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Original paper

Quasi-inverses of monotone functions on complete trellises and their use in the construction of t-norms

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Abstract

This paper focuses on introducing and studying quasi-inverses of monotone functions on complete trellises, and constructing t-norms on complete trellises using monotone functions and their quasi-inverses. We begin by introducing the concepts of quasi-inverse and pseudo-inverse for monotone functions on complete trellises, thus generalizing the corresponding notions for monotone functions on real intervals. The proposed notion of pseudo-inverse also integrates the two distinct concepts of pseudo-inverse on complete lattices into a unified framework. We investigate the monotonicity of quasi-inverses for specialized functions such as order-embeddings and complete inf-homomorphisms. Based on this, we present methods for constructing t-norms on complete trellises using monotone functions and their quasi-inverses. When specialized to complete lattices, our construction subsumes and extends those known results on lattices obtained by analogous methods. In addition, we provide the multiplicative and additive generators for t-norms on complete trellises. Finally, we discuss the Archimedean property of t-norms on complete trellises constructed from injective increasing functions and their quasi-inverses.

Keywords: t-norm, quasi-inverse, order-embedding, complete inf-homomorphism, complete trellis.

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1 Introduction

The theoretical origins of triangular norms can be traced back to 1942, when Menger [13] first introduced related concepts in order to generalize the triangle inequality in statistical metric spaces. Subsequently, Schweizer and Sklar [18] provided an axiomatic definition of triangular norms on the real unit interval $[0, 1]$. T-norms and their dual t-conorms play a fundamental role in various disciplines, including fuzzy set theory [2], multicriteria decision support [8], and probabilistic metric spaces [19]. The study of t-norms on the real unit interval [0,1] has been extensively developed, with research focusing on their construction, characterization, representation, and properties such as distributivity. Latter, this framework has been extended to bounded lattices, forming a significant and active research area in recent years [3, 6, 21].

A significant method for constructing t-norms on [0,1] is the additive and multiplicative generators theorem [16]. This problem can be traced back to Abel [1], who proposed constructing binary operators with desirable algebraic properties using a unary function together with addition or multiplication. Additive generators provide a representation for continuous Archimedean t-norms on [0,1]: a t-norm is continuous Archimedean if and only if it has a continuous additive generator. Dimuro et al. [4] introduced the concepts of interval additive generators of interval t-norms and interval t-conorms. Mayor et al. [12] and Viceník [23, 24] investigated discrete additive generators of discrete t-norms. In 2021, Sun and Liu [22] extended the concepts of the pseudo-inverse of monotone functions and additive generators of t-norms to bounded lattices. In 2024, Dong et al. [5] introduced another concept of the pseudo-inverse for monotone

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functions on complete lattices and constructed t-norms on complete lattices using monotone functions and their quasi-inverses. It is noteworthy that whether on [0,1], bounded lattices, or complete lattices, the construction of t-norms using a monotone function f and its pseudo-inverse $f^{(-1)}$ requires the condition $f \circ f^{(-1)} |_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$ to be satisfied. In 1999, Klement et al. [15] introduced the concept of the quasi-inverse of monotone functions on the real interval. From this concept, it follows that on the real interval, if the pseudo-inverse $f^{(-1)}$ of a monotone function f satisfies $f \circ f^{(-1)} |_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$, then $f^{(-1)}$ is also a quasi-inverse of f. This naturally raises a question: how should pseudo-inverses and quasi-inverses be properly defined on more general order structures? The definition should serve two purposes. First, it should allow some t-norms on order structures to be constructed using monotone functions and their quasi-inverses under weaker conditions. Second, the pseudo-inverses under this definition should unify the two existing notions of pseudo-inverses on complete lattices.

In many practical applications, order relations often fail to satisfy transitivity. For instance, in complex decision-making analysis, individual preferences frequently lack perfect transitivity and may exhibit cyclical preferences. In such cases, models based on non-transitive relations can more accurately capture these intransitive preference structures. These applications demonstrate that when studying relational structures that are more complex and realistic, relaxing the strict requirement of transitivity can lead to more precise and effective mathematical models. This necessity compels us to further study t-norms within more general structures. A pseudo-ordered set (psoset, for short) [9] is a generalization of a partially ordered set, characterized by a binary relation that satisfies reflexivity and antisymmetry but not necessarily transitivity. This leads to two phenomena distinct from those in partially ordered sets: cycles and incomparability. A cycle occurs when, for example, M is better than N, N is better than O, and O is better than M. Two elements are incomparable if neither is better than the other. Due to the absence of transitivity, a further interesting situation may arise: M is better than N and N is better than O, yet M and O are incomparable. Cycles play significant roles in various real-world contexts, such as in tournaments designed to prevent extinction and support biodiversity [14, 17]. Incomparability is another critical aspect, exemplified by the intransitivity of indifference [7]. In 2023, Zedam and De Baets [25] introduced the framework of t-norms on bounded psosets and developed constructive methods for t-norms on bounded trellises. Their work significantly extended the theory of aggregation operators to psosets, stimulating considerable scholarly interest. Subsequently, Zhao and Liu [26] introduced novel constructions of t-norms and t-conorms on bounded trellises. Geng et al. [10] studied ordinal sums of t-norms on a bounded trellis. Beyond these approaches, the fundamental problem of constructing t-norms via monotone functions and their quasi-inverses on psosets represents another vital methodological direction. Inspired by these, we introduces and studies quasi-inverses and pseudo-inverses of monotone functions on complete trellises, and this notion of pseudo-inverse unifies the two existing notions. Building on this, we further provide construction methods for t-norms on complete trellises using monotone functions and their quasi-inverses, which generalize the results in [5] under weaker conditions.

The paper is structured as follows. In section 2, we review some of the basic concepts. In section 3, we introduce the definitions of quasi-inverse and pseudo-inverse for monotone functions on complete trellises. We examine the monotonicity of quasi-inverses associated with particular families of monotone functions. In section 4, we present several methods for constructing t-norms on complete trellises via monotone functions and their quasi-inverses, and introduce both additive and multiplicative generators of t-norms on complete trellises. A comparative analysis with related conclusions on bounded lattices is provided. To facilitate understanding, concrete examples are provided. We also discuss the Archimedean property of t-norms on complete trellises constructed using an injective increasing function f, its quasi-inverse f^*, and a binary operator F. Finally, some conclusions are given.

2 Preliminaries

This section serves to recall relevant concepts and introduce some results on psosets.

Definition 2.1. [9] Let P be a set. A pseudo-order on P is a binary relation $\trianglelefteq$ on P satisfying:

(i) $i \trianglelefteq i$ for any $i \in P$ (reflexive);

(ii) $i \trianglelefteq t$ and $t \trianglelefteq i$ implies $i = t$ for any $i, t \in P$ (antisymmetric).

A set P equipped with a pseudo-order $\trianglelefteq$ is said to be a psoset and is denoted by $(P, \trianglelefteq)$.

Let $(P, \trianglelefteq)$ be a psoset, $A \subseteq P$ and $s, w \in P$. We now introduce some frequently used notations. $\downarrow s = \{x \in P \mid x \trianglelefteq s\}$. $\downarrow A = \{x \in P \mid \exists a \in A,\ x \trianglelefteq a\}$. $\uparrow s = \{x \in P \mid s \trianglelefteq x\}$. $\uparrow A = \{x \in P \mid \exists a \in A,\ a \trianglelefteq x\}$. $s \triangleleft w$ means that $s \trianglelefteq w$ and $s \neq w$. We write $s \ntrianglelefteq w$ to indicate that $s \trianglelefteq w$ does not hold. s and w are said to be incomparable, denoted by $s \parallel w$, whenever $s \ntrianglelefteq w$ and $w \ntrianglelefteq s$. $s \lesssim w$ means that there exists a finite sequence $(i_1, \ldots, i_n)$ of elements from P such that $s \trianglelefteq i_1 \trianglelefteq \cdots \trianglelefteq i_n \trianglelefteq w$. The notions of minimal/maximal element, smallest/greatest element, lower/upper

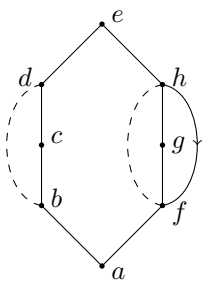


Fig. 1. Hasse diagram of the psoset

Table 1: Meet and join operations in Fig. 1

$\wedge$	a	b	c	d	f	g	h	e
a	a	a	a	a	a	a	a	a
b	a	b	b	a	a	a	a	b
c	a	b	c	c	a	a	a	c
d	a	a	c	d	a	a	a	d
f	a	a	a	a	f	f	h	f
g	a	a	a	a	f	g	g	g
h	a	a	a	a	h	g	h	h
e	a	b	c	d	f	g	h	e

$\vee$	a	b	c	d	f	g	h	e
a	a	b	c	d	f	g	h	e
b	b	b	c	e	e	e	e	e
c	c	c	c	d	e	e	e	e
d	d	e	d	d	e	e	e	e
f	f	e	e	e	f	g	f	e
g	g	e	e	e	g	g	h	e
h	h	e	e	e	f	h	h	e
e	e	e	e	e	e	e	e	e

bound, greatest lower bound (or infimum), smallest upper bound (or supremum) for a psoset are defined in the same way as the corresponding notions for a poset. Similarly as for posets, a finite psoset can be represented by a Hasse diagram with the following difference: if s and w are not related, while in a partially ordered set this would be implied by transitivity, then s and w are joined by a dashed edge. If $s \lesssim w$ and $w \trianglelefteq s$, then s and w are joined by a directed edge going from w to s. [20, 25].

