xrightarrow{T_\alpha} T_C^{\llparenthesis \theta_i \rrparenthesis_{\mathcal{I}}}$ making the corresponding square commute as required. Hence $(C, t_C^{\llparenthesis \theta_i \rrparenthesis_{\mathcal{I}}}) \xrightarrow{\alpha} (A, t_A)$ is an $\mathbb{O}$-structural morphism. Finally, since the functor $\mathbb{OSS} \xrightarrow{U} \mathcal{E}$ is faithful, the universal property of colimits in $\mathcal{E}$ guarantees the uniqueness of α in $\mathbb{OSS}$, which completes the proof.

□

The proof of the above theorem shows that whenever the category $\mathcal{E}$ admits coproducts, coequalizers, and pushouts, the category $\mathbb{OSS}$ also admits these colimits. In the sequel, we focus only on describing the corresponding $\mathbb{O}$-structures on these colimits.

Theorem 3.2. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$.*

a) *Let $\{(X_i, t_i)\}_I$ be a family of objects in $\mathbb{OSS}$, with $t_i : T_i \rightarrowtail \mathsf{T}(X_i)$. If $\llparenthesis \nu_i : X_i \longrightarrow \coprod_{j \in I} X_j \rrparenthesis_I$ is a coproduct sink in $\mathcal{E}$, then $\llparenthesis \nu_i : (X_i, t_i) \longrightarrow (\coprod_{j \in I} X_j, t^{\llparenthesis \nu_i \rrparenthesis_I}) \rrparenthesis_I$ is a coproduct sink in $\mathbb{OSS}$, where $t^{\llparenthesis \nu_i \rrparenthesis_I}$ is the final $\mathbb{O}$-structure coinduced by the sink $\llparenthesis \nu_i : X_i \longrightarrow \coprod_{j \in I} X_j \rrparenthesis_I$.*

b) *Let $(X, t_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, t_Y)$ be a pair of parallel morphisms in $\mathbb{OSS}$. If $X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{q} Q$ is a coequalizer in $\mathcal{E}$, then $(X, t_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, t_Y) \xrightarrow{q} (Q, t_Q^{\llparenthesis q \rrparenthesis})$ is a coequalizer in $\mathbb{OSS}$, where $t_Q^{\llparenthesis q \rrparenthesis}$ is the final $\mathbb{O}$-structure on Q coinduced by q.*

c) *Let* $(X,t_X) \xleftarrow{f} (Z,t_Z) \xrightarrow{g} (Y,t_Y)$ *be a pair of morphisms in* $\mathbb{OSS}$. *If the square*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "p.o."] & Y \arrow[d, "q_2"] \\ X \arrow[r, "q_1"'] & Q \end{tikzcd}$$

is a pushout of f *along* g *in* $\mathcal{E}$, *then the square*

$$\begin{tikzcd} (Z,t_Z) \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "p.o."] & (Y,t_Y) \arrow[d, "q_2"] \\ (X,t_X) \arrow[r, "q_1"'] & (Q,t_Q^{(\!(q_1,q_2)\!)}) \end{tikzcd}$$

is a pushout of f *along* g *in* $\mathbb{OSS}$, *where* $t_Q^{(\!(q_1,q_2)\!)}$ *is the final* $\mathbb{O}$*-structure on* Q *coinduced by* $X \xrightarrow{q_1} Q \xleftarrow{q_2} Y$.

Corollary 3.3. *The converses of Theorems 3.1 and 3.2 hold.*

Proof. By Corollary 2.14, the concrete category $(\mathbb{OSS}, U)$ is topological over $\mathcal{E}$. Hence, by Theorem 21.16 of [1], $\mathbb{OSS}$ is cocomplete if and only if $\mathcal{E}$ is cocomplete. Therefore, if $\mathbb{OSS}$ is cocomplete, then $\mathcal{E}$ is cocomplete, proving the converse of Theorem 3.1. The converse of Theorem 3.2 follows immediately, since coproducts, coequalizers and pushouts are particular colimits and their constructions in $\mathbb{OSS}$ are obtained by lifting the corresponding colimits from $\mathcal{E}$ via the final $\mathbb{O}$-structures described in Theorem 3.2. □

4 Special colimits in fuzzy-type categories

By considering different bases $\mathbb{B}$ and taking various $\mathbb{B}$-operants $\mathbb{O}$, one will get several categories of $\mathbb{O}$-structural spaces. Let us call a category $\mathbb{O}$-structural if it is isomorphic to $[\mathbb{OSS}]$ for some base $\mathbb{B}$ and $\mathbb{B}$-operant $\mathbb{O}$. In this section we provide a vast number of, mostly known, categories that are $\mathbb{O}$-structural and we give simplified equivalent conditions for special colimits in such categories. The $\mathbb{O}$-structurality of each category is either referred to an article where it is proved or left as an easy exercise for the reader to prove it. In the following examples, we use the squaring functor $\mathsf{S} : \mathcal{C} \longrightarrow \mathcal{C}$ and the covariant powerset functor $\mathcal{P} : \mathbf{Sets} \longrightarrow \mathbf{Sets}$, which sends a morphism $f : X \longrightarrow Y$ to $f \times f : X \times X \longrightarrow Y \times Y$ and to the direct image function $\mathcal{P}(f) : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, respectively.

Example 4.1. *One can show that the category* $\mathbf{FTop_I}$ *of type one fuzzy topological spaces, see* [6], *and continuous functions is* $\mathbb{O}$*-structural, see* [11], *by letting* $\mathbb{B}$ *be the base* $\mathbf{Sets}^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{S}}{\rightrightarrows}} \mathbf{Sets}$, *where* T *is the functor that takes* $f : X \longrightarrow Y$ *to the "composition with* f *on the left" function* $- \circ f : I^Y \longrightarrow I^X$ *with* I *being the unit interval,* P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O} = (\beta, \tau, \mu, \nu)$ *be the so-called type one fuzzy topology operant defined for each* $X \in \mathbf{Sets}$ *by,*

$$1 \xrightarrow{\beta_X} I^X, \;\; \textit{taking } 1 \longmapsto \underline{0}$$
$$1 \xrightarrow{\tau_X} I^X, \;\; \textit{taking } 1 \longmapsto \underline{1}$$
$$\mathcal{P}(I^X) \xrightarrow{\mu_X} I^X, \;\; \textit{taking } E \longmapsto \bigvee_{A \in E} A \qquad \textit{and}$$
$$I^X \times I^X \xrightarrow{\nu_X} I^X, \;\; \textit{taking } (A,B) \longmapsto A \wedge B$$

