

A Schur-concave shuffle of M outside the closed convex hull of associative copulas

A. Šeliga ¹

¹*Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 810 05 Bratislava, Slovakia*

adam.seliga@stuba.sk

Abstract

The class of associative copulas is uniformly compact but not convex, and every associative copula is Schur-concave and commutative. Two open problems posed by Durante at FSTA 2010 concern the closed convex hull and the extreme points of that hull. We give a negative answer to the first problem by constructing an explicit family of Schur-concave commutative copulas outside the hull. Each member is a symmetric shuffle of the upper Fréchet-Hoeffding bound. We compute its exact uniform distance from the hull using a star-shapedness property of antidiagonal sections of associative copulas. For the second problem, we show that every extreme point of the hull is associative and reformulate the remaining inclusion question using associative copulas that are extreme among all copulas. We prove that ordinal sums of the lower Fréchet-Hoeffding bound form a proper subclass of these copulas and generate a strictly smaller closed convex hull. Nevertheless, the independence copula admits an integral representation over such ordinal sums and is not extreme in the hull of associative copulas.

Keywords: Associative copulas, Schur concavity, shuffle of M .

1 Introduction

Copulas were introduced by Sklar [20] as a tool to isolate the dependence structure of a multivariate distribution from its marginal behaviour. By Sklar's theorem, the joint distribution of any random pair (X, Y) with continuous marginals factors uniquely as $F(x, y) = C(F_X(x), F_Y(y))$ for a 2-copula $C: [0, 1]^2 \rightarrow [0, 1]$. The class $\mathcal{C}$ of all 2-copulas is convex and uniformly compact; see [8, 17]. Within $\mathcal{C}$, an algebraically rich subclass is the class $\mathcal{C}_a$ of associative copulas, characterized by the identity $C(C(u, v), w) = C(u, C(v, w))$ for all $u, v, w \in [0, 1]$. It is well known that $\mathcal{C}_a$ coincides with the class of 1-Lipschitz triangular norms [2, 10], and that $\mathcal{C}_a$ is uniformly compact; see [11]. Note that $\mathcal{C}_a$ is not convex; the closed convex hull of $\mathcal{C}_a$, denoted by $\overline{\text{co}}(\mathcal{C}_a)$, consists of all copulas obtainable as uniform limits of finite convex combinations of associative copulas.

It is also well known that every associative copula is Schur-concave and commutative; see Alsina [1] and Durante and Sempi [7]. Hence, $\mathcal{C}_a$ is contained in the class $\mathcal{S}_c$ of Schur-concave commutative copulas, and since $\mathcal{S}_c$ is closed and convex, the inclusion

$$\overline{\text{co}}(\mathcal{C}_a) \subseteq \mathcal{S}_c,$$

follows. The exact relationship between $\overline{\text{co}}(\mathcal{C}_a)$ and $\mathcal{S}_c$ was the subject of two open problems posed by Fabrizio Durante at the Tenth International Conference on Fuzzy Set Theory and Applications (FSTA 2010, Liptovský Ján, Slovakia). These problems were recorded by Mesiar and Sarkoci [14], and we restate them here:

(2.5) Is $\overline{\text{co}}(\mathcal{C}_a) = \mathcal{S}_c$?

Corresponding Author: A. Šeliga

Received: May 2026; Revised: September 2026; Accepted: October 2026.

<https://doi.org/10.22111/ijfs.2026.55340.9816>

(2.6) Determine the extremal points of $\overline{\text{co}}(\mathcal{C}_a)$, and decide whether $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \text{Ext}(\mathcal{C})$.

In this paper, we resolve the first of these problems in the negative. For every parameter $\theta \in]0, 1/2[$, we construct a Schur-concave commutative copula C_θ^* that does not belong to $\overline{\text{co}}(\mathcal{C}_a)$. The copula C_θ^* admits the closed-form expression

$$C_\theta^*(u, v) = \max \{0, u + v - 1, \min(u, v) - \theta\}, \quad (u, v) \in [0, 1]^2, \quad (1)$$

and arises as a symmetric shuffle of M in the sense of [15]. The non-membership of C_θ^* in $\overline{\text{co}}(\mathcal{C}_a)$ is established by combining Milman's converse to the Krein-Milman theorem with the elementary observation that C_θ^* is not associative. We also strengthen the qualitative result by computing the exact uniform distance of C_θ^* from $\overline{\text{co}}(\mathcal{C}_a)$, given by

$$d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) = \frac{\theta(1 - 2\theta)}{1 + 2\theta}.$$

The maximum of this distance over $\theta \in]0, 1/2[$ equals $3/2 - \sqrt{2} \approx 0.0858$, attained at $\theta = (\sqrt{2} - 1)/2 \approx 0.207$. Note that the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ depends continuously on the parameter θ and connects the Fréchet-Hoeffding bounds M at $\theta \rightarrow 0^+$ and W at $\theta \rightarrow (1/2)^-$, both of which lie in $\mathcal{C}_a$. Hence, this family is the interior of a topological arc with endpoints $M, W \in \mathcal{C}_a$, and this interior is contained in $\mathcal{S}_c \setminus \overline{\text{co}}(\mathcal{C}_a)$.

For the second problem, we prove the unconditional inclusion $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$, again by Milman's converse. We show, moreover, that this inclusion is strict: the independence copula Π admits the explicit integral representation $\Pi = \int_0^1 C_v dv$, where C_v denotes the ordinal sum of two copies of W with blocks $[0, v]^2$ and $[v, 1]^2$, and consequently Π is not an extremal point of $\overline{\text{co}}(\mathcal{C}_a)$. The related question of whether a non-trivial convex combination of two distinct associative copulas can be associative is discussed in Remark 3.7. We also reformulate the second part of Open Problem 2.6, as follows: $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \text{Ext}(\mathcal{C})$ if and only if $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) = \mathcal{E}_a$, where $\mathcal{E}_a = \mathcal{C}_a \cap \text{Ext}(\mathcal{C})$. The class $\mathcal{C}_a^W$ of ordinal sums of W is a proper subset of $\mathcal{E}_a$, and $\overline{\text{co}}(\mathcal{C}_a^W) \subsetneq \overline{\text{co}}(\mathcal{C}_a)$. An affirmative answer to the second part of Open Problem 2.6 would imply $\overline{\text{co}}(\mathcal{E}_a) = \overline{\text{co}}(\mathcal{C}_a)$; this latter equality is a related open question. The integral representation above shows that $\Pi \in \overline{\text{co}}(\mathcal{C}_a^W)$ despite the strict inclusion of the closed convex hulls.

Our results should be viewed against the background of a growing literature on the geometry of distinguished classes of copulas and on copula transformations. Popović, Ristić and Zografos [18], studied an integral transformation of copulas and their dependence properties; Dolati and Dehghan Nezhad [4] investigated convexity and concavity properties of multivariate copulas; and Durante, Sarkoci and Sempi [6] extended the shuffle construction, which is central to the present paper, from M to arbitrary copulas.

The rest of the paper is organized as follows. In Section 2, we collect the definitions and the cited results from the literature. In Section 3, we prove three structural lemmas: the unconditional inclusion $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$, the integral representation of Π over ordinal sums of W and the reformulation of the second part of Open Problem 2.6. In Section 4, we introduce the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$, and derive its basic properties: satisfaction of the copula axioms, extremality in $\mathcal{C}$, commutativity, the closed-form expression (1), Schur-concavity, and non-associativity. In Section 5, we use these properties, together with a star-shapedness property of the antidiagonal sections of associative copulas (Lemma 5.3), to establish the strict inclusion $\overline{\text{co}}(\mathcal{C}_a) \subsetneq \mathcal{S}_c$ and the exact distance formula. The last section, Section 6, contains a discussion of applications, a comment on higher-dimensional extensions, and some open subquestions.

2 Preliminaries

Throughout this paper, $[0, 1]$ denotes the closed unit interval and λ denotes the Lebesgue measure on $[0, 1]$ or $[0, 1]^2$, with the dimension implicit. The involution exchanging coordinates on $[0, 1]^2$ is denoted by $\sigma: (u, v) \mapsto (v, u)$.

A 2-copula (or simply, a *copula*) is a function $C: [0, 1]^2 \rightarrow [0, 1]$ that is grounded, that is, $C(u, 0) = 0 = C(0, v)$ for all $u, v \in [0, 1]$, has uniform marginals, that is, $C(u, 1) = u$ and $C(1, v) = v$ for all $u, v \in [0, 1]$, and is 2-increasing, that is,

$$C(u_1, v_1) - C(u_2, v_1) - C(u_1, v_2) + C(u_2, v_2) \geq 0,$$

for all $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$. The class of all copulas is denoted by $\mathcal{C}$, and it is equipped with the uniform metric

$$d_\infty(C, D) = \sup_{(u, v) \in [0, 1]^2} |C(u, v) - D(u, v)|.$$

The space $(\mathcal{C}, d_\infty)$ is convex and uniformly compact; see [8, Theorem 1.7.7]. The Fréchet-Hoeffding bounds are given by $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v)$, and the independence copula is $\Pi(u, v) = uv$.

Every copula C uniquely determines a Borel probability measure μ_C on $[0, 1]^2$, given by the formula $\mu_C([0, u] \times [0, v]) = C(u, v)$, and both marginals of μ_C are equal to λ . The collection of these probability measures is referred to as the set of *doubly stochastic measures* on $[0, 1]^2$. We say that μ_C is *concentrated* on a Borel set $A \subseteq [0, 1]^2$ if $\mu_C(A) = 1$.

Definition 2.1. For a convex subset K of a topological vector space, an extreme point of K is a point $x \in K$ that admits no non-trivial decomposition: if $x = ty + (1 - t)z$ for some $t \in]0, 1[$ and $y, z \in K$, then $y = z = x$. The set of extreme points of K is denoted by $\text{Ext}(K)$.

Support criteria for extreme doubly stochastic measures require care; see [5, 9, 12].

Lemma 2.2. Let $S \subseteq [0, 1]^2$ be Borel. If μ_C is the unique doubly stochastic measure concentrated on S , then $C \in \text{Ext}(\mathcal{C})$.

Proof. If $C = tC_1 + (1 - t)C_2$ with $0 < t < 1$, then $\mu_C = t\mu_{C_1} + (1 - t)\mu_{C_2}$. Since $\mu_C(S) = 1$, both μ_{C_1} and μ_{C_2} are concentrated on S . Uniqueness gives $\mu_{C_1} = \mu_{C_2} = \mu_C$, hence $C_1 = C_2 = C$. $\square$

We will also repeatedly use the following classical results from convex analysis, which we state as a single lemma for ease of reference; for the proofs, we refer to [3, 19].

Lemma 2.3. Let E be a locally convex Hausdorff topological vector space.