Since a pseudo-order is not required to satisfy transitivity, two special phenomena may arise, as illustrated in Fig. 1. Fig. 1 illustrates two special cases: $b \trianglelefteq c$, $c \trianglelefteq d$ together with $b \parallel d$, and the case $f \trianglelefteq g$, $g \trianglelefteq h$, and $h \trianglelefteq f$. These two situations cannot occur under a partial order. To illustrate the meet and join operations on psosets, Table 1 provides the corresponding results for the example in Fig. 1.

Definition 2.2. [11] A $\wedge$-semi-trellis (resp. $\vee$-semi-trellis) is a psoset P such that $s \wedge w$ (resp. $s \vee w$) exists, for any $s, w \in P$. A trellis is a psoset that is both a $\wedge$-semi-trellis and a $\vee$-semi-trellis; it is denoted by $(P, \trianglelefteq, \wedge, \vee)$. It is clear that $s \trianglelefteq w$ is defined as $s \wedge w = s$ ($s \vee w = w$) for a trellis P.

A bounded trellis P is a trellis that has a smallest element denoted by 0_P and a greatest element denoted by 1_P and we write $(P, \trianglelefteq, \wedge, \vee, 0_P, 1_P)$. In a bounded trellis P, it holds that $\bigvee \emptyset = 0_P$ and $\bigwedge \emptyset = 1_P$.

Definition 2.3. [20] A complete trellis is a trellis $(P, \trianglelefteq, \wedge, \vee)$ such that $\bigvee R$ and $\bigwedge R$ exist for all $R \subseteq P$.

The complete trellises considered in this paper contain at least three elements.

Definition 2.4. [25] Let P be a psoset. A binary function R on P is called:

(i) commutative, if $R(s, w) = R(w, s)$ for all $s, w \in P$;

(ii) associative, if $R(s, R(w, t)) = R(R(s, w), t)$ for all $s, w, t \in P$;

(iii) increasing, if $R(s_1, w_1) \trianglelefteq R(s_2, w_2)$ for all s_1, s_2, w_1, $w_2 \in P$ with $s_1 \trianglelefteq s_2$ and $w_1 \trianglelefteq w_2$.

Definition 2.5. Let P, Q be psosets. An unary function $f : P \to Q$ is called increasing, if $s_1 \trianglelefteq s_2$ in P implies $f(s_1) \trianglelefteq f(s_2)$ in Q. An unary function $f : P \to Q$ is called decreasing, if $s_1 \trianglelefteq s_2$ in P implies $f(s_2) \trianglelefteq f(s_1)$ in Q.

Definition 2.6. Let R and S be psosets. A function $f : R \to S$ is called an order-embedding if $r_1 \trianglelefteq r_2$ in R if and only if $f(r_1) \trianglelefteq f(r_2)$ in S. A function $f : R \to S$ is called an order-reversing embedding if $r_1 \trianglelefteq r_2$ in R if and only if $f(r_2) \trianglelefteq f(r_1)$ in S.

From the above definitions, it follows that if f is an order-embedding, then it is increasing; if f is an order-reversing embedding, then it is decreasing. Since the orders on psosets satisfy reflexivity, then the order-embedding (order-reversing embedding) defined above is also injective. In fact, if $f : R \to S$ is an order-embedding and there exist $r_1, r_2 \in R$ such that $f(r_1) = f(r_2)$, then $f(r_1) \trianglelefteq f(r_2)$ and $f(r_2) \trianglelefteq f(r_1)$. Since f is an order-embedding, we can obtain $r_1 \trianglelefteq r_2$ and $r_2 \trianglelefteq r_1$. Thus, $r_1 = r_2$ and f is injective.

Definition 2.7. Let R and S be complete trellises.

(i) A function $f : R \to S$ is called a complete inf-homomorphism if $f(\bigwedge M) = \bigwedge\{f(m) \mid m \in M\}$ for any nonempty subset M of R.

(ii) A function $f : R \to S$ is called a complete sup-homomorphism if $f(\bigvee M) = \bigvee\{f(m) \mid m \in M\}$ for any nonempty subset M of R.

(iii) A function $f : R \to S$ is called a complete sup-inf-homomorphism if $f(\bigvee M) = \bigwedge\{f(m) \mid m \in M\}$ for any nonempty subset M of R.

(iv) A function $f : R \to S$ is called a complete inf-sup-homomorphism if $f(\bigwedge M) = \bigvee\{f(m) \mid m \in M\}$ for any nonempty subset M of R.

Let R and S be complete trellises, and $f : R \longrightarrow S$ be a unary function. It is easy to verify that if f is an injective complete inf-homomorphism (injective complete sup-homomorphism), then f is an order embedding. If f is an injective complete sup-inf-homomorphism (injective complete inf-sup-homomorphism), then f is an order-reversing embedding. The converses of these two facts do not necessarily hold.

Definition 2.8. [25] Let P be a bounded psoset. A binary operation $T : P^2 \to P$ is called a t-norm on P if it is commutative, increasing, associative and has 1_P as neutral element, i.e., $T(p, 1_P) = p$ for any $p \in P$.

By virtue of associativity, the t-norm T can be well-defined for n arguments via induction:

$$T(i_1, i_2, \ldots, i_n) = T(T(i_1, i_2, \ldots, i_{n-1}), i_n).$$

Accordingly, the n-th power of an element i with respect to T is defined as

$$i_T^{(n)} = T(i, i, \ldots, i) \quad (n \text{ times}).$$

Definition 2.9. Let P be a bounded psoset. A t-norm T on P is called strictly monotone if $T(s_1, w_1) \triangleleft T(s_2, w_2)$ for all $s_1, s_2, w_1, w_2 \in P$ with $0_P \triangleleft s_1 \trianglelefteq s_2$ and $w_1 \triangleleft w_2$.

Definition 2.10. Let P be a bounded psoset and T be a t-norm on P. An element $s \in P \setminus \{0_P\}$ is called a nilpotent element of T if there exists $k \in \mathbb{N}$ such that $s_T^{(k)} = 0_P$.

From Definitions 2.9 and 2.10, it follows that if a t-norm T on a bounded psoset P is strictly monotone, then T has no nilpotent elements. In fact, if T is strictly monotone and h is a nilpotent element of T, then there exists $n \in \mathbb{N}$ such that $T(h, h_T^{(n-1)}) = T(h, 0_P) = 0_P$. However, since T is strictly monotone, and $0_P \triangleleft h \trianglelefteq h$ as well as $0_P \triangleleft h_T^{(n-1)}$, it follows that $T(h, 0_P) \triangleleft T(h, h_T^{(n-1)})$, which yields a contradiction. Thus, T has no nilpotent elements.

Definition 2.11. Let P be a bounded trellis. A t-norm T on P is called Archimedean if for all $s, t \in P$, $t \trianglelefteq s_T^{(n)}$ holds for every $n \in \mathbb{N}$ implies $s = 1_P$ or $t = 0_P$.

Definition 2.12. [25] Let $[a, b]$ and $[c, d]$ be closed subintervals of $[-\infty, \infty]$ and $f : [a, b] \to [c, d]$ be a non-decreasing function. Each function $f^* : [c, d] \to [a, b]$ satisfying

(i) $f \circ f^* \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$,

(ii) $f^\wedge \leq f^* \leq f^\vee$,

is called a quasi-inverse of f where $f^\wedge(y) = sup f^{-1}([c, y[\,)$ and $f^\vee(y) = inf f^{-1}(\,]y, d])$ for any $y \in [c, d]$.