The functions $\underline{0}$ *and* $\underline{1}$ *are the constant functions with values* 0 *and* 1, *respectively. Since* $\mathcal{E} = \mathbf{Sets}$ *is cocomplete, see* [19], *by Theorems 3.1 and 2.11, the category* $\mathbf{FTop_I}$ *is concretely cocomplete. Now, by Theorem 3.2 part* **(a)**, *the coproduct of the given family* $\{(X_i, T_{X_i})\}_I$ *of type one fuzzy topological spaces is* $(X, T_X^{(\!(\nu_i)\!)_I})$, *where* $X := \coprod_{j \in I} X_j$ *is the disjoint union of the family of sets* $\{X_i\}_I$, $T_X^{(\!(\nu_i)\!)_I} = \{A \in I^X : (\forall i \in I) A \circ \nu_i \in T_{X_i}\}$ *and* $\nu_i : X_i \longrightarrow X$ *is the injection* *of the coproduct, for each* $i \in I$. *Also by Theorem 3.2 part* **(b)**, *one can easily verify that* $(Y, T_Y) \xrightarrow{q} (Q, T_Q^{(\!(q)\!)})$ *is a coequalizer of the parallel type one fuzzy continuous functions* $(X, T_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, T_Y)$, *where* $T_Q^{(\!(q)\!)} = \{A \in I^Q : A \circ q \in T_Y\}$, $Q := Y/\sim$, $\sim$ *is the smallest equivalence relation containing the relation* $y_1 \approx y_2$ *if and only if there exists* $x \in X$ *such that* $y_1 = f(x)$ *and* $y_2 = g(x)$, *for each* $y_1, y_2 \in Y$, *and* q *is the quotient map taking each* $y \in Y$ *to its equivalence class* $[y]$. *Furthermore for type one fuzzy continuous functions* $(X, T_X) \xleftarrow{f} (Z, T_Z) \xrightarrow{g} (Y, T_Y)$ *by Theorem 3.2 part* **(c)**, *it is easy to see that* $(X, T_X) \xrightarrow{q_1} (Q, T_Q^{(\!(q_1,q_2)\!)}) \xleftarrow{q_2} (Y, T_Y)$ *is a pushout of* f *along* g, *where* $T_Q^{(\!(q_1,q_2)\!)} = \{A \in I^Q : A \circ q_1 \in T_X, A \circ q_2 \in T_Y\}$, $Q := X \coprod Y/\sim$, $\sim$ *is the smallest equivalence relation generated by the relation* $\{((f(z), 1), (g(z), 2)) : z \in Z\}$, *and* q_1, q_2 *are the quotient maps taking each* $x \in X, y \in Y$ *to its equivalence classes* $[(x, 1)], [(y, 2)]$, *respectively.*

Example 4.2. *It can be demonstrated that the category* $\mathbf{FTop_{III}}$ *of type three fuzzy topological spaces, see* [7], *and continuous functions is* $\mathbb{O}$*-structural, see* [11], *by letting* $\mathbb{B}$ *be the base* $\mathbf{Sets}^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{S}}{\rightrightarrows}} \mathbf{Sets}$, *where* T *is the functor that takes a set* X *to the set of all intuitionistic fuzzy sets* $II^X := \{(A, B) \in I^X \times I^X : \forall x \in X, 0 \le A(x) + B(x) \le 1\}$, *a morphism* f *to* $(- \circ f, - \circ f)$ *with* I *being the unit interval,* P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O} = (\beta, \tau, \mu, \nu)$ *be to the so-called type three fuzzy topology operant defined for each* $X \in \mathbf{Sets}$ *by,*

$$1 \xrightarrow{\beta_X} II^X, \ \textit{taking } 1 \longmapsto \tilde{0}$$

$$1 \xrightarrow{\tau_X} II^X, \ \textit{taking } 1 \longmapsto \tilde{1}$$

$$\mathcal{P}(II^X) \xrightarrow{\mu_X} II^X, \ \textit{taking } Z \longmapsto \bigvee_{\tilde{A} \in Z} \tilde{A} \qquad \textit{and}$$

$$II^X \times II^X \xrightarrow{\nu_X} II^X, \ \textit{taking } (\tilde{A}, \tilde{B}) \longmapsto \tilde{A} \wedge \tilde{B}$$

In which $\tilde{0} = (\underline{0}, \underline{1})$ *and* $\tilde{1} = (\underline{1}, \underline{0})$ *are the pairs of constant functions with values* $(0, 1)$ *and* $(1, 0)$, *respectively. Now again since the base category* $\mathcal{E} = \mathbf{Sets}$ *is cocomplete, by Theorems 3.1 and 2.11, the category* $\mathbf{FTop_{III}}$ *is concretely cocomplete. Similar to Example 4.1 by computing the special colimits in the category* $\mathbf{Sets}$ *and applying Theorem 3.2, it can be shown that the coproduct of the given family* $\{(X_i, T_i)\}_I$ *of type three fuzzy topological spaces is* $(X, T_X^{(\!(\nu_i)\!)_I})$, *where* $X := \coprod_{j \in I} X_j$, $T_X^{(\!(\nu_i)\!)_I} = \{(A, B) \in II^X : (\forall i \in I)(A \circ \nu_i, B \circ \nu_i) \in T_{X_i}\}$ *and* $\nu_i : X_i \longrightarrow X$ *is the injection of the coproduct, for each* $i \in I$. *Also one can easily verify that* $(Y, T_Y) \xrightarrow{q} (Q, T_Q^{(\!(q)\!)})$ *is a coequalizer of the parallel type three fuzzy continuous functions* $(X, T_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, T_Y)$, *where* $T_Q^{(\!(q)\!)} = \{(A, B) \in II^Q : (A \circ q, B \circ q) \in T_Y\}$, $Q := Y/\sim$. *Moreover, for type three fuzzy continuous functions* $(X, T_X) \xleftarrow{f} (Z, T_Z) \xrightarrow{g} (Y, T_Y)$ *it is easy to see that* $(X, T_X) \xrightarrow{q_1} (Q, T_Q^{(\!(q_1,q_2)\!)}) \xleftarrow{q_2} (Y, T_Y)$ *is a pushout of* f *along* g, *where* $T_Q^{(\!(q_1,q_2)\!)} = \{(A, B) \in II^Q : (A \circ q_1, B \circ q_1) \in T_X, (A \circ q_2, B \circ q_2) \in T_Y\}$, $Q := X \coprod Y/\sim$.