- (i) (Krein-Milman) Every non-empty compact convex subset K of E satisfies $K = \overline{\text{co}}(\text{Ext}(K))$.
- (ii) (Milman) If K is compact and $K = \overline{\text{co}}(F)$ for some compact set $F \subseteq E$, then $\text{Ext}(K) \subseteq F$.
- (iii) (Bauer) Every continuous linear functional on E attains its supremum over a non-empty compact convex set $K \subseteq E$ at some point of $\text{Ext}(K)$.

For two pairs $(u_1, v_1), (u_2, v_2) \in [0, 1]^2$, we say that (u_1, v_1) is majorized by (u_2, v_2) , and we write $(u_1, v_1) \prec (u_2, v_2)$, if $u_1 + v_1 = u_2 + v_2$ and $\max\{u_1, v_1\} \leq \max\{u_2, v_2\}$. A function $C: [0, 1]^2 \rightarrow \mathbb{R}$ is said to be *Schur-concave* if $C(u_1, v_1) \geq C(u_2, v_2)$ whenever $(u_1, v_1) \prec (u_2, v_2)$; see [13]. Note that Schur-concavity implies symmetry, that is, $C(u, v) = C(v, u)$. A symmetric function C is Schur-concave if and only if for every $s \in [0, 2]$, the function $u \mapsto C(u, s - u)$ is non-decreasing on the interval $[\max(0, s - 1), s/2]$. We do not use the analytic characterization of Schur-concavity in terms of partial derivatives, since the copulas considered in this paper are singular and need not be differentiable.

The class of Schur-concave commutative copulas is denoted by $\mathcal{S}_c$. The class $\mathcal{S}_c$ is closed (since both Schur-concavity and symmetry are preserved under uniform limits), is convex, and contains $\mathcal{C}_a$ by [1, 7]. For convexity and concavity properties of multivariate copulas, we refer to [4].

A copula C is said to be *associative* if it satisfies $C(C(u, v), w) = C(u, C(v, w))$ for all $u, v, w \in [0, 1]$. The class of associative copulas, denoted by $\mathcal{C}_a$, coincides with the class of 1-Lipschitz triangular norms; [10]. By the result of [11], the class $\mathcal{C}_a$ is uniformly compact.

By the Mostert-Shields theorem [16] (see also [10, §3.4]), every associative copula admits a unique *ordinal sum* representation: there exist an at most countable family $\{[a_i, b_i]\}_{i \in I}$ of pairwise disjoint open sub-intervals of $]0, 1[$ and a corresponding family of Archimedean copulas $\{A_i\}_{i \in I}$ such that

$$C(u, v) = \begin{cases} a_i + (b_i - a_i)A_i\left(\frac{u - a_i}{b_i - a_i}, \frac{v - a_i}{b_i - a_i}\right), & \text{if } (u, v) \in [a_i, b_i]^2, \\ \min(u, v), & \text{otherwise.} \end{cases} \quad (2)$$

The squares $[a_i, b_i]^2$ are referred to as the *blocks* of the ordinal sum, and the complement of the union $\bigcup_i [a_i, b_i]^2$ in $[0, 1]^2$ is the *off-block region*. Note that on the off-block region, the copula coincides with the Fréchet-Hoeffding upper bound M . We set $\mathcal{E}_a = \mathcal{C}_a \cap \text{Ext}(\mathcal{C})$ and denote by $\mathcal{C}_a^W \subseteq \mathcal{C}_a$ the class of those associative copulas whose Archimedean components A_i in (2) are all equal to W . Observe that the Fréchet-Hoeffding bounds M (corresponding to the empty ordinal sum) and W (corresponding to a single block $[0, 1]^2$) belong to $\mathcal{C}_a^W$.

A measure-preserving bijection $g: [0, 1] \rightarrow [0, 1]$ is a Borel bijection (up to a Lebesgue null set) satisfying $\lambda(g^{-1}(B)) = \lambda(B)$ for every Borel set $B \subseteq [0, 1]$. The *shuffle* of M associated to g , denoted by C_g , is the unique copula whose induced doubly stochastic measure is the pushforward of λ along the map $x \mapsto (x, g(x))$. Equivalently,

$$C_g(u, v) = \lambda(\{x \in [0, u]: g(x) \leq v\}). \quad (3)$$

The class of shuffles of M has been thoroughly studied in [15]; for a modern account, and for an extension of the shuffle construction from M to arbitrary copulas, see [6] and [8, Section 3.6].

3 Extremal points of $\overline{\text{co}}(\mathcal{C}_a)$

In this section, we prove three structural lemmas: the unconditional inclusion $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$, and a reformulation of the second part of Open Problem 2.6. We start with two simple consequences of the uniqueness criterion in Lemma 2.2, both of which will be used in the construction of $\mathcal{C}_\theta^*$ in the next section.

Lemma 3.1. *Let $g: [0, 1] \rightarrow [0, 1]$ be a Borel function. If μ_C is concentrated on the graph $\Gamma_g = \{(x, g(x)): x \in [0, 1]\}$, then $C \in \text{Ext}(\mathcal{C})$.*

Proof. Since g is Borel, its graph Γ_g is Borel. Any doubly stochastic measure concentrated on Γ_g must have distribution function

$$(u, v) \mapsto \lambda(\{x \in [0, u]: g(x) \leq v\}),$$

because the first marginal is λ and the measure is concentrated on the graph of g . Thus this measure is unique, and Lemma 2.2 yields $C \in \text{Ext}(\mathcal{C})$. $\square$

Lemma 3.2. *For every measure-preserving Borel map $g: [0, 1] \rightarrow [0, 1]$, the function C_g , defined by (3), is a copula belonging to $\text{Ext}(\mathcal{C})$.*

Proof. Set $\mu = (\text{id}, g)_* \lambda$. The first marginal of μ is λ itself, as it is the pushforward of λ under the identity map. The second marginal is the pushforward of λ along g , which equals λ since g is measure-preserving. Hence μ is doubly stochastic, so the function C_g given by (3) is a copula. Note that μ is concentrated on Γ_g , so Lemma 3.1 gives $C_g \in \text{Ext}(\mathcal{C})$. $\square$

The third lemma records the relation between $\mathcal{C}_a^W$ and $\mathcal{E}_a$ and distinguishes their closed convex hulls.

Lemma 3.3. *The class $\mathcal{C}_a^W$ is uniformly compact, and*

$$\mathcal{C}_a^W \subsetneq \mathcal{E}_a = \mathcal{C}_a \cap \text{Ext}(\mathcal{C}), \quad \overline{\text{co}}(\mathcal{C}_a^W) \subsetneq \overline{\text{co}}(\mathcal{C}_a).$$

Proof. Let $C \in \mathcal{C}_a^W$ with blocks $[a_i, b_i]^2$, as in (2). Define

$$h(x) = \begin{cases} a_i + b_i - x, & \text{if } x \in]a_i, b_i[, \\ x, & \text{otherwise.} \end{cases}$$

The open block intervals are disjoint, so h is a well-defined involutive Borel bijection preserving λ . The measure μ_C is concentrated on its graph; changing the value of h at the countably many block endpoints does not affect this measure. Lemma 3.1 gives $C \in \text{Ext}(\mathcal{C})$, and hence $\mathcal{C}_a^W \subseteq \mathcal{E}_a$.

To prove compactness, let $C_n \in \mathcal{C}_a^W$ converge uniformly to $C \in \mathcal{C}$. Since $\mathcal{C}_a$ is closed in $\mathcal{C}$ and $\mathcal{C}_a^W \subseteq \mathcal{C}_a$, we have $C \in \mathcal{C}_a$. Fix u with $C(u, u) < u$. For all sufficiently large n , the point u lies in an open block $]a_n, b_n[$ of C_n . Passing to a subsequence, suppose $a_n \rightarrow a$ and $b_n \rightarrow b$. Uniform convergence and the copula Lipschitz bound give $C(a, a) = a$ and $C(b, b) = b$. On $[a, b]^2$, passage to the limit in the block formula gives $C(x, y) = \max\{a, x + y - b\}$. In particular, $a < u < b$, and $C(x, x) < x$ for every $x \in]a, b[$. Thus $]a, b[$ is exactly the component of the non-idempotent set of C containing u . Since u was arbitrary, every Archimedean component of C is W , so $C \in \mathcal{C}_a^W$. Hence, $\mathcal{C}_a^W$ is closed in the uniformly compact class $\mathcal{C}_a$.

For strictness, consider the Archimedean copula A generated by the convex, strictly decreasing function

$$\varphi(u) = \begin{cases} 31/10 - 10u, & \text{if } 0 \leq u \leq 1/5, \\ 3/2 - 2u, & \text{if } 1/5 < u \leq 1/2, \\ 1 - u, & \text{if } 1/2 < u \leq 1. \end{cases}$$

This is the copula used again in Remark 5.4. We verify its extremality directly. In the coordinates $p = \varphi(u)$ and $q = \varphi(v)$, the pushforward of μ_A is concentrated on

$$S = S_{31/10} \cup S_{11/10} \cup S_{1/2} \quad \text{with } S_c = \{(p, q) \in [0, 31/10]^2: p + q = c\}.$$

Indeed, writing $r = \varphi^{(-1)}$, the joint survival function in these coordinates is $r(p+q)$, and r is piecewise affine with slope changes only at $1/2$, $11/10$, and $31/10$. Thus its mixed distributional derivative is concentrated on the three displayed segments.

There is at most one measure on S with the prescribed marginals. To see this, let η be the difference of two such measures, so both marginals of η vanish. On the region $p > 11/10$, only $S_{31/10}$ is present and the first projection is injective, so the first marginal forces η to vanish there. The second marginal similarly forces it to vanish on $q > 11/10$. These two regions cover $S_{31/10}$ because $31/10 > 2(11/10)$. After removing this segment, the same argument with $p > 1/2$ or $q > 1/2$ removes $S_{11/10}$, since $11/10 > 2(1/2)$. The remaining segment $S_{1/2}$ is a graph, and its measure is fixed by its first marginal. Hence $\eta = 0$. The change of coordinates is a Borel bijection, so Lemma 2.2 gives $A \in \mathcal{E}_a$.

Since A is Archimedean, it has no idempotent in $]0, 1[$. If $A \in \mathcal{C}_a^W$, its ordinal sum would therefore consist of the single block $[0, 1]^2$, forcing $A = W$. However, $A(1/2, 1/2) = 1/4 \neq 0 = W(1/2, 1/2)$. Thus $A \notin \mathcal{C}_a^W$ and $\mathcal{C}_a^W \subsetneq \mathcal{E}_a$. Equivalently, its diagonal satisfies $A(u, u) = u - 1/4$ on $[1/2, 3/4]$, which cannot occur on a block of an ordinal sum of W .