Moreover, $f^\wedge$ in Definition 2.12 is called the pseudo-inverse of f and it is denoted $f^{(-1)}$. And if $f^{(-1)}$ satisfies $f \circ f^{(-1)} \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$, then $f^{(-1)}$ is a quasi-inverse of f. According to [15], it is known that using the pseudo-inverse of a monotonic function to construct a t-norm on a unit interval $[0, 1]$ requires the pseudo-inverse to become a quasi-inverse. The following is a method for constructing a t-norm using the quasi-inverse of a monotonic function on a unit interval $[0, 1]$.

Proposition 2.13. [15] *Let $c : [0,1] \to [0,1]$ be an increasing function, c^* be a quasi-inverse of c and T be a t-norm such that for all $x, y \in [0,1]$*

$$T(c(x), c(y)) \in \mathrm{Ran}(c) \cup [0, c(0^+)],$$

and for all $x, y \in [0,1[$

$$c^*(T(c(x), c(y))) \leq x \wedge y.$$

Then the function $T_{[c,c^*]} : [0,1] \to [0,1]$ given by

$$T_{[c,c^*]}(x,y) = \begin{cases} c^*(T(c(x), c(y))), & x, y \in [0,1[, \\ x \wedge y, & \text{otherwise}, \end{cases}$$

is a t-norm.

Furthermore, if $c^* = c^{(-1)}$ is a pseudo-inverse of c in the above proposition, then $c^*(T(c(x), c(y))) \leq x \wedge y$ for all $x, y \in [0,1[$ is satisfied (Proposition 11 of [15]). [15] provided methods for constructing t-norms on the real unit interval $[0,1]$ using monotone functions and their pseudo-inverses or quasi-inverses. [22] and [5] introduced the concept of pseudo-inverse on bounded lattices and complete lattices, respectively, each proposing a different definition. They also correspondingly presented methods for constructing t-norms on bounded lattices and complete lattices using monotone functions and their pseudo-inverses. In the following, we introduce these two definitions of pseudo-inverses in [22] and [5].

Definition 2.14. [22] Let X and Y be posets and $f : X \to Y$ be a non-constant injective monotone function. If f is increasing, then the pseudo-inverse $f^{(-1)_1} : Y \to X$ is given by

$$f^{(-1)_1}(y) = \begin{cases} inf\{x \in X \mid f(x) > y\}, & y \notin \mathrm{Ran}(f), \\ f^{-1}(y), & y \in \mathrm{Ran}(f), \end{cases}$$

where $inf\{x \in X \mid f(x) > y\}$ exists for all $y \in Y \setminus Ran(f)$. If f is decreasing, then the pseudo-inverse $f^{(-1)_1} : Y \to X$ is given by

$$f^{(-1)_1}(y) = \begin{cases} sup\{x \in X \mid f(x) > y\}, & y \notin \mathrm{Ran}(f), \\ f^{-1}(y), & y \in \mathrm{Ran}(f), \end{cases}$$

where $sup\{x \in X \mid f(x) > y\}$ exists for all $y \in Y \setminus Ran(f)$.

Definition 2.15. [5] Let X and Y be complete lattices and $f : X \to Y$ be a non-constant monotone function. If f is increasing, then the pseudo-inverse $f^{(-1)_2} : Y \to X$ is given by

$$f^{(-1)_2}(y) = \begin{cases} sup\{x \in X \mid f(x) < y\}, & y \notin \mathrm{Ran}(f), \\ inf\{x \in X \mid f(x) = y\}, & y \in \mathrm{Ran}(f). \end{cases}$$

If f is decreasing, then the pseudo-inverse $f^{(-1)_2} : Y \to X$ is given by

$$f^{(-1)_2}(y) = \begin{cases} inf\{x \in X \mid f(x) < y\}, & y \notin \mathrm{Ran}(f), \\ sup\{x \in X \mid f(x) = y\}, & y \in \mathrm{Ran}(f). \end{cases}$$

Remark 2.16. Let X and Y be complete lattices and $f : X \to Y$ be a non-constant monotone function. If f is increasing and $X = Y = [0,1]$, then

$$inf\{x \in X \mid f(x) > y\} = sup\{x \in X \mid f(x) < y\},$$

for any $y \in [0,1] \setminus \mathrm{Ran}(f)$. If f is decreasing and $X = Y = [0,1]$, then

$$inf\{x \in X \mid f(x) < y\} = sup\{x \in X \mid f(x) > y\},$$

for any $y \in [0,1] \setminus \mathrm{Ran}(f)$. Nevertheless, these conclusions are not always valid on arbitrary complete lattices. It is precisely for this reason that there exist differently defined pseudo-inverses for monotone functions on complete lattices. As indicated by [22] and [5], constructing t-norms using these two types of pseudo-inverses requires that the pseudo-inverse $f^{(-1)}$ satisfies $f \circ f^{(-1)}|_{\mathrm{Ran}(f)} = \mathrm{id}_{\mathrm{Ran}(f)}$. Inspired by this, we extend the concept of quasi-inverses to complete trellises and provide methods for constructing t-norms on complete trellises using monotone functions and their quasi-inverses.

Definition 2.17. Let P be a psoset, $A, C, D, K \subseteq P$, $f : A \to C$ and $g : D \to K$ be unary operators. We denote by $f \trianglelefteq g$, if $f(p) \trianglelefteq g(q)$ for all $(p,q) \in A \times D$ with $p \trianglelefteq q$.

3 The quasi-inverse of a monotonic function

In this section, we introduce the quasi-inverse of monotone functions on complete trellises and discuss its related properties and monotonicity, thereby setting the stage for the subsequent work on constructing t-norms on complete trellises. The proposed definition of quasi-inverse in this paper aims to unify the different pseudo-inverses within the framework of complete trellis structures, extending beyond the unit interval $[0,1]$, and to enable new definitions of pseudo-inverses for constructing t-norms. We denote by $A \sqcup B$ the disjoint union of A and B, i.e., $A \cup B$ with $A \cap B = \emptyset$.

Definition 3.1. Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant monotone function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$. Each function $f^* : Q \to P$ satisfying

(i) $f \circ f^* \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$,

(ii) $f^\wedge \trianglelefteq f^* \trianglelefteq f^\vee$,

is called a quasi-inverse of f where if f is increasing, then for any $y \in Q$

$$f^\wedge(y) = \begin{cases} inf\{x \in P \mid f(x) = y\}, & y \in \mathrm{Ran}(f), \\ sup\{x \in P \mid f(x) \lhd y\}, & y \in Q_1, \\ inf\{x \in P \mid y \lhd f(x)\}, & y \in Q_2, \end{cases}$$

$$f^\vee(y) = \begin{cases} sup\{x \in P \mid f(x) = y\}, & y \in \mathrm{Ran}(f), \\ sup\{x \in P \mid f(x) \lhd y\}, & y \in Q_1, \\ inf\{x \in P \mid y \lhd f(x)\}, & y \in Q_2, \end{cases}$$

and if f is decreasing, then for any $y \in Q$

$$f^\wedge(y) = \begin{cases} inf\{x \in P \mid f(x) = y\}, & y \in \mathrm{Ran}(f), \\ sup\{x \in P \mid y \lhd f(x)\}, & y \in Q_1, \\ inf\{x \in P \mid f(x) \lhd y\}, & y \in Q_2, \end{cases}$$

$$f^\vee(y) = \begin{cases} sup\{x \in P \mid f(x) = y\}, & y \in \mathrm{Ran}(f), \\ sup\{x \in P \mid y \lhd f(x)\}, & y \in Q_1, \\ inf\{x \in P \mid f(x) \lhd y\}, & y \in Q_2. \end{cases}$$

Moreover, if f is increasing, then $f^\wedge$ is called the pseudo-inverse of f, and it is denoted $f^{(-1)}$; if f is decreasing, then $f^\vee$ is called the pseudo-inverse of f, and it is denoted $f^{(-1)}$.

Remark 3.2. Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant monotone function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. According to the above definition, it can be inferred that if $y \notin \mathrm{Ran}(f)$, then $f^\wedge(y) = f^\vee(y)$. If $y \in \mathrm{Ran}(f)$, then $f^*(y) \in \{x \in P \mid f(x) = y\}$. Since $\trianglelefteq$ does not satisfy transitivity, $f^\wedge(y) \trianglelefteq f^\vee(y)$ cannot be deduced from $f^\wedge(y) \trianglelefteq x \trianglelefteq f^\vee(y)$ for all $y \in \mathrm{Ran}(f)$ and $x \in \{x \in P \mid f(x) = y\}$. However, if $f \circ f^\wedge \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$, then $f^\wedge(y) = min\{x \in P \mid f(x) = y\}$ for all $y \in \mathrm{Ran}(f)$, and thus, $f^\wedge(y) \trianglelefteq f^\vee(y)$. Similarly, if $f \circ f^\vee \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$, then $f^\vee(y) = max\{x \in P \mid f(x) = y\}$ for all $y \in \mathrm{Ran}(f)$, and thus, $f^\wedge(y) \trianglelefteq f^\vee(y)$. Additionally, if $Q \setminus \mathrm{Ran}(f) = Q_1' \sqcup Q_2'$, we can know that the $f^*, f^\wedge, f^\vee$ corresponding to Q_1, Q_2 and the $f^{*'}, f^{\wedge'}, f^{\vee'}$ corresponding to Q_1', Q_2' may not necessarily be equal.