Let us remark that in Examples 4.1 and 4.2, replacing the unit interval I by a complete lattice L, we get similar results, see [9, 10].

Example 4.3. *It can be proven that the category* $\mathbf{FTop_{IV}}$ *of type four fuzzy topological spaces, see* [5], *and continuous functions is* $\mathbb{O}$*-structural, see* [11], *by letting* $\mathbb{B}$ *be the base* $(\mathbf{I^U})^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathbf{Sets}$, *where* U *is a set,* I *is the unit interval, partially ordered set* $\mathbf{I^U}$ *is considered as a category,* T *takes a set* X *to the set* $\mathcal{F}_X$ *of all the fuzzy subsets of* X *and the morphism* $X \le Y$ *to* $- \wedge X$, P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O} = (\beta, \tau, \mu, \nu)$ *be the so-called type four fuzzy topology operant defined for each fuzzy set* $X : U \longrightarrow I$ *by,*

$$1 \xrightarrow{\beta_X} \mathcal{F}_X, \ \textit{taking } 1 \longmapsto \underline{0}$$

$$1 \xrightarrow{\tau_X} \mathcal{F}_X, \ \ taking\ 1 \longmapsto X$$
$$\mathcal{P}(\mathcal{F}_X) \xrightarrow{\mu_X} \mathcal{F}_X, \ \ taking\ E \longmapsto \bigvee_{A\in E} A \qquad and$$
$$\mathcal{F}_X \times \mathcal{F}_X \xrightarrow{\nu_X} \mathcal{F}_X, \ \ taking\ (A,B) \longmapsto A \wedge B$$

Where $\underline{0} : U \longrightarrow I$ *is the constant function with value* 0*. Since* $\mathcal{E} = \mathbf{I^U}$ *is cocomplete, see* [9]*, by Theorems 3.1 and 2.11, the category* $\mathbf{FTop_{IV}}$ *is concretely cocomplete. By Theorem 3.2 part* **(a)***, coproduct of the given family* $\{(X_i, T_{X_i})\}_I$ *of type four fuzzy topological spaces is* $(X, T_X^{(\!(\nu_i)\!)_I})$*, where for each* $t \in U$*,* $X(t) := \bigvee_{j\in I} X_j(t) = \sup_{j\in I}\{X_j(t)\}$*,* $T_X^{(\!(\nu_i)\!)_I} = \big\{A \in \mathcal{F}_X : (\forall i \in I) A\wedge X_i \in T_{X_i}\big\}$*, with the meet* $(A\wedge X_i)(t) := \inf\{A(t), X_i(t)\}$ *defined pointwise for each* $t \in U$*. Also by Theorem 3.2 part* **(b)***, for the parallel type four fuzzy continuous functions* $(X, T_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, T_Y)$ *, a simple calculation shows that* $(Y, T_Y) \xrightarrow{1_Y} (Y, T_Y)$ *is a coequalizer of* f *and* g*. Furthermore for type four fuzzy continuous functions* $(X, T_X) \xleftarrow{f} (Z, T_Z) \xrightarrow{g} (Y, T_Y)$ *by Theorem 3.2 part* **(c)***, clearly* $(X, T_X) \xrightarrow{q_1} (X \vee Y, T_{X\vee Y}^{(\!(q_1,q_2)\!)}) \xleftarrow{q_2} (Y, T_Y)$ *is a pushout of* f *along* g*, where* $T_{X\vee Y}^{(\!(q_1,q_2)\!)} = \big\{A \in \mathcal{F}_{X\vee Y} : A \wedge X \in T_X, A \wedge Y \in T_Y\big\}$*.*

Example 4.4. *It can be demonstrated that the category* **TopMLat** *of topological molecular lattices and continuous generalized order-homomorphisms (or simply continuous GOHs) is* $\mathbb{O}$*-structural, see* [35, 36]*, by letting* $\mathbb{B}$ *be the base* $\mathbf{MLat}^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathbf{Sets}$*, where* **MLat** *is the category of molecular lattices and generalized order-homomorphisms, see* [35]*,* T *is the functor taking* $(L, \leq)$ *to its underlying set* L *and* $f : (L, \leq) \longrightarrow (L', \leq')$ *to its right adjoint function* $\mathsf{T}(f) = \hat{f} : L' \longrightarrow L$*, defined for each* $b \in L'$ *by* $\hat{f}(b) = \bigvee_{f(a)\leq b} a$*,* P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O} = (\beta, \tau, \mu, \nu)$ *be the so-called molecular lattice cotopology operant defined for each* $(L, \leq) \in \mathbf{MLat}$ *by,*

$$1 \xrightarrow{\beta_{(L,\leq)}} L, \ \ taking\ 1 \longmapsto 0$$
$$1 \xrightarrow{\tau_{(L,\leq)}} L, \ \ taking\ 1 \longmapsto 1$$
$$\mathcal{P}(L) \xrightarrow{\mu_{(L,\leq)}} L, \ \ taking\ A \longmapsto \bigwedge_{a\in A} a \qquad and$$
$$L \times L \xrightarrow{\nu_{(L,\leq)}} L, \ \ taking\ (a,b) \longmapsto a \vee b$$

Since $\mathcal{E} = \mathbf{MLat}$ *is cocomplete, see* [21]*, by Theorems 3.1 and 2.11, the category* **TopMLat** *is concretely cocomplete. By Theorem 3.2 part* **(a)***, coproduct of the given family* $\{\big((L_i, \leq_i), \eta_i\big)\}_I$ *of topological molecular lattices is* $\big((\prod_{j\in I} L_j, \leq), \eta^{(\!(\nu_i)\!)_I}\big)$*, where the order* $\leq$ *in* $\prod_{j\in I} L_j$ *is the pointwise order,* $\big(\!\big((L_i, \leq_i) \xrightarrow{\nu_i} (\prod_{j\in I} L_j, \leq)\big)\!\big)_I$ *is the coproduct of* $\{(L_i, \leq_i)\}_I$ *in* **MLat***, see* [21]*, the mapping* $\nu_i : (L_i, \leq_i) \longrightarrow (\prod_{j\in I} L_j, \leq)$ *is defined by* $\nu_i(x) = \{x_j\}_I$*, and*