Finally, if $A \in \overline{\text{co}}(\mathcal{C}_a^W)$, its extremality in $\mathcal{C}$ would imply extremality in $\overline{\text{co}}(\mathcal{C}_a^W)$. Compactness of $\mathcal{C}_a^W$ and Milman's converse (Lemma 2.3(ii)) would then give $A \in \mathcal{C}_a^W$, a contradiction. Since $A \in \mathcal{C}_a$, the inclusion $\overline{\text{co}}(\mathcal{C}_a^W) \subsetneq \overline{\text{co}}(\mathcal{C}_a)$ follows. $\square$

We now turn to the partial answer to Open Problem 2.6.

Theorem 3.4. $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$.

Proof. By the result of [11], the class $\mathcal{C}_a$ is a compact subset of the locally convex Hausdorff space $C([0, 1]^2)$ in the uniform topology. Hence $\overline{\text{co}}(\mathcal{C}_a)$ is a compact convex set whose generator $\mathcal{C}_a$ is compact, and the conclusion follows directly from Milman's converse to the Krein-Milman theorem (Lemma 2.3(ii)). $\square$

The next proposition identifies a concrete subset of $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$.

Proposition 3.5. $\mathcal{C}_a^W \subsetneq \mathcal{E}_a \subseteq \text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \cap \text{Ext}(\mathcal{C})$.

Proof. By definition, $\mathcal{E}_a = \mathcal{C}_a \cap \text{Ext}(\mathcal{C}) \subseteq \text{Ext}(\mathcal{C})$, and Lemma 3.3 gives $\mathcal{C}_a^W \subsetneq \mathcal{E}_a$. Note that if $K \subseteq L$ are convex subsets of a vector space and $x \in K \cap \text{Ext}(L)$, then $x \in \text{Ext}(K)$, since any decomposition $x = tx_1 + (1-t)x_2$ with $t \in]0, 1[$ and $x_1, x_2 \in K \subseteq L$ forces $x_1 = x_2 = x$ by the extremality of x in L . Applying this observation with $K = \overline{\text{co}}(\mathcal{C}_a)$ and $L = \mathcal{C}$, and using $\mathcal{E}_a \subseteq \mathcal{C}_a \subseteq \overline{\text{co}}(\mathcal{C}_a)$, we conclude that $\mathcal{E}_a \subseteq \text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$. $\square$

Example 3.6. For $v \in [0, 1]$, let C_v denote the ordinal sum of two copies of W with blocks $[0, v]^2$ and $[v, 1]^2$, that is, by (2),

$$C_v(u, w) = \begin{cases} \max(0, u + w - v), & \text{if } (u, w) \in [0, v]^2, \\ v + \max(0, u + w - 1 - v), & \text{if } (u, w) \in [v, 1]^2, \\ \min(u, w), & \text{otherwise,} \end{cases} \quad (4)$$

with the degenerate cases $C_0 = C_1 = W$. The measure μ_{C_v} is concentrated on the union of the antidiagonals of the two blocks. After removal of the finitely many endpoints, this union is the graph of the involution given by $x \mapsto v - x$ on $]0, v[$ and $x \mapsto 1 + v - x$ on $]v, 1[$; the endpoints may be fixed. By Lemma 3.3, $C_v \in \text{Ext}(\mathcal{C})$, and by Proposition 3.5, $C_v \in \text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$. Note also that the map $v \mapsto C_v$ is 1-Lipschitz from $[0, 1]$ to $(\mathcal{C}, d_\infty)$, since for every fixed (u, w) the right-hand side of (4) is piecewise affine in v with slopes in $[-1, 1]$.

Remark 3.7. The reverse inclusion $\mathcal{C}_a \subseteq \text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$ asserts that no associative copula can be written as a non-trivial convex combination of two distinct copulas in $\overline{\text{co}}(\mathcal{C}_a)$. Restricting the two copulas to $\mathcal{C}_a$ gives a weaker question: do there exist distinct $C_1, C_2 \in \mathcal{C}_a$ and $t \in]0, 1[$ such that $tC_1 + (1-t)C_2$ is associative? In Corollary 3.11 below, we show that the reverse inclusion fails: the independence copula Π admits a non-trivial convex decomposition within $\overline{\text{co}}(\mathcal{C}_a)$. This does not settle the weaker question, since the two copulas appearing in this decomposition are not associative. The determination of the full set $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$ also remains open.

We next give a reformulation of the second part of Open Problem 2.6.

Theorem 3.8. The following two conditions are equivalent:

1. $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \text{Ext}(\mathcal{C})$.
2. $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) = \mathcal{E}_a$.

Moreover, either of these conditions implies $\overline{\text{co}}(\mathcal{E}_a) = \overline{\text{co}}(\mathcal{C}_a)$. No converse to this last implication is asserted.

Proof. The implication (2) $\Rightarrow$ (1) is immediate from the definition $\mathcal{E}_a = \mathcal{C}_a \cap \text{Ext}(\mathcal{C})$. For the converse, Theorem 3.4 gives $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$. Combining this with condition 1, we obtain $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a \cap \text{Ext}(\mathcal{C}) = \mathcal{E}_a$. The reverse inclusion is exactly Proposition 3.5.

For the final claim, assume condition 2. The Krein-Milman Theorem (Lemma 2.3(i)) gives $\overline{\text{co}}(\mathcal{C}_a) = \overline{\text{co}}(\text{Ext}(\overline{\text{co}}(\mathcal{C}_a))) = \overline{\text{co}}(\mathcal{E}_a)$. $\square$

Remark 3.9. Theorem 3.8 reformulates the second part of Open Problem 2.6 in terms of $\mathcal{E}_a$. By Lemma 3.3, the smaller class $\mathcal{C}_a^W$ cannot generate $\overline{\text{co}}(\mathcal{C}_a)$. Nevertheless, individual associative copulas outside $\mathcal{C}_a^W$ may belong to $\overline{\text{co}}(\mathcal{C}_a^W)$. A natural example is the independence copula Π : it is associative. It has a two-dimensional density, whereas every copula in $\mathcal{C}_a^W$ is concentrated on the block antidiagonals and the off-block diagonal. The next proposition shows that Π does belong to $\overline{\text{co}}(\mathcal{C}_a^W)$.

Proposition 3.10. Let $\{C_v\}_{v \in [0,1]}$ be the family of Example 3.6. Then

$$\Pi(u, w) = \int_0^1 C_v(u, w) dv, \quad (u, w) \in [0, 1]^2. \quad (5)$$

In particular, $\Pi \in \overline{\text{co}}(\mathcal{C}_a^W)$.

Proof. Both sides of (5) are symmetric in (u, w) , so we may assume $u \leq w$. Write $s = u + w$. By (4), the integrand equals $v + \max(0, s - 1 - v)$ for $v \in [0, u]$, equals u for $v \in [u, w]$, and equals $\max(0, s - v)$ for $v \in [w, 1]$. We distinguish two cases.

If $s \leq 1$, then $s - 1 - v \leq 0$ on $[0, u]$ and $\max(0, s - v) = s - v$ precisely for $v \leq s$, so that

$$\int_0^1 C_v(u, w) dv = \int_0^u v dv + u(w - u) + \int_w^s (s - v) dv = \frac{u^2}{2} + uw - u^2 + \frac{(s - w)^2}{2} = uw,$$

where we used $s - w = u$.

If $s > 1$, then $0 \leq s - 1 \leq u$, and

$$\int_0^1 C_v(u, w) dv = \int_0^u v dv + \int_0^{s-1} (s - 1 - v) dv + u(w - u) + \int_w^1 (s - v) dv.$$

The four terms evaluate to $u^2/2$, $(u + w - 1)^2/2$, $uw - u^2$, and $(u + w)(1 - w) - (1 - w^2)/2$, respectively, and using the identity $(u + w - 1)^2 - (1 - w)^2 = u^2 + 2u(w - 1)$ shows that the sum again equals uw . This proves (5).

For the second claim, recall from Example 3.6 that $v \mapsto C_v$ is 1-Lipschitz in d_∞ . Hence the Riemann sums

$$\frac{1}{k} \sum_{j=1}^k C_{(2j-1)/(2k)},$$

which are convex combinations of elements of $\mathcal{C}_a^W$, converge uniformly to the right-hand side of (5) as $k \rightarrow \infty$. Therefore, $\Pi \in \overline{\text{co}}(\mathcal{C}_a^W)$. $\square$

Corollary 3.11. The copula Π is not an extremal point of $\overline{\text{co}}(\mathcal{C}_a)$. Consequently, the inclusion of Theorem 3.4 is strict, that is, $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subsetneq \mathcal{C}_a$, and the reverse inclusion $\mathcal{C}_a \subseteq \text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$ of Remark 3.7 fails.

Proof. Splitting the integral (5) at $v = 1/2$ yields $\Pi = \frac{1}{2}D_1 + \frac{1}{2}D_2$ with

$$D_1 = 2 \int_0^{1/2} C_v dv, \quad D_2 = 2 \int_{1/2}^1 C_v dv,$$

and the Riemann sum argument of Proposition 3.10 shows $D_1, D_2 \in \overline{\text{co}}(\mathcal{C}_a^W) \subseteq \overline{\text{co}}(\mathcal{C}_a)$. It remains to observe that $D_1 \neq D_2$. At $(u, w) = (1/4, 1/4)$, we compute as follows: for $v \in [0, 1/4]$, (4) gives $C_v(1/4, 1/4) = v + \max(0, -1/2 - v) = v$, while for $v \in [1/4, 1]$, it gives $C_v(1/4, 1/4) = \max(0, 1/2 - v)$. Hence

$$D_1(1/4, 1/4) = 2 \int_0^{1/4} v dv + 2 \int_{1/4}^{1/2} \left(\frac{1}{2} - v\right) dv = \frac{1}{16} + \frac{1}{16} = \frac{1}{8},$$

whereas $D_2(1/4, 1/4) = 2 \int_{1/2}^1 \max(0, 1/2 - v) dv = 0$. Thus, Π admits a non-trivial decomposition within $\overline{\text{co}}(\mathcal{C}_a)$, so $\Pi \notin \text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$, while $\Pi \in \mathcal{C}_a$. $\square$

Remark 3.12. Proposition 3.10 shows that $\Pi \in \overline{\text{co}}(\mathcal{C}_a^W)$, but Lemma 3.3 gives $\overline{\text{co}}(\mathcal{C}_a^W) \subsetneq \overline{\text{co}}(\mathcal{C}_a)$. A related question is whether $\overline{\text{co}}(\mathcal{E}_a) = \overline{\text{co}}(\mathcal{C}_a)$. This equality holds if and only if every Archimedean copula belongs to $\overline{\text{co}}(\mathcal{E}_a)$. The necessity is trivial. For the sufficiency, let $C \in \mathcal{C}_a$ have the representation (2); discarding all blocks with $b_i - a_i < \varepsilon$ changes C by at most ε in d_∞ , so we may assume that I is finite. If each component A_i is approximated by a convex combination of elements of $\mathcal{E}_a$, then, using the affinity of (2) in the components and taking product weights, we can approximate C by convex combinations of ordinal sums of elements of $\mathcal{E}_a$. Such an ordinal sum belongs to $\mathcal{E}_a$: in any convex decomposition into copulas, the zero mass outside its blocks and its off-block diagonal is preserved; after rescaling each block, extremality of its component forces both summands to agree there, and the diagonal remainder is fixed by its marginals. Thus every associative copula belongs to $\overline{\text{co}}(\mathcal{E}_a)$. We return to this question in Open Question 2 of Section 6. The converse of the final implication in Theorem 3.8 would require an additional argument, since compactness of $\mathcal{E}_a$ has not been established.