Remark 3.3. Let P be a complete lattice, Q be a poset and $f : P \to Q$ be a non-constant monotone function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$.

(i) According to the above definition, it can be inferred that if $y \notin Ran(f)$, then $f^\wedge(y) = f^\vee(y)$ and if $y \in Ran(f)$, then $f^\wedge(y) \leq f^\vee(y)$.

(ii) Let $P = [a, b]$, $Q = [c, d]$, f be a non-constant increasing function. Obviously, $f^\wedge(y) = sup\, f^{-1}([c, y[)$ and $f^\vee(y) = inf\, f^{-1}(]y, d])$ for any $y \in [c, d]$. Similarly, if f is a non-constant decreasing function, then $f^\wedge(y) = sup\, f^{-1}(]y, d])$ and $f^\vee(y) = inf\, f^{-1}([c, y[)$ for any $y \in [c, d]$. This indicates that the above definition extends the concepts of quasi-inverse and pseudo-inverse from the real unit interval to complete trellises.

Table 2: The functions f_1, $f_{11}^{\wedge}$, $f_{11}^{\vee}$ and f_{11}^{*} of Example 3.4.

f_1	0_P	a	b	c	d	v	g	1_P
	s	n	m	h	z	z	z	1_Q

$f_{11}^{\wedge}$	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	0_P	b	a	1_P	a	1_P	1_P

$f_{11}^{\vee}$	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	0_P	b	a	1_P	1_P	1_P	1_P

f_{11}^{*}	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	0_P	b	a	1_P	d	1_P	1_P

Table 3: The functions $f_{12}^{\wedge}$, $f_{12}^{\vee}$ and f_{12}^{*} of Example 3.4.

$f_{12}^{\wedge}$	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	c	b	a	0_P	a	1_P	1_P

$f_{12}^{\vee}$	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	c	b	a	0_P	1_P	1_P	1_P

f_{12}^{*}	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	c	b	a	0_P	v	1_P	1_P

(iii) Specially, if f is a non-constant increasing function and $Q_2 = \emptyset$, then $f^{\wedge}(y) = f^{(-1)_2}(y)$ for any $y \in Q$. If f is a non-constant decreasing function and $Q_1 = \emptyset$, then $f^{\vee}(y) = f^{(-1)_2}(y)$ for any $y \in Q$. If f is an injective increasing function and $Q_1 = \emptyset$, then $f^{\wedge}(y) = f^{\vee}(y) = f^{(-1)_1}(y)$ for any $y \in Q$, and thus, $f^{(-1)_1}$ is a quasi-inverse of f. If f is an injective decreasing function and $Q_2 = \emptyset$, then $f^{\wedge}(y) = f^{\vee}(y) = f^{(-1)_1}(y)$ for any $y \in Q$, and thus, $f^{(-1)_1}$ is a quasi-inverse of f. Similarly, if f is an injective increasing (decreasing) function and $Q_2 = \emptyset$ ($Q_1 = \emptyset$), then $f^{(-1)_2}$ is a quasi-inverse of f.

Example 3.4. Let $P = (\{0_P, a, b, c, d, v, g, 1_P\}, \trianglelefteq)$ be a complete trellis and $Q = (\{0_Q, s, t, h, k, m, n, u, z, w, 1_Q\}, \trianglelefteq)$ be a psoset, whose Hasse diagrams are given in Fig. 2(a) and Fig. 2(b), respectively. Let $f_1 : P \to Q$ be given in Table 2, $Q_1 = \{0_Q, t\}$, $Q_2 = \{k, u, w\}$, $Q_1^{'} = \{k, u, w\}$ and $Q_2^{'} = \{0_Q, t\}$. According to Definition 3.1 and $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, Table 2 presents three functions from Q to P, denoted as $f_{11}^{\wedge}$, $f_{11}^{\vee}$ and f_{11}^{*}. According to Definition 3.1 and $Q \setminus \mathrm{Ran}(f) = Q_1^{'} \sqcup Q_2^{'}$, Table 3 presents three functions from Q to P, denoted as $f_{12}^{\wedge}$, $f_{12}^{\vee}$ and f_{12}^{*}. It can be verified that f_{11}^{*} and f_{12}^{*} are the quasi-inverses of f_1.



(a) (b)

Fig. 2. Hasse diagrams of a complete trellis (left) and a psoset (right) in Example 3.4.

Proposition 3.5. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant monotone function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$ and $f^{'} : Q \to P$ be a function satisfying $f^{\wedge} \trianglelefteq f^{'} \trianglelefteq f^{\vee}$. Then*

(i) $f \circ f^{'} \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$ *if and only if* $f^{'}(y) \in \{x \mid f(x) = y\}$ *for any* $y \in \mathrm{Ran}(f)$*;*

(ii) $f^{'} \circ f = id_P$ *if and only if* f *is injective.*

Proof. This proof is clear and straightforward. □

Especially, if f is a complete inf-homomorphism, or a complete inf-sup-homomorphism, then $f \circ f^{\wedge} \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$. In fact, if f is a complete inf-homomorphism, then for any $y \in \mathrm{Ran}(f)$,

$$f(f^{\wedge}(y)) = f(\bigwedge\{x \in P \mid f(x) = y\}) = \bigwedge\{f(x) \mid f(x) = y, x \in P\} = y.$$

If f is a complete inf-sup-homomorphism, then for any $y \in \mathrm{Ran}(f)$,

$$f(f^{\wedge}(y)) = f(\bigwedge\{x \in P \mid f(x) = y\}) = \bigvee\{f(x) \mid f(x) = y, x \in P\} = y.$$

Similarly, if f is a complete sup-homomorphism, or a complete sup-inf-homomorphism, then $f \circ f^{\vee} \mid_{\mathrm{Ran}(f)} = id_{\mathrm{Ran}(f)}$. In the following, we examine the monotonicity properties of quasi-inverses of certain special monotone functions.

Proposition 3.6. *Let* P *be a complete lattice,* Q *be a poset,* $f : P \to Q$ *be an order-embedding,* $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$ *and* $f^* : Q \to P$ *be a quasi-inverse of* f*. If* $[\, y_1, y_2\,] \cap \mathrm{Ran}(f) \neq \emptyset$ *for any* $y_1 \in Q_2$ *and* $y_2 \in Q_1$ *with* $y_1 < y_2$*, then* f^* *is increasing.*

Proof. If $h \in Q \setminus \mathrm{Ran}(f)$ and $m, n \in P$ with $f(m) < h < f(n)$, then $m < n$ by f is an order-embedding. Thus, $m < n$ for any $m \in \{x \in P \mid f(x) < h\}$ and $n \in \{x \in P \mid f(x) > h\}$, and $\bigvee\{x \in P \mid f(x) < h\} \leq \bigwedge\{x \in P \mid f(x) > h\}$. Let a, $b \in Q$ with $a \leq b$. We divided the following cases to prove $f^*(a) \leq f^*(b)$:

(i) $a, b \in \mathrm{Ran}(f)$. Then there exist $c, d \in P$ such that $a = f(c) \leq f(d) = b$. Since f is an order-embedding, we can obtain $f^*(a) = c \leq d = f^*(b)$.

(ii) $a, b \notin \mathrm{Ran}(f)$. If $a, b \in Q_1$, then

$$f^*(a) = \bigvee\{x \in P \mid f(x) < a\} \leq \bigvee\{x \in P \mid f(x) < b\} = f^*(b),$$

by $\{x \in P \mid f(x) < a\} \subseteq \{x \in P \mid f(x) < b\}$. If $a, b \in Q_2$, then

$$f^*(a) = \bigwedge\{x \in P \mid f(x) > a\} \leq \bigwedge\{x \in P \mid f(x) > b\} = f^*(b),$$

by $\{x \in P \mid f(x) > b\} \subseteq \{x \in P \mid f(x) > a\}$. If $a \in Q_1$ and $b \in Q_2$, then

$$f^*(a) = \bigvee\{x \in P \mid f(x) < a\} \leq \bigwedge\{x \in P \mid f(x) > a\} \leq \bigwedge\{x \in P \mid f(x) > b\} = f^*(b).$$

If $a \in Q_2$ and $b \in Q_1$, then there exits $z_1 \in P$ such that $f(z_1) \in [\, a, b\,]$. Then

$$f^*(a) = \bigwedge\{x \in P \mid f(x) > a\} \leq z_1 \leq \bigvee\{x \in P \mid f(x) < b\} = f^*(b),$$

by $z_1 \in \{x \in P \mid f(x) > a\} \bigcap \{x \in P \mid f(x) < b\}$.