$$x_j = \begin{cases} 0, & if\ j \neq i, \\ x, & if\ j = i. \end{cases}$$

Furthermore, we have $\eta^{(\!(\nu_i)\!)_I} = \{x \in \prod_{j\in I} L_j : (\forall i \in I) \widehat{\nu_i}(x) \in \eta_i\}$*. Also by Theorem 3.2 part* **(b)***, for the continuous GOHs* $\big((L_1, \leq_1), \eta_1\big) \overset{f}{\underset{g}{\rightrightarrows}} \big((L_2, \leq_2), \eta_2\big)$ *, it can be routinely verified that* $\big((L_2, \leq_2), \eta_2\big) \xrightarrow{q} \big((Q, \leq_2), \eta^{(\!(q)\!)_I}\big)$ *is a coequalizer of* f *and* g *with* $Q = \{a \in L_2 : \widehat{f}(a) = \widehat{g}(a)\}$ *the molecular sublattice of* L_2*,* $\eta^{(\!(q)\!)_I} = \{a \in Q : \widehat{q}(a) \in \eta_2\} = \{a \in L_2 : \widehat{f}(a) = \widehat{g}(a), \widehat{q}(a) \in \eta_2\}$ *and for each* $x \in L_2$*,* q *is defined by* $q(x) = \bigwedge\{a \in Q : x \leq_2 a\} = \bigwedge\{a \in L_2 : \widehat{f}(a) = \widehat{g}(a), x \leq_2 a\}$*, see* [21]*. Furthermore for continuous GOHs* $\big((L_1, \leq_1), \eta_1\big) \xleftarrow{f} \big((L, \leq), \eta\big) \xrightarrow{g} \big((L_2, \leq_2), \eta_2\big)$ *by Theorem 3.2 part* **(c)***, one can verify that* $\big((L_1, \leq_1), \eta_1\big) \xrightarrow{q\nu_1} \big((Q, \leq'), \eta^{(\!(q\nu_1, q\nu_2)\!)}\big) \xleftarrow{q\nu_2} \big((L_2, \leq_2), \eta_2\big)$ *is a pushout* *of f along g, where* $(L_1,\leq_1) \xrightarrow{\nu_1} (L_1\times L_2,\leq') \xleftarrow{\nu_2} (L_2,\leq_2)$ *is the coproduct of molecular lattices* $(L_1,\leq_1)$ *and* $(L_2,\leq_2)$ *in* **MLat**, $(L,\leq) \overset{\nu_1 f}{\underset{\nu_2 g}{\rightrightarrows}} (L_1\times L_2,\leq') \xrightarrow{q} (Q,\leq')$ *is a coequalizer in* **MLat** *with* $Q=\{(a,b)\in L_1\times L_2 : \widehat{\nu_1 f}(a,b)=\widehat{\nu_2 g}(a,b)\}=\{(a,b)\in L_1\times L_2 : \widehat{f}(a)=\widehat{g}(b)\}$, $\eta^{(\!(q\nu_1,q\nu_2)\!)}=\big\{(a,b)\in Q : \widehat{q\nu_1}(a,b)\in\eta_1, \widehat{q\nu_2}(a,b)\in\eta_2\big\}=\big\{(a,b)\in L_1\times L_2 : \widehat{f}(a)=\widehat{g}(b), \widehat{q\nu_1}(a,b)\in\eta_1, \widehat{q\nu_2}(a,b)\in\eta_2\big\}$ *and for each* $(a,b)\in L_1\times L_2$, q *is defined by* $q(a,b)=\bigwedge\big\{(x,y)\in Q:(x,y)\leq'(a,b)\big\}=\bigwedge\big\{(x,y)\in L_1\times L_2 : \widehat{f}(x)=\widehat{g}(y), x\leq_1 a, y\leq_2 b\big\}$.

Example 4.5. *It can be proven that the category* **TopDMLat** *of topological De Morgan molecular lattices and $*$-generalized order-homomorphisms, see* [21], *is $\mathbb{O}$-structural by letting $\mathbb{B}$ be the base* $\mathbf{DMLat}^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathbf{Sets}$, *where* **DMLat** *is the category with objects De Morgan molecular lattices and $*$-generalized order-homomorphisms as morphisms, see* [21], T *is the functor taking a $*$-generalized order-homomorphism* $f:(L,\leq)\longrightarrow(L',\leq')$ *to the function* $\mathsf{T}(f)=f^{-1}(-):L'\longrightarrow L$, *where* $f^{-1}(b)=\bigvee_{f(a)\leq b} a$, *for each* $b\in L'$, P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O}=(\beta,\tau,\mu,\nu)$ *be the so-called De Morgan molecular lattice topology operant defined for each* $(L,\leq)\in\mathbf{DMLat}$ *by,*

$$1 \xrightarrow{\beta_{(L,\leq)}} L, \ \text{taking } 1\longmapsto 0$$
$$1 \xrightarrow{\tau_{(L,\leq)}} L, \ \text{taking } 1\longmapsto 1$$
$$\mathcal{P}(L) \xrightarrow{\mu_{(L,\leq)}} L, \ \text{taking } A\longmapsto \bigvee_{a\in A} a \qquad \text{and}$$
$$L\times L \xrightarrow{\nu_{(L,\leq)}} L, \ \text{taking } (a,b)\longmapsto a\wedge b$$

According to the discussion on coproducts and coequalizers in **DMLat**, *see* [21], *the category* **TopDMLat** *is concretely cocomplete by Theorems 3.1 and 2.11. In particular, by Theorem 3.2, coproducts in* **TopDMLat** *are obtained by equipping the product lattice* $\prod_{j\in I} L_j$ *(with the pointwise order and componentwise pseudocomplement) with the final structure coinduced by the canonical coproduct injections* ν_i. *Similarly, given two continuous $*$-GOHs* $f,g:(L_1,\eta_1)\rightrightarrows(L_2,\eta_2)$, *the coequalizer is the De Morgan molecular sublattice* $Q=\{a\in L_2 : \widehat{f}(a)=\widehat{g}(a)\}$ *of* L_2 *endowed with the topology inherited via the quotient map* q. *Finally, pushouts in* **TopDMLat** *are described by combining these constructions: the coproduct* $(L_1\times L_2,\nu_1,\nu_2)$ *in* **DMLat**, *together with the coequalizer of* $\nu_1 f$ *and* $\nu_2 g$, *yields the desired pushout object endowed with the corresponding final structure.*

Example 4.6. *One can show that the category* **TopFuzz** *of topological fuzzes and continuous order-homomorphisms, see* [13, 35], *is $\mathbb{O}$-structural, by letting $\mathbb{B}$ be the base* $\mathbf{Fuzz}^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathbf{Sets}$, *where* **Fuzz** *is the category of fuzzes and order-homomorphisms, see* [35], T *is the functor that takes an order-homomorphism* $f:\big((L_1,\leq_1),'_{L_1}\big)\longrightarrow\big((L_2,\leq_2),'_{L_2}\big)$ *to its right adjoint function* $\mathsf{T}(f)=\hat{f}:L_2\longrightarrow L_1$, *defined for each* $b\in L_2$ *by* $\hat{f}(b)=\bigvee_{f(a)\leq b} a$, P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O}=(\beta,\tau,\mu,\nu)$ *to be the so-called topology fuzz operant defined for each* $\big((L,\leq),'_L\big)\in\mathbf{Fuzz}$, *similar to Example 4.5, see* [13, 35].