4 A Schur-concave shuffle of M

In this section, we construct the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ of copulas and derive their basic properties: satisfaction of the copula axioms, extremality in $\mathcal{C}$, commutativity, the closed-form expression (1), Schur-concavity, and non-associativity. The construction is based on a piecewise affine involution of the unit interval, defined as follows.

Definition 4.1. For every $\theta \in]0, 1/2[$, let $f_\theta: [0, 1] \rightarrow [0, 1]$ be the function given by

$$f_\theta(x) = \begin{cases} 1 - x, & \text{if } x \in [0, \theta] \cup [1 - \theta, 1], \\ x, & \text{otherwise.} \end{cases} \quad (6)$$

denotes the shuffle of M associated with f_θ via (3) C_θ^* , and we write Γ_θ for the graph of f_θ , that is,

$$\Gamma_\theta = \{(x, f_\theta(x)) : x \in [0, 1]\} = L_- \cup L_0 \cup L_+,$$

where the three line segments $L_-, L_0, L_+ \subseteq [0, 1]^2$ are given by

$$\begin{aligned} L_- &= \{(x, 1 - x) : x \in [0, \theta]\}, \\ L_0 &= \{(x, x) : x \in]\theta, 1 - \theta[\}, \\ L_+ &= \{(x, 1 - x) : x \in [1 - \theta, 1]\}. \end{aligned}$$

The next lemma collects the basic properties of f_θ and Γ_θ .

Lemma 4.2. For every $\theta \in]0, 1/2[$, the function f_θ is a measure-preserving bijection of $[0, 1]$ with $f_\theta \circ f_\theta = \text{id}_{[0, 1]}$, and the graph Γ_θ is invariant under the swap $\sigma: (u, v) \mapsto (v, u)$.

Proof. By (6), the function f_θ maps $[0, \theta]$ bijectively onto $[1 - \theta, 1]$ via the affine isometry $x \mapsto 1 - x$, fixes $]\theta, 1 - \theta[$ pointwise, and maps $[1 - \theta, 1]$ bijectively onto $[0, \theta]$ via the same affine isometry. Hence, f_θ is a bijection of $[0, 1]$ that, being a piecewise affine isometry, preserves the Lebesgue measure. The relation $f_\theta \circ f_\theta = \text{id}_{[0, 1]}$ is immediate from the construction. For the swap-invariance of Γ_θ , observe that $\sigma(L_0) = L_0$ trivially, and

$$\sigma(L_-) = \{(1 - x, x) : x \in [0, \theta]\} = \{(y, 1 - y) : y \in [1 - \theta, 1]\} = L_+,$$

and analogously $\sigma(L_+) = L_-$. Therefore, $\sigma(\Gamma_\theta) = \Gamma_\theta$. □

We now use Lemma 4.2 together with the structural lemmas of Section 3 to verify the basic properties of C_θ^* .

Proposition 4.3. For every $\theta \in]0, 1/2[$, the function C_θ^* is a copula belonging to $\text{Ext}(\mathcal{C})$ and is commutative.

Proof. Lemma 4.2 together with Lemma 3.2 immediately shows that C_θ^* is a copula and belongs to $\text{Ext}(\mathcal{C})$. For commutativity, it suffices to show that $\sigma_* \mu_{C_\theta^*} = \mu_{C_\theta^*}$. Note that, by definition, $\mu_{C_\theta^*} = (\text{id}, f_\theta)_* \lambda$, and using $f_\theta \circ f_\theta = \text{id}$ from Lemma 4.2, we obtain

$$\sigma_* \mu_{C_\theta^*} = (f_\theta, \text{id})_* \lambda = (\text{id}, f_\theta)_* ((f_\theta)_* \lambda) = (\text{id}, f_\theta)_* \lambda = \mu_{C_\theta^*},$$

where we have used $(f_\theta)_* \lambda = \lambda$. □

The next lemma gives the closed-form expression for C_θ^* .

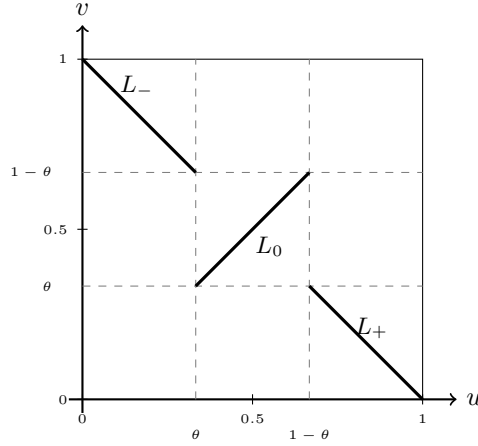


Figure 1: The set Γ_θ on which the doubly stochastic measure $\mu_{C_\theta^*}$ is concentrated, drawn for $\theta = 1/3$. The set is the disjoint union of three line segments, with the endpoints of the central segment omitted, and the mass distribution on Γ_θ is the pushforward of the Lebesgue measure on $[0, 1]$ along the map $x \mapsto (x, f_\theta(x))$.

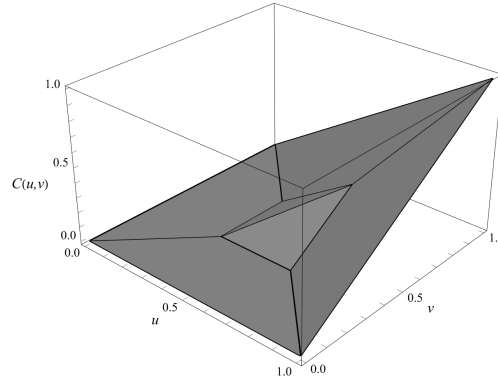


Figure 2: The graph of C_θ^* for $\theta = 1/3$, drawn in an oblique projection. By (1), the graph consists of four affine pieces: the copula vanishes on the region $\{\min(u, v) \leq \theta, u + v \leq 1\}$, equals $\min(u, v) - \theta$ on the central square $[\theta, 1 - \theta]^2$ (that is, $u - \theta$ on the half with $u \leq v$, and $v - \theta$ on the half with $v \leq u$), and equals $u + v - 1$ on the region $\{\max(u, v) \geq 1 - \theta, u + v \geq 1\}$. The singular copula measure is concentrated on the central diagonal and the two tail antidiagonals shown in Figure 1.

Lemma 4.4. *For every $\theta \in]0, 1/2[$ and every $(u, v) \in [0, 1]^2$,*

$$C_\theta^*(u, v) = \max \{0, u + v - 1, \min(u, v) - \theta\}. \quad (7)$$

Proof. By the definition (3) and the partition of f_θ into three pieces, we have $C_\theta^*(u, v) = \Lambda_- + \Lambda_0 + \Lambda_+$, where

$$\begin{aligned} \Lambda_- &= \lambda(\{x \in [0, \theta] \cap [0, u]: 1 - x \leq v\}) = [\min(u, \theta) - \max(0, 1 - v)]^+, \\ \Lambda_0 &= \lambda(\{x \in [\theta, 1 - \theta] \cap [0, u]: x \leq v\}) = [\min\{u, v, 1 - \theta\} - \theta]^+, \\ \Lambda_+ &= \lambda(\{x \in [1 - \theta, 1] \cap [0, u]: 1 - x \leq v\}) = [\min(u, 1) - \max(1 - \theta, 1 - v)]^+, \end{aligned}$$

each understood as the non-negative length of an interval, vanishing whenever the interval is empty (i.e., the notation $[x]^+$ means $\max(x, 0)$). Let $I_1 = [0, \theta]$, $I_2 = [\theta, 1 - \theta]$, and $I_3 = [1 - \theta, 1]$. By the symmetry $C_\theta^*(u, v) = C_\theta^*(v, u)$ is established in Proposition 4.3, it suffices to compute the components Λ_- , Λ_0 , and Λ_+ in the six regions $u \in I_p$, $v \in I_q$ with $p \leq q$, and the values are tabulated in Table 1. It remains to verify that the sum of the three components in each region equals the right-hand side of (7). For example, in Region VI, we have $\Lambda_- + \Lambda_0 + \Lambda_+ = u + v - 1$, while $\min(u, v) - \theta = u - \theta \geq 0$ and $(u + v - 1) - (u - \theta) = v + \theta - 1 \geq 0$, so the maximum of the three candidates in (7) equals $u + v - 1$. The remaining rows are verified analogously, and the values in Regions IV, VII and VIII follow by symmetry of C_θ^* and of the right-hand side of (7). $\square$

region	$u \in$	$v \in$	Λ_-	Λ_0	Λ_+
I	I_1	I_1	0	0	0
II	I_1	I_2	0	0	0
III	I_1	I_3	$[u + v - 1]^+$	0	0
V	I_2	I_2	0	$\min(u, v) - \theta$	0
VI	I_2	I_3	$v + \theta - 1$	$u - \theta$	0
IX	I_3	I_3	$v + \theta - 1$	$1 - 2\theta$	$u + \theta - 1$

Table 1: The components Λ_- , Λ_0 , and Λ_+ of $C_\theta^*(u, v)$ in the six regions $u \in I_p$, $v \in I_q$ with $p \leq q$, used in the proof of Lemma 4.4; the values in the remaining three regions follow from the symmetry of C_θ^* .

Example 4.5. For $\theta = 1/3$ and $(u, v) = (1/2, 1/2)$, the formula (7) gives

$$C_{1/3}^*(1/2, 1/2) = \max\{0, 0, 1/2 - 1/3\} = 1/6.$$

This is consistent with a direct computation: the only segment of $\Gamma_{1/3}$ that intersects $[0, 1/2]^2$ is the portion of L_0 lying in $[1/3, 1/2]^2$, whose projection onto the first coordinate has Lebesgue length $1/6$, and which therefore has copula mass $1/6$. Note also that, by (7), $C_{1/3}^*(1/3, 2/3) = 0$, even though $(1/3, 2/3)$ lies on the antidiagonal where one might expect non-trivial mass. A three-dimensional rendering of the graph of $C_{1/3}^*$ is shown in Figure 2.