(iii) $a \in \mathrm{Ran}(f)$ and $b \notin \mathrm{Ran}(f)$. If $b \in Q_1$, then

$$f^*(a) \leq \bigvee\{x \in P \mid f(x) < b\} = f^*(b),$$

by $f^*(a) \in \{x \in P \mid f(x) = a\} \subseteq \{x \in P \mid f(x) < b\}$. If $b \in Q_2$, then

$$f^*(a) \leq \bigwedge\{x \in P \mid f(x) > b\} = f^*(b),$$

by $f^*(a) \leq b_1$ for any $b_1 \in \{x \in P \mid f(x) > b\}$ and $f^*(a) \in \{x \in P \mid f(x) = a\}$.

(iv) $a \notin \mathrm{Ran}(f)$ and $b \in \mathrm{Ran}(f)$. If $a \in Q_1$, then

$$f^*(a) = \bigvee\{x \in P \,|\, f(x) < a\} \leq f^*(b),$$

by $a_1 \leq f^*(b)$ for any $a_1 \in \{x \in P \,|\, f(x) < a\}$ and $f^*(b) \in \{x \in P \,|\, f(x) = b\}$. If $a \in Q_2$, then

$$f^*(a) = \bigwedge\{x \in P \,|\, f(x) > a\} \leq f^*(b),$$

by $f^*(b) \in \{x \in P \,|\, f(x) = b\} \subseteq \{x \in P \,|\, f(x) > a\}$.

Therefore, f^* is increasing. □

Let P be a complete trellis, Q be a psoset and $f : P \to Q$ be a function. We denote by $K_1 = \{k \in Q \setminus \mathrm{Ran}(f) \mid \forall x \in P \setminus \{0_P\},\ f(x) \not\trianglelefteq k\}$, $K_2 = \{k \in Q \setminus \mathrm{Ran}(f) \mid \exists A \subseteq P,\ f(A) \subseteq\!\uparrow k,\ \wedge A = 0_P\}$, $H_1 = \{h \in Q \setminus \mathrm{Ran}(f) \mid \exists A \subseteq P,\ f(A) \subseteq\!\downarrow h,\ \vee A = 1_P\}$, $H_2 = \{h \in Q \setminus \mathrm{Ran}(f) \mid \forall x \in P \setminus \{1_P\},\ h \not\trianglelefteq f(x)\}$. According to the above definition, it follows that $\bigvee\{x \in P \mid f(x) \lhd k\} = 0_P$ for all $k \in K_1$, $\bigwedge\{x \in P \mid k \lhd f(x)\} = 0_P$ for all $k \in K_2$, $\bigvee\{x \in P \mid f(x) \lhd h\} = 1_P$ for all $h \in H_1$, and $\bigwedge\{x \in P \mid h \lhd f(x)\} = 1_P$ for all $h \in H_2$. We say that $f : P \longrightarrow Q$ is injective at $a \in P$ if, whenever there exists $k \in P$ such that $f(k) = f(a)$, it follows that $k = a$.

Proposition 3.7. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant increasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1 \subseteq Q_1$, $K_2 \setminus K_1 \subseteq Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. If f is injective at 0_P and f^* is increasing on $\mathrm{Ran}(f) \cup K_1 \cup K_2$, then $K_2 \setminus K_1 = \emptyset$.*

Proof. The conclusion holds trivially when K_2 is the empty set. We now consider the case when K_2 is nonempty. Assume that $K_2 \setminus K_1 \neq \emptyset$. Then there exists $h \in K_2 \setminus K_1$ such that $f(x) \lhd h$ for some $x \in P \setminus \{0_P\}$. Since f^* is increasing on $\mathrm{Ran}(f) \cup K_1 \cup K_2$, we can obtain

$$0_P = f^*(f(x)) \trianglelefteq f^*(h) = 0_P,$$

for some $x \in P \setminus \{0_P\}$. Furthermore, the injectivity of f at 0_P implies that $x = 0_P$ for some $x \in P \setminus \{0_P\}$. But this is a contradiction. Thus, $K_2 \setminus K_1 = \emptyset$. □

Proposition 3.8. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant increasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $H_1 \setminus H_2 \subseteq Q_1$, $H_2 \subseteq Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. If f is injective at 1_P and f^* is increasing on $\mathrm{Ran}(f) \cup H_1 \cup H_2$, then $H_1 \setminus H_2 = \emptyset$.*

Proof. The proof is analogous to that of Proposition 3.7. □

Next, we discuss the monotonicity of the pseudo-inverses of some special monotone functions on complete trellises. For a psoset Q, we write $Q^{\mathrm{mtr}} = \{q \in Q \mid \text{for all } x, y \in Q,\ \text{if } x \trianglelefteq q \trianglelefteq y,\ \text{then } x \trianglelefteq y\}$.

Proposition 3.9. *Let P, Q be complete trellises, $f : P \to Q$ be a non-constant function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, and $f^* : Q \to P$ be a quasi-inverse of f. Then the following statements hold.*

(i) *If f is an order-embedding and $Q_1 \subseteq Q^{\mathrm{mtr}}$, then f^* is increasing on $\mathrm{Ran}(f) \cup Q_1$;*

(ii) *If f is an order-embedding and $Q_2 \subseteq Q^{\mathrm{mtr}}$, then f^* is increasing on $\mathrm{Ran}(f) \cup Q_2$;*

(iii) *If f is a complete inf-homomorphism, $K_1 \subseteq Q_1$ and $f^* = f^\wedge$, then $f^\wedge$ is increasing on $\mathrm{Ran}(f) \cup K_1$;*

(iv) *If f is a complete inf-homomorphism, $H_2 \subseteq Q_2$, $f^\wedge(f(1_P)) = 1_P$ and $f^* = f^\wedge$, then $f^\wedge$ is increasing on $\mathrm{Ran}(f) \cup H_2$;*

(v) *If f is a complete sup-homomorphism, $K_1 \subseteq Q_1$, $f^\vee(f(0_P)) = 0_P$ and $f^* = f^\vee$, then $f^\vee$ is increasing on $\mathrm{Ran}(f) \cup K_1$;*

(vi) *If f is a complete sup-homomorphism, $H_2 \subseteq Q_2$ and $f^* = f^\vee$, then $f^\vee$ is increasing on $\mathrm{Ran}(f) \cup H_2$.*

Proof. Since the proof is similar to that of Proposition 3.6, we only provide the proof for the case that differs from Proposition 3.6 below.

(i) Let $y_1 \in Q_1$ and $y_2 \in \mathrm{Ran}(f)$ with $y_1 \trianglelefteq y_2$. Since $y_1 \in Q^{\mathrm{mtr}}$, we can obtain $f(x) \trianglelefteq y_2$ for any $x \in \{x \in P \,|\, f(x) \lhd y_1\}$. Then $x \trianglelefteq h$ for any $x \in \{x \in P \,|\, f(x) \lhd y_1\}$ and $h \in \{x \in P \,|\, f(x) = y_2\}$ by f is an order-embedding. Thus,

$$f^*(y_1) = \bigvee\{x \in P \,|\, f(x) \lhd y_1\} \trianglelefteq f^*(y_2).$$

(ii) Let $y_1 \in \mathrm{Ran}(f)$ and $y_2 \in Q_2$ with $y_1 \trianglelefteq y_2$. Since $y_2 \in Q^{\mathrm{mtr}}$, we can obtain $y_1 \trianglelefteq f(x)$ for any $x \in \{x \in P \,|\, y_2 \lhd f(x)\}$. Then $h \trianglelefteq x$ for any $x \in \{x \in P \,|\, y_2 \lhd f(x)\}$ and $h \in \{x \in P \,|\, f(x) = y_1\}$ by f is an order-embedding. Thus,

$$f^*(y_1) \trianglelefteq \bigwedge\{x \in P \,|\, y_2 \lhd f(x)\} = f^*(y_2).$$