Since $\mathcal{E}=\mathbf{Fuzz}$ *is cocomplete, see* [2], *by Theorems 3.1 and 2.11, the category* **TopFuzz** *is concretely cocomplete. Similar to Example 4.5 by computing the special colimits in the category* **MLat** *and applying Theorem 3.2, one can show that coproduct of the given family* $\{\big((L_i,\leq_i),'_{L_i},\tau_i\big)\}_I$ *of topological fuzzes is* $\big((\prod_{j\in I} L_j,\leq),',\tau^{(\!(\nu_i)\!)_I}\big)$, *where the order* $\leq$ *in* $\prod_{j\in I} L_j$ *and order-reversing involution* $':\prod_{j\in I} L_j\longrightarrow\prod_{j\in I} L_j$ *are defined pointwise,* $\Big(\!\!\Big((L_i,\leq_i)\xrightarrow{\nu_i}(\prod_{j\in I} L_j,\leq)\Big)\!\!\Big)_I$ *is the coproduct of* $\{(L_i,\leq_i)\}_I$ *in* **MLat**, *see* [21]. *Moreover we have* $\tau^{(\!(\nu_i)\!)_I}=\{x\in\prod_{j\in I} L_j : (\forall i\in I)\widehat{\nu_i}(x)\in\tau_i\}$.

Also for the continuous order-homomorphisms $\big((L_1,\leq_1),'_{L_1},\tau_1\big) \overset{f}{\underset{g}{\rightrightarrows}} \big((L_2,\leq_2),'_{L_2},\tau_2\big)$, *it can be routinely verified that* $\big((L_2,\leq_2),\tau_2\big)\xrightarrow{q}\big((Q,\leq_2),\tau^{(\!(q)\!)}\big)$ *is a coequalizer of* f *and* g *with* $Q=\{a\in L_2 : \widehat{f}(a)=\widehat{g}(a)\}$ *the molecular sublattice of* L_2, $\tau^{(\!(q)\!)}=\{a\in Q : \widehat{q}(a)\in\tau_2\}$, *see* [21]. *Furthermore for continuous order-homomorphisms*

$\big((L_1,\leq_1),'_{L_1},\tau_1\big) \xleftarrow{f} \big((L,\leq),'_L,\tau\big) \xrightarrow{g} \big((L_2,\leq_2),'_{L_2},\tau_2\big)$ *by a similar argument as in Example 4.5 it is straightforward to verify that the 2-sink* $\big((L_1,\leq_1),'_{L_1},\tau_1\big) \xrightarrow{q\nu_1} \big((Q,\leq'),'_{L_1\times L_2},\tau^{\langle\!\langle q\nu_1,q\nu_2\rangle\!\rangle}\big) \xleftarrow{q\nu_2} \big((L_2,\leq_2),'_{L_2},\tau_2\big)$ *is a pushout of* f *along* g, *where* $(L_1,\leq_1) \xrightarrow{\nu_1} (L_1\times L_2,\leq') \xleftarrow{\nu_2} (L_2,\leq_2)$ *is the coproduct of molecular lattices* $(L_1,\leq_1)$ *and* $(L_2,\leq_2)$ *in* **MLat**, $(L,\leq) \overset{\nu_1 f}{\underset{\nu_2 g}{\rightrightarrows}} (L_1\times L_2,\leq') \xrightarrow{q} (Q,\leq')$ *is a coequalizer in* **MLat** *with* $Q=\{(a,b)\in L_1\times L_2:\widehat{\nu_1 f}(a,b)=\widehat{\nu_2 g}(a,b)\}$, $\tau^{\langle\!\langle q\nu_1,q\nu_2\rangle\!\rangle}=\big\{(a,b)\in Q:\widehat{q\nu_1}(a,b)\in\tau_1,\widehat{q\nu_2}(a,b)\in\tau_2\big\}$.

Example 4.7. *One can show that the category* **STop** *of soft topological spaces in Shabir and Naz-type and soft continuous mappings, see* [8, 31], *is* $\mathbb{O}$*-structural by letting* $\mathbb{B}$ *be the base* $(\mathbf{Sets}\times\mathbf{Sets})^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathbf{Sets}$, *where* T *takes a pair of sets* (X,A) *to* $S(X,A)$, *the family of all soft sets over* X *and a pair of mappings* $(X,A) \xrightarrow{(f,e)} (Y,B)$, *to the induced soft inverse image mapping* $\mathsf{T}(f,e)=f^{-1}\circ-\circ e: S(Y,B)\longrightarrow S(X,A)$, *where* $f:X\longrightarrow Y$ *and* $e:A\longrightarrow B$, *see* [17], P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O}=(\beta,\tau,\mu,\nu)$ *be the so-called soft topology operant defined for each* $(X,A)\in\mathbf{Sets}\times\mathbf{Sets}$ *by,*

$$1 \xrightarrow{\beta_{(X,A)}} S(X,A)\ ,\ \textit{taking}\ 1\longmapsto(\tilde{\Phi},A)$$

$$1 \xrightarrow{\tau_{(X,A)}} S(X,A)\ ,\ \textit{taking}\ 1\longmapsto(\tilde{X},A)$$

$$\mathcal{P}(S(X,A)) \xrightarrow{\mu_{(X,A)}} S(X,A)\ ,\ \textit{taking}\ T\longmapsto \bigvee_{(F,A)\in T}(F,A)\ \textit{and}$$

$$S(X,A)\times S(X,A) \xrightarrow{\nu_{(X,A)}} S(X,A)\ ,\ \textit{taking}\ ((F_1,A),(F_2,A))\longmapsto(F_1,A)\wedge(F_2,A)$$