Remark 4.6. The closed form (7) makes the limiting behaviour of the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ transparent. As $\theta \rightarrow 0^+$, we have

$$C_\theta^*(u, v) \rightarrow \max\{0, u + v - 1, \min(u, v)\} = \min(u, v) = M(u, v),$$

since $\min(u, v) \geq 0$ and $\min(u, v) \geq u + v - 1$. As $\theta \rightarrow (1/2)^-$, the term $\min(u, v) - \theta$ becomes dominated by $u + v - 1$ on the region $u + v \geq 1$ and is non-positive on $u + v < 1$, which gives $C_\theta^*(u, v) \rightarrow W(u, v)$. Hence, the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ is a continuous one-parameter family in the uniform metric that connects M at $\theta \rightarrow 0^+$ and W at $\theta \rightarrow (1/2)^-$.

We now verify that C_θ^* is Schur-concave by a direct application of the closed form (7).

Theorem 4.7. For every $\theta \in]0, 1/2[$, the copula C_θ^* is Schur-concave; consequently, $C_\theta^* \in \mathcal{S}_c$.

Proof. By Proposition 4.3, C_θ^* is symmetric. Schur-concavity is equivalent to the property that for every $s \in [0, 2]$, the function $\varphi_s(u) = C_\theta^*(u, s - u)$ is non-decreasing on the interval $[\max(0, s - 1), s/2]$. From Lemma 4.4,

$$\varphi_s(u) = \max\{0, s - 1, \min(u, s - u) - \theta\}.$$

For $u \in [\max(0, s - 1), s/2]$, we have $u \leq s - u$, so $\min(u, s - u) = u$ and

$$\varphi_s(u) = \max\{0, s - 1, u - \theta\}.$$

Note that the constants 0 and $s - 1$ are independent of u , while the function $u \mapsto u - \theta$ is non-decreasing. Hence, the maximum φ_s is also non-decreasing on $[\max(0, s - 1), s/2]$, completing the proof. $\square$

Example 4.8. Schur-concavity is not automatic for commutative shuffles of M , so Theorem 4.7 genuinely depends on the choice of f_θ . Consider the measure-preserving bijection $g(x) = x + 1/2 \pmod{1}$, whose graph

$$\Gamma_g = \{(x, x + 1/2) : x \in [0, 1/2[\} \cup \{(x, x - 1/2) : x \in [1/2, 1] \},$$

is invariant under the swap σ up to finitely many endpoints, and $g \circ g = \text{id}$ up to a null set, so the argument of Proposition 4.3 shows that the shuffle C_g is a commutative member of $\text{Ext}(\mathcal{C})$. From (3), we compute $C_g(1/2, 1/2) = 0$ and $C_g(1/4, 3/4) = 1/4$. Since $(1/2, 1/2) \prec (1/4, 3/4)$, Schur-concavity would require $C_g(1/2, 1/2) \geq C_g(1/4, 3/4)$, and hence fails: $C_g \notin \mathcal{S}_c$.

The last property of C_θ^* that we need is its non-associativity.

Lemma 4.9. For every $\theta \in]0, 1/2[$, $C_\theta^* \notin \mathcal{C}_a$.

Proof. Let $u = 1/2$, $v = 1 - \theta$, and $w = 1/2 + \theta$. First, we compute $C_\theta^*(v, w) = C_\theta^*(1 - \theta, 1/2 + \theta)$. Since $(1 - \theta) + (1/2 + \theta) - 1 = 1/2$ and

$$\min(1 - \theta, 1/2 + \theta) - \theta \leq 1/2 + \theta - \theta = 1/2,$$

we obtain $C_\theta^*(v, w) = 1/2$. Consequently,

$$C_\theta^*(u, C_\theta^*(v, w)) = C_\theta^*(1/2, 1/2) = 1/2 - \theta.$$

On the other hand,

$$C_\theta^*(u, v) = C_\theta^*(1/2, 1 - \theta) = \max\{0, 1/2 - \theta, \min(1/2, 1 - \theta) - \theta\} = 1/2 - \theta.$$

Then,

$$C_\theta^*(C_\theta^*(u, v), w) = C_\theta^*(1/2 - \theta, 1/2 + \theta) = \max\{0, 0, 1/2 - 2\theta\} = \max\{0, 1/2 - 2\theta\}.$$

Since $\theta \in]0, 1/2[$, we have $\max\{0, 1/2 - 2\theta\} < 1/2 - \theta$. Thus, $C_\theta^*(u, C_\theta^*(v, w)) \neq C_\theta^*(C_\theta^*(u, v), w)$, proving that C_θ^* is non-associative for every $\theta \in]0, 1/2[$. $\square$

5 The strict inclusion and the distance formula

We are now in a position to combine the results of the previous two sections to settle Open Problem 2.5 and to compute the exact uniform distance from C_θ^* to $\overline{\text{co}}(\mathcal{C}_a)$.

Theorem 5.1. $\overline{\text{co}}(\mathcal{C}_a) \subsetneq \mathcal{S}_c$.

Proof. The inclusion $\overline{\text{co}}(\mathcal{C}_a) \subseteq \mathcal{S}_c$ is immediate, since $\mathcal{C}_a \subseteq \mathcal{S}_c$ and $\mathcal{S}_c$ is closed and convex. To show that the inclusion is strict, fix any $\theta \in]0, 1/2[$. By Theorem 4.7, $C_\theta^* \in \mathcal{S}_c$. Suppose, for the sake of contradiction, that $C_\theta^* \in \overline{\text{co}}(\mathcal{C}_a)$. Proposition 4.3 gives $C_\theta^* \in \text{Ext}(\mathcal{C})$, and the proof of Proposition 3.5 (applied with $K = \overline{\text{co}}(\mathcal{C}_a)$ and $L = \mathcal{C}$) shows that $C_\theta^* \in \text{Ext}(\overline{\text{co}}(\mathcal{C}_a))$. Theorem 3.4 then gives $C_\theta^* \in \mathcal{C}_a$, contradicting Lemma 4.9. Hence, $C_\theta^* \notin \overline{\text{co}}(\mathcal{C}_a)$, and the inclusion is strict. $\square$

Remark 5.2. The proof of Theorem 5.1 produces, for every $\theta \in]0, 1/2[$, a Schur-concave commutative copula in $\mathcal{S}_c \setminus \overline{\text{co}}(\mathcal{C}_a)$. By Remark 4.6, the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ depends continuously on θ in the uniform metric and connects the associative copulas M at $\theta \rightarrow 0^+$ and W at $\theta \rightarrow (1/2)^-$. Note that, despite both endpoints being in $\mathcal{C}_a$, every interior point of this curve lies strictly outside $\overline{\text{co}}(\mathcal{C}_a)$. Hence, $\mathcal{S}_c \setminus \overline{\text{co}}(\mathcal{C}_a)$ contains the interior of a topological arc whose endpoints belong to $\mathcal{C}_a$.

We now sharpen Theorem 5.1 by computing the exact uniform distance from C_θ^* to $\overline{\text{co}}(\mathcal{C}_a)$. The proof matches an upper bound, obtained by best approximation within the one-parameter family $\{(1 - c)M + cW\}_{c \in [0, 1]} \subseteq \overline{\text{co}}(\mathcal{C}_a)$, with a lower bound provided by a separating two-point continuous linear functional. The key to the lower bound is the following property of the antidiagonal sections of associative copulas, which may be of independent interest: the section $t \mapsto C(t, 1 - t)$ is star-shaped from above with respect to the origin.

Lemma 5.3. Let $C \in \mathcal{C}_a$, and let $F(t) = C(t, 1 - t)$ denote its antidiagonal section. Then the function $t \mapsto F(t)/t$ is non-increasing on $]0, 1/2[$. In particular,

$$C(t, 1 - t) \geq 2t C(1/2, 1/2), \quad t \in [0, 1/2]. \quad (8)$$

Proof. Consider the ordinal sum representation (2) of C . The open block intervals are pairwise disjoint, so at most one block $[a, b]^2 = [a_i, b_i]^2$ satisfies $a < 1/2 < b$. If no such block exists, then for $t < 1/2$ the point $(t, 1 - t)$ cannot lie in a common block: at $t = 1/2$, it is either off-block or a block endpoint, and $C(1/2, 1/2) = 1/2$. Thus $F(t) = t$ on $[0, 1/2]$, including the case of two blocks abutting at $1/2$, and the claim is trivial. Otherwise, only the block $[a, b]^2$ can contain an antidiagonal point, and this happens exactly for $t \in [t_0, 1/2]$, where $t_0 = \max(a, 1 - b)$. With $\ell = b - a$ and $A := A_i$, we have

$$F(t) = \begin{cases} t, & \text{if } t \in [0, t_0], \\ a + \ell g(t), & \text{if } t \in [t_0, 1/2], \end{cases} \quad \text{where } g(t) = A(x_t, y_t), \quad x_t = \frac{t - a}{\ell}, \quad y_t = \frac{1 - t - a}{\ell}.$$

Note that F is Lipschitz, being an antidiagonal section of the 1-Lipschitz function C , so F/t is locally Lipschitz on $]0, 1/2[$; on $]0, t_0]$, it is constant (equal to 1), and it therefore suffices to verify that $tF'(t) \leq F(t)$ at almost every $t \in]t_0, 1/2[$.

Let $\varphi: [0, 1] \rightarrow [0, \infty]$ be a convex additive generator of the Archimedean copula A , so that φ is convex and strictly decreasing, with $\varphi(1) = 0$, and $A(x, y) = \varphi^{(-1)}(\varphi(x) + \varphi(y))$; see [10] and [17, Chapter 4]. Recall that the one-sided derivatives of φ exist on $]0, 1[$, that $\varphi' \leq 0$ is non-decreasing (so that $|\varphi'|$ is non-increasing), and that φ' exists off a countable set.

First, consider the set $Z = \{t \in]t_0, 1/2[: g(t) = 0\}$. On Z , the Lipschitz function F takes the constant value a , so $F'(t) = 0$ at almost every $t \in Z$, and $tF'(t) = 0 \leq a \leq F(t)$ there.