(iii) We need to prove that if $y_1, y_2 \in \mathrm{Ran}(f) \cup K_1$ with $y_1 \trianglelefteq y_2$, then $f^\wedge(y_1) \trianglelefteq f^\wedge(y_2)$. If $y_1 \in K_1$ and $y_2 \in \mathrm{Ran}(f) \cup K_1$, then $0_P = f^\wedge(y_1) \trianglelefteq f^\wedge(y_2)$. If $y_1 \in \mathrm{Ran}(f)$ and $y_2 \in K_1$, then

$$f^\wedge(y_1) = f^\wedge(f(0_P)) = 0_P = f^\wedge(y_2).$$

If $y_1, y_2 \in Ran(f)$ with $y_1 \trianglelefteq y_2$, then

$$f(f^\wedge(y_1) \wedge f^\wedge(y_2)) = f(f^\wedge(y_1)) \wedge f(f^\wedge(y_2)) = y_1 \wedge y_2 = y_1.$$

That is $f^\wedge(y_1) \wedge f^\wedge(y_2) \in \{x \in P \,|\, f(x) = y_1\}$ and $f^\wedge(y_1) \trianglelefteq f^\wedge(y_1) \wedge f^\wedge(y_2) \trianglelefteq f^\wedge(y_1)$. Thus, $f^\wedge(y_1) \wedge f^\wedge(y_2) = f^\wedge(y_1)$ and $f^\wedge(y_1) \trianglelefteq f^\wedge(y_2)$.

(iv) The proof is analogous to that of (iii).

(v) We need to prove that if $y_1, y_2 \in Ran(f) \cup K_1$ with $y_1 \trianglelefteq y_2$, then $f^\vee(y_1) \trianglelefteq f^\vee(y_2)$. If $y_1 \in K_1$ and $y_2 \in \mathrm{Ran}(f) \cup K_1$, then $f^\vee(y_1) = 0_P \trianglelefteq f^\vee(y_2)$. If $y_1, y_2 \in \mathrm{Ran}(f)$ with $y_1 \trianglelefteq y_2$, then

$$f(f^\vee(y_1) \vee f^\vee(y_2)) = f(f^\vee(y_1)) \vee f(f^\vee(y_2)) = y_1 \vee y_2 = y_2.$$

That is $f^\vee(y_1) \vee f^\vee(y_2) \in \{x \in P \,|\, f(x) = y_2\}$ and $f^\vee(y_2) \trianglelefteq f^\vee(y_1) \vee f^\vee(y_2) \trianglelefteq f^\vee(y_2)$. Thus, $f^\vee(y_1) \vee f^\vee(y_2) = f^\vee(y_2)$ and $f^\vee(y_1) \trianglelefteq f^\vee(y_2)$. If $y_1 \in \mathrm{Ran}(f)$ and $y_2 \in K_1$ with $y_1 \trianglelefteq y_2$, then

$$f^\vee(y_1) = f^\vee(f(0_P)) = 0_P = f^\vee(y_2).$$

(vi) The proof is analogous to that of (v).

□

The following propositions concern the monotonicity of the quasi-inverse for decreasing functions. Their proofs are analogous to those for the case of increasing functions and are therefore omitted here. Let P be a complete trellis, Q be a psoset and $f : P \to Q$ be a function. We denote by $K_1^{'} = \{k \in Q \setminus \mathrm{Ran}(f) \mid \forall x \in P \setminus \{0_P\}, k \nsubseteq f(x)\}$, $K_2^{'} = \{k \in Q \setminus \mathrm{Ran}(f) \mid \exists A \subseteq P, f(A) \subseteq\downarrow k, \wedge A = 0_P\}$, $H_1^{'} = \{h \in Q \setminus \mathrm{Ran}(f) \mid \exists A \subseteq P, f(A) \subseteq\uparrow h, \vee A = 1_P\}$, $H_2^{'} = \{h \in Q \setminus \mathrm{Ran}(f) \mid \forall x \in P \setminus \{1_P\}, f(x) \nsubseteq h\}$. According to the above definition, it follows that $\bigvee\{x \in P \mid k \lhd f(x)\} = 0_P$ for all $k \in K_1^{'}$, $\bigwedge\{x \in P \mid f(x) \lhd k\} = 0_P$ for all $k \in K_2^{'}$, $\bigvee\{x \in P \mid h \lhd f(x)\} = 1_P$ for all $h \in H_1^{'}$, and $\bigwedge\{x \in P \mid f(x) \lhd h\} = 1_P$ for all $h \in H_2^{'}$.

Proposition 3.10. *Let P be a complete lattice, Q be a poset, $f : P \to Q$ be a non-constant order-reversing embedding, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. If $[y_1, y_2] \bigcap \mathrm{Ran}(f) \neq \emptyset$ for any $y_1 \in Q_1$ and $y_2 \in Q_2$ with $y_1 < y_2$, then f^* is decreasing.*

Proposition 3.11. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant decreasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1^{'} \subseteq Q_1$, $K_2^{'} \setminus K_1^{'} \subseteq Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. If f is injective at 0_P and f^* is decreasing on $\mathrm{Ran}(f) \cup K_1^{'} \cup K_2^{'}$, then $K_2^{'} \setminus K_1^{'} = \emptyset$.*

Proposition 3.12. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant decreasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $H_1^{'} \setminus H_2^{'} \subseteq Q_1$, $H_2^{'} \subseteq Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. If f is injective at 1_P and f^* is decreasing on $\mathrm{Ran}(f) \cup H_1^{'} \cup H_2^{'}$, then $H_1^{'} \setminus H_2^{'} = \emptyset$.*

Proposition 3.13. *Let P, Q be complete trellises, $f : P \to Q$ be a non-constant function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$ and $f^* : Q \to P$ be a quasi-inverse of f. Then the following statements hold.*

(i) *If f is an order-reversing embedding and $Q_1 \subseteq Q^{\mathrm{mtr}}$, then f^* is decreasing on* $\mathrm{Ran}(f) \cup Q_1$*;*

(ii) *If f is an order-reversing embedding and $Q_2 \subseteq X^{\mathrm{mtr}}$, then f^* is decreasing on* $\mathrm{Ran}(f) \cup Q_2$*;*

(iii) *If f is a complete inf-sup-homomorphism, $K_1^{'} \subseteq Q_1$ and $f^* = f^{\wedge}$, then $f^{\wedge}$ is decreasing on* $\mathrm{Ran}(f) \cup K_1^{'}$*;*

(iv) *If f is a complete inf-sup-homomorphism, $H_2^{'} \subseteq Q_2$, $f^{\wedge}(f(1_P)) = 1_P$ and $f^* = f^{\wedge}$, then $f^{\wedge}$ is decreasing on* $\mathrm{Ran}(f) \cup H_2^{'}$*;*

(v) *If f is a complete sup-inf-homomorphism, $K_1^{'} \subseteq Q_1$, $f^{\vee}(f(0_P)) = 0_P$ and $f^* = f^{\vee}$, then $f^{\vee}$ is decreasing on* $\mathrm{Ran}(f) \cup K_1^{'}$*;*

(vi) *If f is a complete sup-inf-homomorphism, $H_2^{'} \subseteq Q_2$ and $f^* = f^{\vee}$, then $f^{\vee}$ is decreasing on* $\mathrm{Ran}(f) \cup H_2^{'}$.

4 Constructing a t-norm using a monotonic function and its quasi-inverse

In this section, we propose methods of constructing t-norms on complete trellises using a monotonic function and its quasi-inverse. Firstly, we present the following proposition for the convenience of subsequent proof.

Proposition 4.1. *Let P be a bounded psoset and F be a binary operator on P. Then the binary operator $T : P^2 \to P$ defined by*

$$T(x,y) = \begin{cases} F(x,y), & x,y \in P \setminus \{1_P\}, \\ x, & x \in P, y = 1_P, \\ y, & y \in P, x = 1_P \end{cases}$$

is a t-norm on P if and only if F is commutative, associative, increasing, and $F(x,z) \trianglelefteq y$ for any $x,y,z \in P$ with $x \trianglelefteq y$.

Proof. This proof is clear and straightforward. □

In the above proposition, if P is a bounded poset, then T is a t-norm on P if and only if F is commutative, associative, increasing and $F(x,y) \leq x$ for any $x,y \in P$.

4.1 Constructing a t-norm using an increasing function and its quasi-inverse

From the analysis in Section 3, for a function $f : P \to Q$, there exist two associated special sets K_1 and K_2. Based on the properties of elements in these two sets, a reasonable partition of $Q \setminus \mathrm{Ran}(f)$ can be chosen such that the quasi-inverse f^* of f takes the value 0_P on $K_1 \cup K_2$. This facilitates our further construction of t-norms on P. Based on the above discussion, the following theorem provides a method for constructing a t-norm on a complete trellis using a non-constant increasing function and its quasi-inverse.