In which $(\tilde{\Phi},A)$, $(\tilde{X},A)$, $\bigvee$ *and* $\wedge$ *are the null soft set, absolute soft set, soft union and soft intersection, respectively, see* [31]. *Now again since the base category* $\mathcal{E}=\mathbf{Sets}\times\mathbf{Sets}$ *is cocomplete, see* [19], *by Theorems 3.1 and 2.11, the category* **STop** *is concretely cocomplete. By the fact that the colimits in* $\mathbf{Sets}\times\mathbf{Sets}$ *are computed pointwise, so by computing the special colimits in the category* **Sets** *and applying Theorem 3.2, it can be shown that the coproduct of the given family* $\{(X_i,A_i,\tau_i)\}_I$ *of soft topological spaces is* $(X,A,\tau^{\langle\!\langle\nu_i,\nu'_i\rangle\!\rangle_I})$, $\tau^{\langle\!\langle\nu_i,\nu'_i\rangle\!\rangle_I}=\big\{(F,A)\in S(X,A):(\forall i\in I)(\nu_i^{-1}\circ F\circ\nu'_i,A_i)\in\tau_i\big\}$, $\nu_i:X_i\longrightarrow X$ *and* $\nu'_i:A_i\longrightarrow A$ *are the injections of coproducts, for each* $i\in I$, *where* $X:=\coprod_{j\in I}X_j$ *and* $A:=\coprod_{j\in I}A_j$ *are disjoint unions in* **Sets**. *Also one can easily verify that* $(Y,B,\tau_Y) \xrightarrow{(q,l)} (Q,C,\tau_Q^{\langle\!\langle q,l\rangle\!\rangle})$ *is a coequalizer of the parallel soft continuous mappings* $(X,A,\tau_X) \overset{(f,e)}{\underset{(g,k)}{\rightrightarrows}} (Y,B,\tau_Y)$, *where* $\tau_Q^{\langle\!\langle q,l\rangle\!\rangle}=\big\{(F,C)\in S(Q,C):(q^{-1}\circ F\circ l,B)\in\tau_Y\big\}$, $Q:=Y/\!\sim$ *and* $C:=B/\!\sim'$, $\sim$ *and* $\sim'$ *are the smallest equivalence relation on* Y *and* B *such that* $f(x)\sim g(x)$ *and* $e(a)\sim' k(a)$ *for all* $x\in X, a\in A$, *respectively. Moreover, for soft continuous mappings* $(X,A,\tau_X) \xleftarrow{(f,e)} (Z,D,\tau_Z) \xrightarrow{(g,k)} (Y,B,\tau_Y)$ *it is easy to see that*

$$(X,A,\tau_X) \xrightarrow{(q_1,l_1)} (Q,C,\tau_Q^{\langle\!\langle(q_1,l_1),(q_2,l_2)\rangle\!\rangle}) \xleftarrow{(q_2,l_2)} (Y,B,\tau_Y)\ ,$$

is a pushout of (f,e) *along* (g,k), *where* $\tau_Q^{\langle\!\langle(q_1,l_1),(q_2,l_2)\rangle\!\rangle}=\big\{(F,C)\in S(Q,C):(q_1^{-1}\circ F\circ l_1,A)\in\tau_X,(q_2^{-1}\circ F\circ l_2,B)\in\tau_Y\big\}$, $Q:=X\coprod Y/\!\sim$ *and* $C:=A\coprod B/\!\sim'$.

Example 4.8. *One can show that the category* $\mathbf{FSTop_I}$ *of type I fuzzy soft topological spaces in Roy and Samanta-type and type I fuzzy soft continuous mappings between them, see* [25, 34], *is* $\mathbb{O}$*-structural by letting* $\mathbb{B}$ *be the base* $(\mathbf{Sets}\times\mathbf{Sets})^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathbf{Sets}$, *where* T *taking a pair of sets* (X,A) *to* $FS(X,A)$, *the family of all type I fuzzy soft sets over* X *and a pair of mappings* $(X,A) \xrightarrow{(f,e)} (Y,B)$, *to the induced backward type I fuzzy powerset operator* $\mathsf{T}(f,e)=f_I^{\leftarrow}\circ-\circ e: FS(Y,B)\longrightarrow FS(X,A)$, *see* [16, 20, 24], P *is the covariant powerset functor,* Q *is the squaring functor* S *and the so-called type I fuzzy soft topology operant* $\mathbb{O}=(\beta,\tau,\mu,\nu)$ *is defined for each* $(X,A)\in\mathbf{Sets}\times\mathbf{Sets}$, *similar to Example 4.7, see* [3, 16].

Similar to Example 4.7, one can compute the special colimits in the category $\mathbf{FSTop_I}$. *Since the base category* $\mathcal{E} = \mathbf{Sets} \times \mathbf{Sets}$ *is cocomplete, see* [19], *by Theorems 3.1 and 2.11, the category* $\mathbf{FSTop_I}$ *is concretely cocomplete. As colimits in* $\mathbf{Sets} \times \mathbf{Sets}$ *are computed pointwise, Theorem 3.2 immediately provides explicit descriptions of coproducts, coequalizers and pushouts in* $\mathbf{FSTop_I}$. *For instance, the coproduct of a family* $\{(X_i, A_i, \tau_i)\}_{i\in I}$ *is obtained by taking the disjoint unions* $X = \coprod_j X_j$, $A = \coprod_j A_j$ *in* $\mathbf{Sets}$, *together with the type I fuzzy soft topology generated pointwise from the* τ_i. *Likewise, coequalizers and pushouts are obtained in the same way as in Example 4.7, namely as the corresponding quotients on sets and parameters, equipped with the coinduced type I fuzzy soft topologies.*

Example 4.9. *We can establish that the category* $\mathbf{NTop}$ *of neutrosophic topological spaces and neutrosophic continuous functions between them, see* [29, 30], *is* $\mathbb{O}$*-structural by letting* $\mathbb{B}$ *be the base* $\mathbf{Sets}^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} \mathbf{Sets}$, *where* T *takes a set* X *to the family of all neutrosophic sets,* $\mathcal{N}(X)$, *in* X *and a function* $X \xrightarrow{f} Y$ *to the preimage function* $\mathsf{T}(f) = f^{-1} : \mathcal{N}(Y) \longrightarrow \mathcal{N}(X)$, *where for each* $A_Y = (T, I, F) \in \mathcal{N}(Y)$, $f^{-1}(A_Y) = (T \circ f, I \circ f, F \circ f)$, *see* [12], P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O} = (\beta, \tau, \mu, \nu)$ *be the so-called neutrosophic topology operant defined for each* $X \in \mathbf{Sets}$ *by,*

$$1 \xrightarrow{\beta_X} \mathcal{N}(X)\ ,\ \textit{taking}\ 1 \longmapsto 0_{X,N}$$

$$1 \xrightarrow{\tau_X} \mathcal{N}(X)\ ,\ \textit{taking}\ 1 \longmapsto 1_{X,N}$$

$$\mathcal{P}(\mathcal{N}(X)) \xrightarrow{\mu_X} \mathcal{N}(X)\ ,\ \textit{taking}\ T \longmapsto \bigcup_{B_X \in T} B_X\ \textit{and}$$

$$\mathcal{N}(X) \times \mathcal{N}(X) \xrightarrow{\nu_X} \mathcal{N}(X)\ ,\ \textit{taking}\ (B_X, C_X) \longmapsto B_X \cap C_X$$