Next, fix $t \in]t_0, 1/2[$ with $g := g(t) > 0$ such that $F'(t)$ exists and φ is differentiable at $x := x_t$, at $y = y_t$, and at g ; by the countability of the exceptional sets (and by discarding the null set of t for which $g(t)$ is a non-differentiability point of φ and $F'(t) \neq 0$), almost every t with $g(t) > 0$ and $F'(t) \neq 0$ is of this form, while points with $F'(t) = 0$ satisfy $tF'(t) = 0 \leq F(t)$ trivially. Since $g > 0$, we have $\varphi(g) = \varphi(x) + \varphi(y)$, and differentiating in t yields

$$F'(t) = \ell \frac{d}{dt} A(x_t, y_t) = \frac{\varphi'(x) - \varphi'(y)}{\varphi'(g)},$$

where $\varphi'(g) < 0$ because φ is convex and strictly decreasing. Multiplying the desired inequality $tF'(t) \leq F(t) = a + \ell g$ by $\varphi'(g) < 0$, we see that it is equivalent to

$$t(\varphi'(x) - \varphi'(y)) \geq (a + \ell g)\varphi'(g). \quad (9)$$

Note that $x \leq y$ (because $t \leq 1/2$) and $g = A(x, y) \leq \min(x, y) = x$. Using $t = a + \ell x$, the monotonicity of $|\varphi'|$ together with $g \leq x$, the bound

$$(x - g)|\varphi'(x)| \leq \int_g^x |\varphi'(\tau)| d\tau = \varphi(g) - \varphi(x) = \varphi(y),$$

and the supporting-line inequality $\varphi(y) \leq |\varphi'(y)|(1 - y)$, which follows from the convexity of φ and $\varphi(1) = 0$, we obtain:

$$\begin{aligned} t(\varphi'(x) - \varphi'(y)) - (a + \ell g)\varphi'(g) &= -t|\varphi'(x)| + t|\varphi'(y)| + (a + \ell g)|\varphi'(g)| \\ &\geq -t|\varphi'(x)| + t|\varphi'(y)| + (a + \ell g)|\varphi'(x)| \\ &= t|\varphi'(y)| - \ell(x - g)|\varphi'(x)| \\ &\geq t|\varphi'(y)| - \ell\varphi(y) \\ &\geq |\varphi'(y)|(t - \ell(1 - y)) \\ &= |\varphi'(y)|(1 - b) \\ &\geq 0, \end{aligned}$$

where we used $\ell(1 - y) = \ell - (1 - t - a) = t - (1 - b)$ and $b \leq 1$. This proves (9).

Altogether, the locally Lipschitz function F/t has a non-positive derivative at almost every point of $]t_0, 1/2[$ and is constant on $]0, t_0]$; hence it is non-increasing on $]0, 1/2]$. The inequality (8) follows by comparing the values at t and at $1/2$, and it extends to $t = 0$ trivially. $\square$

Remark 5.4. Lemma 5.3 is sharp, and star-shapedness cannot be strengthened to concavity of the antidiagonal section. Consider the Archimedean copula A whose generator φ is piecewise affine with $\varphi(0) = 3.1$ and slopes -10 , -2 , -1 on $[0, 0.2]$, $[0.2, 0.5]$, $[0.5, 1]$, respectively (being convex, φ generates a copula; see [17, Chapter 4]). A direct computation gives

$$F(t) = A(t, 1 - t) = \begin{cases} 0.9t, & \text{if } t \in [0, 0.2], \\ 0.16 + 0.1t, & \text{if } t \in [0.2, 0.4], \\ 0.5t, & \text{if } t \in [0.4, 0.5]. \end{cases}$$

The slope of F increases from 0.1 to 0.5 at $t = 0.4$, so F is not concave on $[0, 1/2]$; nevertheless, $F(t)/t$ is non-increasing, and (8) holds with equality on the whole interval $[0.4, 0.5]$. Note also that Lemma 5.3 yields an alternative proof of Lemma 4.9: by (7), the antidiagonal section of C_θ^* is $F_\theta(t) = \max(0, t - \theta)$ for $t \in [0, 1/2]$, so $F_\theta(\theta) = 0 < 2\theta(1/2 - \theta) = 2\theta F_\theta(1/2)$, violating (8).

Theorem 5.5. For every $\theta \in]0, 1/2[$,

$$d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) = \frac{\theta(1 - 2\theta)}{1 + 2\theta}, \quad (10)$$

and the infimum is attained by the explicit copula

$$D_\theta^* = \frac{1 - 2\theta}{1 + 2\theta} M + \frac{4\theta}{1 + 2\theta} W \in \overline{\text{co}}(\mathcal{C}_a). \quad (11)$$

Proof. We first establish the upper bound. For $c \in [0, 1]$, set $D_c = (1 - c)M + cW \in \overline{\text{co}}(\mathcal{C}_a)$ and $\Delta_c(u, v) = D_c(u, v) - C_\theta^*(u, v)$. We compute the supremum of $|\Delta_c|$ on $[0, 1]^2$ by using the closed form (7) and partitioning the unit square into three regions, according to which of the three candidates in (7) attains the maximum.

Region 1: $\min(u, v) \leq \theta$ and $u + v \leq 1$, where $C_\theta^* = 0$. Here $\Delta_c(u, v) = (1 - c)\min(u, v)$, with the supremum attained at $\min(u, v) = \theta$. Thus, $\sup \Delta_c = (1 - c)\theta$ on this region.

Region 2: $u, v \in [\theta, 1 - \theta]$, where $C_\theta^*(u, v) = \min(u, v) - \theta$. A direct case distinction based on the sign of $u + v - 1$ gives

$$\Delta_c(u, v) = \begin{cases} c(\max(u, v) - 1) + \theta, & \text{if } u + v \geq 1, \\ \theta - c\min(u, v), & \text{if } u + v < 1. \end{cases}$$

If $u + v \geq 1$, then $\max(u, v) \in [1/2, 1 - \theta]$; if $u + v < 1$, then $\min(u, v) \in [\theta, 1/2]$. In either case, $\Delta_c \in [\theta - c/2, \theta - c\theta]$. Hence,

$$\sup |\Delta_c| = \max\{|\theta - c/2|, \theta(1 - c)\}.$$

Region 3: $\max(u, v) \geq 1 - \theta$ and $u + v \geq 1$, where $C_\theta^*(u, v) = u + v - 1$. A direct calculation gives $\Delta_c(u, v) = (1 - c)(1 - \max(u, v)) \in [0, (1 - c)\theta]$, so $\sup |\Delta_c| = (1 - c)\theta$ on this region as well.

Combining the three regions, we obtain

$$d_\infty(C_\theta^*, D_c) = \max\{|\theta - c/2|, \theta(1 - c)\}. \quad (12)$$

We minimize the right-hand side over $c \in [0, 1]$. For $c \leq 2\theta$, $|\theta - c/2| = \theta - c/2 \leq \theta(1 - c)$, so $\sup |\Delta_c| = \theta(1 - c)$, which is decreasing in c . Its minimum on $[0, 2\theta]$ is therefore attained at $c = 2\theta$, with the value $\theta(1 - 2\theta)$. For $c > 2\theta$, $\sup |\Delta_c| = \max\{c/2 - \theta, \theta(1 - c)\}$. Note that the two candidates are equal at the unique solution

$$c^* = \frac{4\theta}{1 + 2\theta} \in [2\theta, 1],$$

of $c/2 - \theta = \theta(1 - c)$, where the common value is

$$\frac{c^*}{2} - \theta = \frac{2\theta}{1 + 2\theta} - \theta = \frac{\theta(1 - 2\theta)}{1 + 2\theta} = \theta(1 - c^*).$$

For $c < c^*$, the maximum equals $\theta(1 - c) > \theta(1 - c^*)$, and for $c > c^*$, it equals $c/2 - \theta > c^*/2 - \theta$. Hence, the global minimum of (12) on $[0, 1]$ is attained at $c = c^*$, with the value $\theta(1 - 2\theta)/(1 + 2\theta)$. Setting $D_\theta^* = D_{c^*}$ yields (11) and the upper bound

$$d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) \leq d_\infty(C_\theta^*, D_\theta^*) = \frac{\theta(1 - 2\theta)}{1 + 2\theta}.$$

For the lower bound, define a continuous linear functional $\psi: C([0, 1]^2) \rightarrow \mathbb{R}$ by

$$\psi(C) = C(1/2, 1/2) - \frac{1}{2\theta}C(\theta, 1 - \theta). \quad (13)$$

Note that both terms are point evaluations, hence continuous on $(\mathcal{C}, d_\infty)$, and the operator norm of ψ satisfies

$$\|\psi\|_{\text{op}} \leq 1 + \frac{1}{2\theta} = \frac{1 + 2\theta}{2\theta}.$$

Substituting $(u, v) = (1/2, 1/2)$ and $(u, v) = (\theta, 1 - \theta)$ into (7), we obtain

$$\psi(C_\theta^*) = (1/2 - \theta) - (1/(2\theta)) \cdot 0 = 1/2 - \theta.$$

Note that both points $(1/2, 1/2)$ and $(\theta, 1 - \theta)$ lie on the antidiagonal $u + v = 1$. By Lemma 5.3, every $C \in \mathcal{C}_a$ satisfies $C(\theta, 1 - \theta) \geq 2\theta C(1/2, 1/2)$, that is, $\psi(C) \leq 0$, with equality, for example, when $C = M$, whose antidiagonal section is $F(t) = t$. Hence,

$$\sup_{C \in \mathcal{C}_a} \psi(C) = 0 = \sup_{D \in \overline{\text{co}}(\mathcal{C}_a)} \psi(D),$$

where the second equality follows from linearity on finite convex combinations and continuity under uniform limits. Combining this with the bound on $\|\psi\|_{\text{op}}$, we obtain

$$d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) \geq \frac{\psi(C_\theta^*) - \sup_{D \in \overline{\text{co}}(\mathcal{C}_a)} \psi(D)}{\|\psi\|_{\text{op}}} \geq \frac{1/2 - \theta}{(1 + 2\theta)/(2\theta)} = \frac{\theta(1 - 2\theta)}{1 + 2\theta}.$$

The upper and lower bounds match, completing the proof. $\square$

Corollary 5.6. *The function $\theta \mapsto d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) = \theta(1 - 2\theta)/(1 + 2\theta)$ on $]0, 1/2[$ attains its supremum uniquely at $\theta^* = (\sqrt{2} - 1)/2 \approx 0.2071$, with the maximum value*

$$d_\infty(C_{\theta^*}^*, \overline{\text{co}}(\mathcal{C}_a)) = \frac{3}{2} - \sqrt{2} \approx 0.0858.$$

The corresponding nearest copula is given by $D_{\theta^*}^* = (\sqrt{2} - 1)M + (2 - \sqrt{2})W$.