Theorem 4.2. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant increasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1 \subseteq Q_1$, $K_2 \setminus K_1 \subseteq Q_2$, $f^* : Q \to P$ be a quasi-inverse of f and $F : Q^2 \to Q$ be a commutative and associative function such that for any $x,y,z,t \in P \setminus \{1_P\}$ the following conditions are satisfied:*

(i) *$x \trianglelefteq y$ implies $f^*(F(f(x),f(z))) \trianglelefteq y$,*

(ii) *$F(f(x),f(y)) \in \mathrm{Ran}(f) \cup K_1 \cup K_2$,*

(iii) *$x \trianglelefteq y$, $z \trianglelefteq t$ and $F(f(x),f(z)) \in \mathrm{Ran}(f)$ implies $f^*(F(f(x),f(z))) \trianglelefteq f^*(F(f(y),f(t)))$.*

Then the binary operator $T_{f^*} : P^2 \to P$ defined by

$$T_{f^*}(x,y) = \begin{cases} f^*(F(f(x),f(y))), & x,y \in P \setminus \{1_P\}, \\ x \wedge y, & \text{otherwise} \end{cases} \tag{1}$$

is a t-norm on P.

Proof. According to (1), T_{f^*} is commutative and 1_P is the neutral element of T_{f^*}. Since $K_1 \subseteq Q_1$ and $K_2 \setminus K_1 \subseteq Q_2$, we can conclude that $f^*(y) = 0_P$ for all $y \in K_1 \cup K_2$.

Let m, n, h, $u \in P$ with $m \trianglelefteq n$ and $h \trianglelefteq u$. We divided the following cases to prove $T_{f^*}(m,h) \trianglelefteq T_{f^*}(n,u)$:

(i) $1_P \in \{m, n, h, u\}$. Then $T_{f^*}(m, h) \trianglelefteq T_{f^*}(n, u)$ by condition (i).

(ii) $1_P \notin \{m, n, h, u\}$. If $F(f(m), f(h)) \in K_1 \cup K_2$, then

$$T_{f^*}(m, h) = f^*(F(f(m), f(h))) = 0_P \trianglelefteq T_{f^*}(n, u).$$

If $F(f(m), f(h)) \in \mathrm{Ran}(f)$, then

$$T_{f^*}(m, h) = f^*(F(f(m), f(h))) \trianglelefteq f^*(F(f(n), f(u))) = T_{f^*}(n, u),$$

by condition (iii).

Therefore, T_{f^*} is increasing. Let m, n, $h \in P$. We divided the following cases to prove $T_{f^*}(T_{f^*}(m, n), h) = T_{f^*}(m, T_{f^*}(n, h))$:

(i) $1_P \in \{m, n, h\}$. Then $T_{f^*}(T_{f^*}(m, n), h) = T_{f^*}(m, T_{f^*}(n, h))$.

(ii) $1_P \notin \{m, n, h\}$. Then

$$T_{f^*}(T_{f^*}(m, n), h) = T_{f^*}(f^*(F(f(m), f(n))), h) = f^*(F(f(f^*(F(f(m), f(n)))), f(h))).$$

If $F(f(m), f(n)) \in \mathrm{Ran}(f)$, then

$$T_{f^*}(T_{f^*}(m, n), h) = f^*(F(F(f(m), f(n)), f(h))) = f^*(F(f(m), F(f(n), f(h)))),$$

because F is associative. Whenever $F(f(n), f(h)) \in \mathrm{Ran}(f)$, then

$$\begin{aligned} T_{f^*}(m, T_{f^*}(n, h)) &= f^*(F(f(m), f(f^*(F(f(n), f(h)))))) \\ &= f^*(F(f(m), F(f(n), f(h)))) \\ &= T_{f^*}(T_{f^*}(m, n), h). \end{aligned}$$

Whenever $F(f(n), f(h)), F(f(m), F(f(n), f(h))) \in K_1 \cup K_2$, it follows that

$$\begin{aligned} T_{f^*}(m, T_{f^*}(n, h)) &= f^*(F(f(m), f(f^*(F(f(n), f(h)))))) \\ &= f^*(F(f(m), f(0_P))) \\ &= f^*(F(f(m), F(f(n), f(h)))) \\ &= 0_P, \end{aligned}$$

by $f^*(F(f(m), f(0_P))) \trianglelefteq 0_P$. Whenever $F(f(n), f(h)) \in K_1 \cup K_2$ and $F(F(f(m), f(n)), f(h)) \in \mathrm{Ran}(f)$, it follows that $f^*(F(f(n), f(h))) = 0_P$. Since $f^*(F(f(m), f(n))) \trianglelefteq n$, $F(f(f^*(F(f(m), f(n)))), f(h)) \in \mathrm{Ran}(f)$ and condition (iii), it can be obtain that

$$f^*(F(f(f^*(F(f(m), f(n)))), f(h))) \trianglelefteq f^*(F(f(n), f(h))) = 0_P.$$

Thus,

$$T_{f^*}(T_{f^*}(m, n), h) = 0_P = T_{f^*}(m, T_{f^*}(n, h)).$$

If $F(f(m), f(n)) \in K_1 \cup K_2$, then

$$T_{f^*}(T_{f^*}(m, n), h) = T_{f^*}(f^*(F(f(m), f(n))), h) = f^*(F(f(0_P), f(h))) = 0_P,$$

by $f^*(F(f(0_P), f(h))) \trianglelefteq 0_P$.

Whenever $F(f(n), f(h)), F(f(m), F(f(n), f(h))) \in \mathrm{Ran}(f)$, it follows that

$$f^*(F(f(m), f(f^*(F(f(n), f(h)))))) \trianglelefteq f^*(F(f(m), f(n))) = 0_P,$$

by $f^*(F(f(n), f(h))) \trianglelefteq n$. Thus,

$$T_{f^*}(T_{f^*}(m, n), h) = 0_P = T_{f^*}(m, T_{f^*}(n, h)).$$

Whenever $F(f(n), f(h)) \in \mathrm{Ran}(f)$ and $F(f(m), F(f(n), f(h))) \in K_1 \bigcup K_2$, it follows that

$$\begin{aligned} T_{f^*}(m, T_{f^*}(n,h)) &= f^*(F(f(m), F(f(n), f(h)))) \\ &= T_{f^*}(T_{f^*}(m,n), h) \\ &= 0_P. \end{aligned}$$

Whenever $F(f(n), f(h)) \in K_1 \cup K_2$, it follows that

$$\begin{aligned} T_{f^*}(m, T_{f^*}(n,h)) &= f^*(F(f(m), f(f^*(F(f(n), f(h)))))) \\ &= f^*(F(f(m), f(0_P))) \\ &= 0_P \\ &= T_{f^*}(T_{f^*}(m,n), h), \end{aligned}$$

by $f^*(F(f(m), f(0_P))) \trianglelefteq 0_P$.

Therefore, T_{f^*} is a t-norm on P. □

Remark 4.3. In Theorem 4.2, if $P = Q = [0,1]$, then $\mathrm{Ran}(f) \cup K_1 \cup K_2 = \mathrm{Ran}(f) \cup [0, f(0^+)]$ and the condition (iii) becomes that F is increasing. Thus, Proposition 2.13 can be inferred. In Theorem 4.2, if P is a complete lattice and Q is a poset, then (i) − (iii) become the following:

(i) $f^*(F(f(x), f(y))) \leq x$,

(ii) $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1 \cup K_2$,

(iii) $x \leq y$ and $F(f(x), f(z)) \in \mathrm{Ran}(f)$ implies $f^*(F(f(x), f(z))) \leq f^*(F(f(y), f(z)))$.

This theorem reveals conditions that are as minimal as possible for constructing a t-norm on a complete trellis using an order-preserving function f, its pseudo-inverse g, and a commutative associative binary function F. However, although these conditions are weak, the construction process is relatively complicated. In the subsequent results, we further strengthen the requirements on f and F to make the construction more convenient.