In which $0_{X,N}$, $1_{X,N}$, $\bigcup$ *and* $\cap$ *are the neutrosophic empty set, neutrosophic universal set in* X, *neutrosophic union and neutrosophic intersection, respectively, see* [12, 14]. *By Theorems 3.1 and 2.11, the category* $\mathbf{NTop}$ *is concretely cocomplete. Similar to Example 4.1 by computing the special colimits in the category* $\mathbf{Sets}$ *and applying Theorem 3.2, it can be shown that the coproduct of the given family* $\{(X_i, \tau_{X_i})\}_I$ *of neutrosophic topological spaces is* $(X, \tau_X^{(\!(\nu_i)\!)_I})$, *where* $X := \coprod_{j\in I} X_j$, $\tau_X^{(\!(\nu_i)\!)_I} = \{A_X = (T, I, F) \in \mathcal{N}(X) : (\forall i \in I) f^{-1}(A_X) = (T \circ f, I \circ f, F \circ f) \in \tau_{X_i}\}$ *and* $\nu_i : X_i \longrightarrow X$ *is the injection of coproduct, for each* $i \in I$. *Also one can easily verify that* $(Y, \tau_Y) \xrightarrow{q} (Q, \tau_Q^{(\!(q)\!)})$ *is a coequalizer of the parallel neutrosophic continuous functions* $(X, \tau_X) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \tau_Y)$, *where* $\tau_Q^{(\!(q)\!)} = \{A_Q = (T, I, F) \in \mathcal{N}(Q) : q^{-1}(A_Q) = (T \circ q, I \circ q, F \circ q) \in \tau_Y\}$, $Q := Y/\sim$. *Moreover, for neutrosophic continuous functions* $(X, \tau_X) \xleftarrow{f} (Z, \tau_Z) \xrightarrow{g} (Y, \tau_Y)$ *it is easy to see that* $(X, \tau_X) \xrightarrow{q_1} (Q, \tau_Q^{(\!(q_1,q_2)\!)}) \xleftarrow{q_2} (Y, \tau_Y)$ *is a pushout of* f *along* g, *where* $\tau_Q^{(\!(q_1,q_2)\!)} = \{A_Q = (T, I, F) \in \mathcal{N}(Q) : q_1^{-1}(A_Q) = (T \circ q_1, I \circ q_1, F \circ q_1) \in \tau_X, q_2^{-1}(A_Q) = (T \circ q_2, I \circ q_2, F \circ q_2) \in \tau_Y\}$, $Q := X \coprod Y/\sim$.

Example 4.10. *One can verify that the category* $\mathbf{HSTop}$ *of hypersoft topological spaces, see* [23], *and hypersoft continuous mappings between them, see* [26], *is* $\mathbb{O}$*-structural by letting* $\mathbb{B}$ *be the base* $(\mathbf{Sets} \times \mathbf{Sets}_n)^{op} \xrightarrow{\mathsf{T}} \mathbf{Sets} \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} \mathbf{Sets}$, *where* $\mathbf{Sets}_n$ *has as objects the Cartesian product* $A_1 \times A_2 \times ... \times A_n$ *of n-tuples of sets* $(A_1, A_2, ..., A_n)$ *denoted by* $\mathcal{A}$, *and as morphisms the Cartesian product of functions* $A_1 \times A_2 \times ... \times A_n \xrightarrow{\theta_1 \times \theta_2 \times ... \times \theta_n} B_1 \times B_2 \times ... \times B_n$ *denoted by* $\mathcal{A} \xrightarrow{\theta} \mathcal{B}$. T *takes a pair* $(X, \mathcal{A})$ *to* $\Omega(X, \mathcal{A})$, *the family of all hypersoft sets over* X *and a mapping* $(X, \mathcal{A}) \xrightarrow{(\Delta,\theta)} (Y, \mathcal{B})$ *to the induced hypersoft inverse image mapping* $\mathsf{T}(\Delta, \theta) = \Delta^{-1} \circ - \circ \theta : \Omega(Y, \mathcal{B}) \longrightarrow \Omega(X, \mathcal{A})$, *where* $\Delta : X \longrightarrow Y$ *and* $\theta : \mathcal{A} \longrightarrow \mathcal{B}$, *see* [26, 32], P *is the covariant powerset functor,* Q *is the squaring functor* S *and by taking* $\mathbb{O} = (\beta, \tau, \mu, \nu)$ *be the so-called hypersoft topology operant defined for each* $(X, \mathcal{A}) \in \mathbf{Sets} \times \mathbf{Sets}_n$ *by,*

$$1 \xrightarrow{\beta_{(X,\mathcal{A})}} \Omega(X, \mathcal{A})\ ,\ \textit{taking}\ 1 \longmapsto (\Phi, \mathcal{A})$$

$$1 \xrightarrow{\tau_{(X,\mathcal{A})}} \Omega(X, \mathcal{A})\ ,\ \textit{taking}\ 1 \longmapsto (\Psi, \mathcal{A})$$

$$\mathcal{P}(\Omega(X, \mathcal{A})) \xrightarrow{\mu_{(X,\mathcal{A})}} \Omega(X, \mathcal{A})\ ,\ \textit{taking}\ \mathcal{T} \longmapsto \bigvee_{(\mathbb{F},\mathcal{A})\in\mathcal{T}} (\mathbb{F}, \mathcal{A})\ \textit{and}$$

$$\Omega(X,\mathcal{A})\times\Omega(X,\mathcal{A})\xrightarrow{\nu_{(X,\mathcal{A})}}\Omega(X,\mathcal{A})\ ,\ taking\ ((\mathbb{F}_1,\mathcal{A}),(\mathbb{F}_2,\mathcal{A}))\longmapsto(\mathbb{F}_1,\mathcal{A})\wedge(\mathbb{F}_2,\mathcal{A})$$