Proof. Differentiating $f(\theta) = \theta(1 - 2\theta)/(1 + 2\theta)$ yields $f'(\theta) = (1 - 4\theta - 4\theta^2)/(1 + 2\theta)^2$. The derivative is positive before and negative after the unique root $\theta^* = (\sqrt{2} - 1)/2$ of $1 - 4\theta - 4\theta^2 = 0$ in $]0, 1/2[$, proving uniqueness of the maximizer. At this point, $1 - 2\theta^* = 2 - \sqrt{2}$ and $1 + 2\theta^* = \sqrt{2}$. Then,

$$f(\theta^*) = \frac{3}{2} - \sqrt{2}.$$

The optimal mixing parameter is $c^* = 4\theta^*/(1 + 2\theta^*) = 2 - \sqrt{2}$, which gives $D_{\theta^*}^* = (\sqrt{2} - 1)M + (2 - \sqrt{2})W$. $\square$

Remark 5.7. *The distance function $\theta \mapsto \theta(1 - 2\theta)/(1 + 2\theta)$ extends continuously to the closed interval $[0, 1/2]$ and vanishes at both endpoints, in accordance with Remark 4.6: as $\theta \rightarrow 0^+$ and $\theta \rightarrow (1/2)^-$, the copulas C_θ^* converge uniformly to M and W , respectively, and both limits belong to $\mathcal{C}_a \subseteq \overline{\text{co}}(\mathcal{C}_a)$. Near the endpoints the decay is linear,*

$$\begin{aligned} d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) &= \theta + O(\theta^2) \quad \text{as } \theta \rightarrow 0^+, \\ d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) &= \frac{1/2 - \theta}{2} + O((1/2 - \theta)^2) \quad \text{as } \theta \rightarrow (1/2)^-. \end{aligned}$$

Hence, along the closed arc $\{C_\theta^*\}_{\theta \in [0, 1/2]}$, membership in $\overline{\text{co}}(\mathcal{C}_a)$ is discontinuous (the endpoints belong to $\overline{\text{co}}(\mathcal{C}_a)$, while all interior points do not), whereas the distance to $\overline{\text{co}}(\mathcal{C}_a)$ varies continuously. Extremality in $\mathcal{C}$, on the other hand, persists along the whole closed arc, since $M, W \in \mathcal{C}_a^W \subseteq \text{Ext}(\mathcal{C})$ by Lemma 3.3.

Example 5.8. *We tabulate selected values of the distance formula in Table 2.*

θ	$d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a))$	c^*	D_θ^*
1/4	1/12	2/3	$(1/3)M + (2/3)W$
$(\sqrt{2} - 1)/2$	$3/2 - \sqrt{2}$	$2 - \sqrt{2}$	$(\sqrt{2} - 1)M + (2 - \sqrt{2})W$
1/3	1/15	4/5	$(1/5)M + (4/5)W$
2/5	2/45	8/9	$(1/9)M + (8/9)W$

Table 2: Selected values of the distance $d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a))$ from (10), together with the optimal mixing parameter c^* and the corresponding nearest copula D_θ^* from (11).

The geometry established in this section is summarized in Figure 3. The family $\{C_\theta^*\}_{\theta \in [0, 1/2]}$ is the interior of a topological arc with endpoints $M, W \in \mathcal{C}_a$. Every point of this interior lies outside $\overline{\text{co}}(\mathcal{C}_a)$, at the exactly known distance (10); the left panel of Figure 3 shows the profile of this distance along the arc, and the right panel shows the resulting picture of the inclusions $\mathcal{C}_a \subseteq \overline{\text{co}}(\mathcal{C}_a) \subsetneq \mathcal{S}_c \subseteq \mathcal{C}$.

Remark 5.9. *By Corollary 5.6, the family $\{C_\theta^*\}_{\theta \in [0, 1/2]} \subseteq \mathcal{S}_c \setminus \overline{\text{co}}(\mathcal{C}_a)$ from Remark 5.2 lies in a uniform shell of width at most $3/2 - \sqrt{2} \approx 0.086$ around $\overline{\text{co}}(\mathcal{C}_a)$. Our family is a proper subset of $\mathcal{S}_c \setminus \overline{\text{co}}(\mathcal{C}_a)$. Indeed, for fixed $\theta \in]0, 1/2[$ and $\alpha \in]0, 1[$, set $H_{\alpha, \theta} = \alpha C_\theta^* + (1 - \alpha)\Pi \in \mathcal{S}_c$. The separating functional (13) satisfies*

$$\psi(H_{\alpha, \theta}) = \alpha(1/2 - \theta) - (1 - \alpha)(1 - 2\theta)/4 = \frac{1 - 2\theta}{4}(3\alpha - 1).$$

Hence $H_{\alpha, \theta} \notin \overline{\text{co}}(\mathcal{C}_a)$ whenever $\alpha > 1/3$. Its measure has a non-zero absolutely continuous component of mass $1 - \alpha$, whereas every C_θ^* is singular, proving properness. These mixtures have both singular and absolutely continuous components; they are not themselves absolutely continuous. The value of $\sup_{C \in \mathcal{S}_c} d_\infty(C, \overline{\text{co}}(\mathcal{C}_a))$ remains unknown and may be strictly larger than $3/2 - \sqrt{2}$. A universal upper bound is, however, easy to obtain: since $W \leq C \leq M$ pointwise for every $C \in \mathcal{C}$, and since $\frac{1}{2}(M + W) \in \overline{\text{co}}(\mathcal{C}_a)$, every copula C satisfies

$$d_\infty(C, \overline{\text{co}}(\mathcal{C}_a)) \leq \frac{1}{2} \sup_{(u, v) \in [0, 1]^2} (M(u, v) - W(u, v)) = \frac{1}{4}.$$

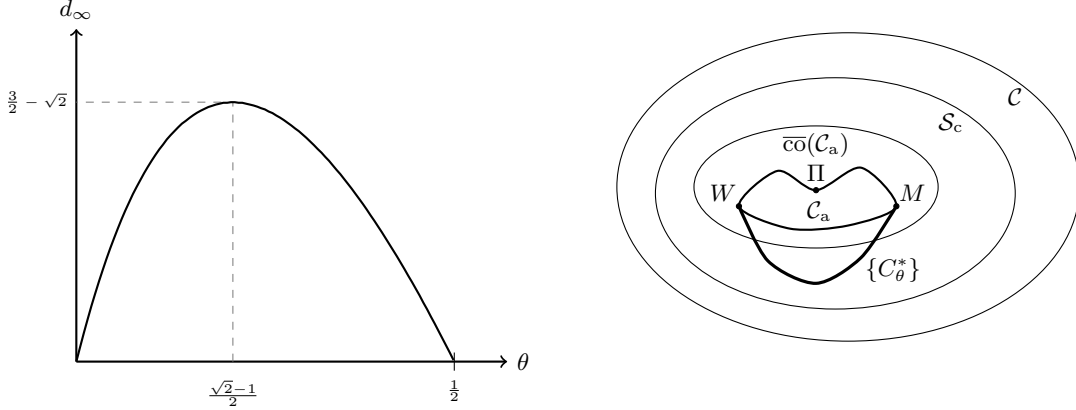


Figure 3: Left: the distance $d_\infty(C_\theta^*, \overline{\text{co}}(\mathcal{C}_a)) = \theta(1-2\theta)/(1+2\theta)$ as a function of θ , with the maximum $3/2 - \sqrt{2}$ attained at $\theta = (\sqrt{2} - 1)/2$ (Corollary 5.6). Right: a schematic (not to scale) view of the inclusions $\mathcal{C}_a \subseteq \overline{\text{co}}(\mathcal{C}_a) \subsetneq \mathcal{S}_c \subseteq \mathcal{C}$. The family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ is the interior of an arc with endpoints M and W ; this interior lies in $\mathcal{S}_c \setminus \overline{\text{co}}(\mathcal{C}_a)$; the independence copula Π lies in $\mathcal{C}_a$, and, by Proposition 3.10, even in $\overline{\text{co}}(\mathcal{C}_a^W)$.

Consequently, $3/2 - \sqrt{2} \leq \sup_{C \in \mathcal{S}_c} d_\infty(C, \overline{\text{co}}(\mathcal{C}_a)) \leq 1/4$. We also note that the nearest copula D_θ^* of (11) is a convex combination of M and W , hence an element of $\overline{\text{co}}(\mathcal{C}_a^W)$; since $\overline{\text{co}}(\mathcal{C}_a^W) \subseteq \overline{\text{co}}(\mathcal{C}_a)$, the distance formula (10) remains valid verbatim with $\overline{\text{co}}(\mathcal{C}_a)$ replaced by $\overline{\text{co}}(\mathcal{C}_a^W)$.

6 Concluding remarks

In this paper, we have resolved Open Problem 2.5 by constructing, for every $\theta \in]0, 1/2[$, an explicit Schur-concave commutative copula C_θ^* outside $\overline{\text{co}}(\mathcal{C}_a)$. The closed-form expression $C_\theta^*(u, v) = \max\{0, u + v - 1, \min(u, v) - \theta\}$ allowed us to verify Schur-concavity directly from the majorization definition and to compute the exact uniform distance from C_θ^* to $\overline{\text{co}}(\mathcal{C}_a)$, namely $\theta(1 - 2\theta)/(1 + 2\theta)$; the key ingredient in the lower bound is the star-shapedness of the antidiagonal sections of associative copulas established in Lemma 5.3. As a consequence, the maximum distance between an element of our family and $\overline{\text{co}}(\mathcal{C}_a)$ equals $3/2 - \sqrt{2}$ and is attained at $\theta = (\sqrt{2} - 1)/2$. For Open Problem 2.6, we have established the unconditional inclusion $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$, and we have reformulated the second part of the problem as the equality $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) = \mathcal{E}_a$, which would imply $\overline{\text{co}}(\mathcal{E}_a) = \overline{\text{co}}(\mathcal{C}_a)$. Lemma 3.3 shows that $\mathcal{C}_a^W \subsetneq \mathcal{E}_a$ and $\overline{\text{co}}(\mathcal{C}_a^W) \subsetneq \overline{\text{co}}(\mathcal{C}_a)$. Nevertheless, the integral representation $\Pi = \int_0^1 C_v dv$ of Proposition 3.10 shows that $\Pi \in \overline{\text{co}}(\mathcal{C}_a^W)$, and that the inclusion $\text{Ext}(\overline{\text{co}}(\mathcal{C}_a)) \subseteq \mathcal{C}_a$ is strict. We close with a discussion of applications, a comment on higher dimensions, and several open subquestions whose resolution would either complete Open Problem 2.6 or further sharpen the geometry of $\overline{\text{co}}(\mathcal{C}_a)$.

Applications. Beyond settling Open Problem 2.5, the family $\{C_\theta^*\}_{\theta \in]0, 1/2[}$ admits a direct probabilistic interpretation that may be of independent modelling interest. If U is uniform on $[0, 1]$ and $V = f_\theta(U)$, then, by construction, the pair (U, V) has joint distribution function C_θ^* : the dependence is countermonotone in the two tail regions $U \in [0, \theta] \cup [1 - \theta, 1]$ and comonotone in the central region $U \in [\theta, 1 - \theta]$. Dependence structures of this regime-switching type arise, for instance, in quantitative risk management, where the joint behaviour of risk factors under stress differs qualitatively from their behaviour in the central range; see [8, 17] for general background on copula-based dependence modelling. In this context, the distance formula (10) yields an explicit lower bound on the approximation error incurred by any model built from associative copulas: since Archimedean copulas and their ordinal sums exhaust $\mathcal{C}_a$, every $D \in \overline{\text{co}}(\mathcal{C}_a)$ (in particular, every finite mixture of Archimedean copulas) misestimates at least one joint probability $C_\theta^*(u, v) = P(U \leq u, V \leq v)$ by at least $\theta(1 - 2\theta)/(1 + 2\theta)$. For $\theta = (\sqrt{2} - 1)/2$, this error is at least $3/2 - \sqrt{2} \approx 0.086$, that is, more than eight percentage points, uniformly over all such models.

Higher dimensions. All results of this paper are formulated for 2-copulas, which is the setting of Open Problems 2.5 and 2.6. Let us briefly indicate the obstacles to a d -dimensional extension for $d \geq 3$. First, the lower Fréchet-Hoeffding bound $W_d(u_1, \dots, u_d) = \max(0, u_1 + \dots + u_d - d + 1)$ is not a copula for $d \geq 3$ [17], so the class $\mathcal{C}_a^W$ and the antidiagonal supports that underlie Sections 3-5 have no direct analogue. Second, the construction of C_θ^* rests on the measure-preserving involution f_θ of $[0, 1]$; copulas concentrated on graphs $x \mapsto (x, g_2(x), \dots, g_d(x))$ of measure-preserving bijections still exist in higher dimensions, but the commutativity of the resulting copula amounts to the full

exchangeability of the underlying measure, which imposes severe compatibility constraints on $g_2, \dots, g_d$. Third, the separating functional (13) evaluates copulas at two points on the antidiagonal $u + v = 1$, and its analysis in Lemma 5.3 rests on the Mostert-Shields ordinal sum representation (2) of associative binary operations; for $d \geq 3$, associativity can be imposed directly on the d -copula in the sense of Post, as in [14, Section 2.1 and Problem 2.1]. The binary representation used here does not by itself supply the corresponding higher-dimensional structure. A d -dimensional analogue of Open Problem 2.5 would also have to specify the appropriate counterparts of $\mathcal{C}_a$ and $\mathcal{S}_c$; for convexity and concavity properties available in the multivariate setting, we refer to [4]. We leave this direction for future work.

Open Question 1. Can a non-trivial convex combination of two distinct associative copulas be associative?

This is the weaker question discussed in Remark 3.7. Corollary 3.11 expresses the independence copula as a non-trivial convex combination of two distinct copulas in $\overline{\text{co}}(\mathcal{C}_a^W)$, but those two copulas are not associative. Natural test cases remain the strict Archimedean copulas of Frank, Clayton, and Gumbel.

Open Question 2. Can every associative copula be approximated uniformly by finite convex combinations of associative copulas that are extreme among all copulas?

In our notation, the question is whether $\overline{\text{co}}(\mathcal{E}_a) = \overline{\text{co}}(\mathcal{C}_a)$. By Remark 3.12, it is enough to consider Archimedean copulas. An affirmative answer to the second part of Open Problem 2.6 would imply this equality; the converse is not established here. Lemma 3.3 shows that replacing $\mathcal{E}_a$ by $\mathcal{C}_a^W$ makes the answer negative.

Finite-grid computations suggest that some familiar Archimedean copulas nevertheless belong to $\overline{\text{co}}(\mathcal{C}_a^W)$. We considered the 75 025 ordinal sums of W whose blocks have endpoints in the grid $\{0, 1/12, \dots, 1\}$. A linear program minimizing the maximal deviation at the grid points reproduces the values of the Clayton copula $uv/(u+v-uv)$ and of the Gumbel copula with parameter 2 to within a solver tolerance of 10^{-9} . The same holds on all coarser grids. These finite computations do not establish membership in the closed convex hull. By the 1-Lipschitz property of copulas, an exact match on $\{0, 1/n, \dots, 1\}^2$ for every n would suffice. As a consistency check, the same linear program applied to $\mathcal{C}_{1/4}^*$ returns the distance $1/12$, according to (10) and the final remark of Section 5.

Open Question 3. Does every member of the EFGM family belong to $\overline{\text{co}}(\mathcal{C}_a)$?

Recall that

$$\text{EFGM}_\theta(u, v) = uv + \theta uv(1-u)(1-v), \quad \theta \in [-1, 1].$$

All these copulas are Schur-concave and commutative. The case $\theta = 0$ corresponds to the independence copula, which belongs to $\overline{\text{co}}(\mathcal{C}_a^W)$. For every parameter value, we have

$$d_\infty(\text{EFGM}_\theta, \overline{\text{co}}(\mathcal{C}_a)) \leq d_\infty(\text{EFGM}_\theta, \Pi) = \frac{|\theta|}{16} \leq \frac{1}{16} < \frac{3}{2} - \sqrt{2}.$$

Thus no member is as far from $\overline{\text{co}}(\mathcal{C}_a)$ as the most distant member of our shuffle family.

The linear program in Open Question 2 also reproduces the grid values of EFGM_1 to within the solver tolerance. This motivates testing the stronger conjecture that the entire EFGM family belongs to $\overline{\text{co}}(\mathcal{C}_a^W)$. A member outside $\overline{\text{co}}(\mathcal{C}_a)$ would provide a Schur-concave commutative copula whose measure is entirely absolutely continuous, unlike the mixtures in the final remark of Section 5, which retain a singular component. If all members belong to $\overline{\text{co}}(\mathcal{C}_a)$, then the identity

$$\Pi = \frac{1}{2}\text{EFGM}_\varepsilon + \frac{1}{2}\text{EFGM}_{-\varepsilon},$$

for $0 < |\varepsilon| < 1$ gives another non-trivial decomposition of Π within $\overline{\text{co}}(\mathcal{C}_a)$, this time into two copulas with polynomial densities.

Open Question 4. Can the three-segment construction C_θ^* be extended to shuffles with more segments while retaining Schur-concavity and an explicit distance formula?

Such an extension would give a richer family of Schur-concave commutative copulas at a controlled distance from $\overline{\text{co}}(\mathcal{C}_a)$. For the present family, what is the exact distance $d_\infty(C_\theta^*, \mathcal{C}_a)$ to the associative copulas themselves? The distance formula (10) yield a lower bound. For an upper bound, equation(7) gives $d_\infty(C_\theta^*, M) = \theta$ and $d_\infty(C_\theta^*, W) = 1/2 - \theta$, with the respective deviations attained at $(\theta, 1 - \theta)$ and $(1/2, 1/2)$. Since $M, W \in \mathcal{C}_a$, we obtain

$$\frac{\theta(1-2\theta)}{1+2\theta} \leq d_\infty(C_\theta^*, \mathcal{C}_a) \leq \min \left\{ \theta, \frac{1}{2} - \theta \right\}.$$

The exact value and the parameter that maximizes it remain unknown.

Acknowledgement

ADAM ŠELIGA acknowledges the support of the VEGA 1/0036/23 grant.

References

- [1] C. Alsina, *On Schur-concave t -norms and triangle functions*, General Inequalities 4, **71** of Internat. Schriftenreihe Numer. Math., Birkhäuser, Basel, (1984), 241-248. https://doi.org/10.1007/978-3-0348-6259-2_22
- [2] C. Alsina, M. J. Frank, B. Schweizer, *Associative functions: Triangular norms and copulas*, World Scientific, Hackensack, NJ, (2006). <https://doi.org/10.1142/6036>
- [3] J. B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, Springer, New York, **96** (2007). <https://doi.org/10.1007/978-1-4757-4383-8>
- [4] A. Dolati, A. Dehghan Nezhad, *Some results on convexity and concavity of multivariate copulas*, Iranian Journal of Mathematical Sciences and Informatics, **9**(2) (2014), 87-100. <https://doi.org/10.7508/ijmsi.2014.02.008>
- [5] R. G. Douglas, *On extremal measures and subspace density*, Michigan Mathematical Journal, **11**(3) (1964), 243-246. <https://doi.org/10.1307/mmj/1028999137>
- [6] F. Durante, P. Sarkoci, C. Sempi, *Shuffles of copulas*, Journal of Mathematical Analysis and Applications, **352**(2) (2009), 914-921. <https://doi.org/10.1016/j.jmaa.2008.11.064>
- [7] F. Durante, C. Sempi, *Copulae and Schur-concavity*, International Mathematical Journal, **3** (2003), 893-905.
- [8] F. Durante, C. Sempi, *Principles of copula theory*, CRC Press, Boca Raton, FL, Chapman and Hall/CRC, 2015. <https://doi.org/10.1201/b18674>
- [9] K. Hestir, S. C. Williams, *Supports of doubly stochastic measures*, Bernoulli, **1**(3) (1995), 217-243. <https://doi.org/10.3150/bj/1193667816>
- [10] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000. <https://doi.org/10.1007/978-94-015-9540-7>
- [11] E. P. Klement, R. Mesiar, E. Pap, *Uniform approximation of associative copulas by strict and non-strict copulas*, Illinois Journal of Mathematics, **45**(4) (2001), 1393-1400. <https://doi.org/10.1215/ijm/1258138075>
- [12] J. Lindenstrauss, *A remark on extreme doubly stochastic measures*, The American Mathematical Monthly, **72**(4) (1965), 379-382. <https://doi.org/10.2307/2313497>
- [13] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of majorization and its applications*, 2nd ed., Springer Series in Statistics, Springer, New York, 2011. <https://doi.org/10.1007/978-0-387-68276-1>
- [14] R. Mesiar, P. Sarkoci, *Open problems posed at the tenth International Conference on Fuzzy Set Theory and Applications (FSTA 2010, Liptovský Ján, Slovakia)*, Kybernetika, **46**(4) (2010), 585-599.
- [15] P. Mikusiński, H. Sherwood, M. D. Taylor, *Shuffles of $\min$* , Stochastica, **13**(1) (1992), 61-74.
- [16] P. S. Mostert, A. L. Shields, *On the structure of semigroups on a compact manifold with boundary*, Annals of Mathematics, **65**(1) (1957), 117-143. <https://doi.org/10.2307/1969668>
- [17] R. B. Nelsen, *An introduction to copulas*, 2nd ed., Springer Series in Statistics, Springer, New York, 2006. <https://doi.org/10.1007/0-387-28678-0>
- [18] B. V. Popović, M. M. Ristić, K. Zografos, *Integral transformation of a copula function*, Communications in Statistics - Theory and Methods, **52**(18) (2023), 6341-6354. <https://doi.org/10.1080/03610926.2022.2028837>
- [19] H. L. Royden, *Real analysis*, 3rd ed., Macmillan, New York, 1988.
- [20] M. Sklar, *Fonctions de répartition à n dimensions et leurs marges*, Publ. Inst. Statist. Univ. Paris, **8**(3) (1959), 229-231.