In which $(\Phi,\mathcal{A})$, $(\Psi,\mathcal{A})$, $\bigvee$ *and* $\wedge$ *are the relative null hypersoft set, relative whole hypersoft set over* X, *hypersoft union and hypersoft intersection, respectively, see* [27, 28]. *Now one can show that colimits in the category* $\mathbf{Sets}_n$ *exist and are computed pointwise, so the base category* $\mathcal{E}=\mathbf{Sets}\times\mathbf{Sets}_n$ *is cocomplete, by Theorems 3.1 and 2.11, the category* **HSTop** *is concretely cocomplete. Since colimits in* $\mathbf{Sets}\times\mathbf{Sets}_n$ *are computed pointwise, by computing the special colimits in the category* **Sets** *and applying Theorem 3.2, it can be shown that the coproduct of the given family* $\{(X_i,\mathcal{A}^i,\tau_i)\}_I$ *of hypersoft topological spaces is* $(X,\mathcal{A},\tau_X^{(\!(\nu_i,\nu_i')\!)_I})$, $\tau_X^{(\!(\nu_i,\nu_i')\!)_I}=\{(\mathbb{F},\mathcal{A})\in\Omega(X,\mathcal{A}):(\forall i\in I)(\nu_i^{-1}\circ\mathbb{F}\circ\nu_i',\mathcal{A}^i)\in\tau_i\}$, $\nu_i:X_i\longrightarrow X$ *and* $\nu_i':\mathcal{A}^i\longrightarrow\mathcal{A}$ *are the injections of coproducts, for each* $i\in I$, *where* $X:=\coprod_{j\in I}X_j$ *and* $\mathcal{A}:=\coprod_{j\in I}\mathcal{A}^j=\coprod_{j\in I}A_1^j\times\coprod_{j\in I}A_2^j\times\ldots\times\coprod_{j\in I}A_n^j$, *in which* $\coprod_{j\in I}X_j$ *and* $\coprod_{j\in I}A_i^j$ *for* $i=1,2,\ldots,n$ *are disjoint unions in* **Sets**. *Also one can easily verify that* $(Y,\mathcal{B},\tau_Y)\xrightarrow{(\Upsilon,\iota)}(Q,\mathcal{D},\tau_Q^{(\!(\Upsilon,\iota)\!)})$ *is a coequalizer of the parallel hypersoft continuous mappings* $(X,\mathcal{A},\tau_X)\overset{(\Delta,\theta)}{\underset{(\Gamma,\kappa)}{\rightrightarrows}}(Y,\mathcal{B},\tau_Y)$, *where* $\tau_Q^{(\!(\Upsilon,\iota)\!)}=\{(\mathbb{F},\mathcal{D})\in\Omega(Q,\mathcal{D}):(\Upsilon^{-1}\circ\mathbb{F}\circ\iota,\mathcal{B})\in\tau_Y\}$, $Q:=Y/\!\sim$ *and* $\mathcal{D}:=\mathcal{B}/\!\sim'$, $\sim$ *and* $\sim'$ *are the smallest equivalence relations on* Y *and* $\mathcal{B}$ *such that* $\Delta(x)\sim\Gamma(x)$ *and* $\theta_1(b_1)\sim'\kappa_1(b_1),\theta_2(b_2)\sim'\kappa_2(b_2),\ldots,\theta_n(b_n)\sim'\kappa_n(b_n)$ *for all* $x\in X,(b_1,b_2,\ldots,b_n)\in\mathcal{B}=B_1\times B_2\times\ldots\times B_n$, *respectively. Moreover, for hypersoft continuous mappings* $(X,\mathcal{A},\tau_X)\xleftarrow{(\Delta,\theta)}(Z,\mathcal{D},\tau_Z)\xrightarrow{(\Gamma,\kappa)}(Y,\mathcal{B},\tau_Y)$ *it is easy to see that* $(X,\mathcal{A},\tau_X)\xrightarrow{(\Upsilon_1,\iota_1)}(Q,\mathcal{D}',\tau_Q^{(\!((\Upsilon_1,\iota_1),(\Upsilon_2,\iota_2))\!)})\xleftarrow{(\Upsilon_2,\iota_2)}(Y,\mathcal{B},\tau_Y)$ *is a pushout of* (Δ,θ) *along* (Γ,κ), *where* $\tau_Q^{(\!((\Upsilon_1,\iota_1),(\Upsilon_2,\iota_2))\!)}=\{(\mathbb{F},\mathcal{D}')\in\Omega(Q,\mathcal{D}'):(\Upsilon_1^{-1}\circ\mathbb{F}\circ\iota_1,\mathcal{A})\in\tau_X,(\Upsilon_2^{-1}\circ\mathbb{F}\circ\iota_2,\mathcal{B})\in\tau_Y\}$, $Q:=X\coprod Y/\!\sim$ *and* $\mathcal{D}':=\mathcal{A}\coprod\mathcal{B}/\!\sim'=\big((A_1\times A_2\times\ldots\times A_n)\coprod(B_1\times B_2\times\ldots\times B_n)\big)/\!\sim'=\big((A_1\coprod B_1)\times(A_2\coprod B_2)\times\ldots\times(A_n\coprod B_n)\big)/\!\sim'$, *in which* $X\coprod Y$ *and* $A_i\coprod B_i$ *for* $i=1,2,\ldots,n$ *are disjoint unions in* **Sets**.

5 Conclusions

We establish that the category of $\mathbb{O}$-structural spaces, $\mathbb{OSS}$, admits all small colimits, with explicit descriptions of coproducts, coequalizers, and pushouts. These constructions apply to a variety of previously studied $\mathbb{O}$-structural categories, including several classes of fuzzy, soft, and neutrosophic topological spaces. Our results demonstrate that colimits in these settings can be systematically recovered through the underlying $\mathbb{O}$-structural framework, thereby providing categorical tools for further investigations of generalized topological structures. These results suggest several directions for future research. In particular, an important direction is the study of limits in fuzzy-type categories within the $\mathbb{O}$-structural setting. Moreover, the colimit constructions introduced in this paper are expected to find applications in areas such as uncertainty modeling, fuzzy control, and decision-making, where aggregation and combination of structures play a central role. These potential applications will be explored in future work. Another interesting direction is to investigate further categorical properties of generalized topological structures, such as connectedness and compactness, within this framework.

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