Example 4.4. Let $P = (\{0_P, a, b, c, d, v, g, 1_P\}, \trianglelefteq)$ and $Q = (\{0_Q, s, t, h, k, m, n, u, z, w, 1_Q\}, \trianglelefteq)$ and f_1 are given in Example 3.4. Let $Q_1 = \{0_Q, t, u\}$ and $Q_2 = \{k, w\}$. Then $K_1 = \{0_Q, t, u\} \subseteq Q_1$ and $K_2 \setminus K_1 = \{k\} \subseteq Q_2$. We take f_1^* in Table 3 and the binary function F_1 on Q such that

$$F_1(x,y) = \begin{cases} n, & (x,y) \in \{(n,n),(z,z),(z,n),(n,z)\}, \\ m, & (x,y) \in \{(m,m),(w,w),(m,w),(w,m)\}, \\ min(x,y), & x = 1_Q \text{ or } y = 1_Q, \\ 0_Q, & \text{otherwise.} \end{cases}$$

It is easy to verify that F_1 is a t-norm on Q and the conditions (i)-(iii) are satisfied in Theorem 4.2. Then we can construct a t-norm T_1 on P using Theorem 4.2 as shown in Table 4.

The following two corollaries are obtained by combining Theorem 4.2 with items (iii) and (v) of Proposition 3.9, respectively.

Corollary 4.5. *Let P and Q be complete trellises, $f : P \to Q$ be a complete inf-homomorphism, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1 \subseteq Q_1$, $f^\wedge : Q \to P$ be a quasi-inverse of f and $F : Q^2 \to Q$ be a commutative, increasing and associative function such that for any $x, y, z \in P \setminus \{1_P\}$ the following conditions are satisfied:*

(i) $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1$,

(ii) $x \trianglelefteq y$ *implies* $f^\wedge(F(f(x), f(z))) \trianglelefteq y$.

Then the binary operator $T_{f^\wedge} : P^2 \to P$ defined by

$$T_{f^\wedge}(x,y) = \begin{cases} f^\wedge(F(f(x), f(y))), & x, y \in P \setminus \{1_P\}, \\ x \wedge y, & \text{otherwise} \end{cases}$$

is a t-norm on P.

Table 4: f_1^* and T_1 in Example 4.4.

f_1^*	0_Q	s	t	h	k	m	n	u	z	w	1_Q
	0_P	0_P	0_P	c	0_P	b	a	0_P	d	1_P	1_P

T_1	0_P	a	b	c	d	v	g	1_P
0_P	0_P	0_P	0_P	0_P	0_P	0_P	0_P	0_P
a	0_P	a	0_P	0_P	a	a	a	a
b	0_P	0_P	b	0_P	0_P	0_P	0_P	b
c	0_P	0_P	0_P	0_P	0_P	0_P	0_P	c
d	0_P	a	0_P	0_P	a	a	a	d
v	0_P	a	0_P	0_P	a	a	a	v
g	0_P	a	0_P	0_P	a	a	a	g
1_P	0_P	a	b	c	d	v	g	1_P

In Corollary 4.5, let P and Q be complete lattices, $Q_2 = \emptyset$, and F be a t-norm on Q. Then condition (ii) holds naturally. In fact, by Proposition 3.9 (iii), if $x, y \in P \setminus \{1_P\}$ with $x \leq y$, then $f^\wedge(F(f(x), f(z))) \leq f^\wedge(F(f(y), 1_Q)) \leq y$. Thus, Theorem 4.3 in [5] is obtained. Moreover, Theorem 4.3 in [5] requires that $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup [0_Q, f(0_P)[\cup \bigcup_{a\in\mathcal{A}}[0_Q, a[(\mathcal{A} = \{a \in Q \mid a \parallel f(0_P)\})$ for any $x, y \in P \setminus \{1_P\}$, while in Corollary 4.5, it is required that $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1$ for any $x, y \in P \setminus \{1_P\}$. Clearly, $[0_Q, f(0_P)[\cup \bigcup_{a\in\mathcal{A}}[0_Q, a[= [0_Q, f(0_P)[\cup \mathcal{A}$ and $[0_Q, f(0_P)[\cup \mathcal{A} \subseteq K_1$ holds, since $K_1 = [0_Q, f(0_P)[\cup \mathcal{A} \cup \{x \in Q \setminus \mathrm{Ran}(f) \mid\!\downarrow x \cap \mathrm{Ran}(f) = \{f(0_P)\}\}$. In summary, Theorem 4.3 in [5] is a special case of Corollary 4.5, and in Corollary 4.5, the range of possible values of $F(f(x), f(y))$ is larger for $x, y \in P \setminus \{1_P\}$.

Corollary 4.6. *Let P and Q be complete trellises, $f : P \to Q$ be a complete sup-homomorphism, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1 \subseteq Q_1$, $f^\vee : Q \to P$ be a quasi-inverse of f, $f^\vee(f(0_P)) = 0_P$ and $F : Q^2 \to Q$ be a commutative, increasing and associative function such that for any $x, y, z \in P \setminus \{1_P\}$ the following conditions are satisfied:*

(i) $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1$,

(ii) $x \trianglelefteq y$ *implies* $f^\vee(F(f(x), f(z))) \trianglelefteq y$.

Then the binary operator $T_{f^\vee} : P^2 \to P$ defined by

$$T_{f^\vee}(x, y) = \begin{cases} f^\vee(F(f(x), f(y))), & x, y \in P \setminus \{1_P\}, \\ x \wedge y, & \text{otherwise} \end{cases}$$

is a t-norm on P.

Additionally, if we require f to be an injective function, we can obtain the following method for constructing a t-norm on a complete trellis.

Theorem 4.7. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be an injective increasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^* : Q \to P$ be a quasi-inverse of f, $K_1 \subseteq Q_1$, $K_2 \setminus K_1 \subseteq Q_2$ and $F : Q^2 \to Q$ be a commutative and associative function with neutral element $f(1_P)$ such that for any $x, y, z, t \in P$ the following conditions are satisfied:*

(i) $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1 \cup K_2$,

(ii) $x \trianglelefteq y$, $z \trianglelefteq t$ *and* $F(f(x), f(z)) \in \mathrm{Ran}(f)$ *implies* $f^*(F(f(x), f(z))) \trianglelefteq f^*(F(f(y), f(t)))$.

Then the binary operator $T_{f^*} : P^2 \to P$ defined by

$$T_{f^*}(x, y) = f^*(F(f(x), f(y))), \tag{2}$$

is a t-norm on P.

Proof. According to (2) and Theorem 4.2, to prove that T_{f^*} is a t-norm on P, we just need to prove that 1_P is the neutral element of T_{f^*}, and $x \trianglelefteq y$ implies $f^*(F(f(x), f(z))) \trianglelefteq y$. Let $x \in P$. Since F with neutral element $f(1_P)$ and f is an injective function, we can get

$$T_{f^*}(x, 1_P) = f^*(F(f(x), f(1_P))) = f^*(f(x)) = x.$$

Table 5: The functions f_2 and f_2^* of Example 4.9.

f_2	0_P	a	b	c	d	1_P
	e	m	n	r	k	1_Q

f_2^*	0_Q	e	h	k	l	m	n	r	1_Q
	0_P	0_P	0_P	d	0_P	a	b	c	1_P

Since F is commutative, we can know that 1_P is the neutral element of T_{f^*}. If $x, y \in P$ with $x \trianglelefteq y$, then

$$T_{f^*}(x,y) = f^*(F(f(x), f(z))) \trianglelefteq f^*(F(f(y), f(1_P))) = y.$$

Therefore, T_{f^*} is a t-norm on P. □

Remark 4.8. (i) In Theorem 4.7, if $P = Q = [0,1]$, $f(0) = f(0^+)$, $f(1) = 1$, $F = T_P$ and $f^* = f^\wedge$, then f become a multiplicative generator of T_{f^*} on $[0,1]$ and Theorem 4.7 corresponds to the method of constructing a t-norm on $[0,1]$ using a multiplication generator in [16].

(ii) Based on Theorem 4.7, we propose the multiplicative generators theorem of t-norms on complete trellises. Let P be a complete trellis, $f : P \to [0,1]$ be an injective increasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^{(-1)} : [0,1] \longrightarrow P$ be a quasi-inverse of f, $K_1 \subseteq Q_1$ and $f(1_P) = 1$, such that for any $x, y, z, t, p, q \in P$ the following conditions are satisfied:

(i) $f(x) \cdot f(y) \in \mathrm{Ran}(f) \cup K_1$,

(ii) $x \trianglelefteq y$, $z \trianglelefteq t$, $f(p) = f(x) \cdot f(z)$ and $f(q) = f(y) \cdot f(t)$ implies $p \trianglelefteq q$.

Then the following binary operator $T : P^2 \to P$ defined by

$$T(x,y) = f^{(-1)}(f(x) \cdot f(y)),$$

is a t-norm on P and we call f a multiplicative generator of T. Under these conditions, $K_1 = [0, f(0_P)[\cup \{x \in [0,1] \mid \forall z \in P \setminus \{0_P\},\ f(0_P) < x < f(z)\}$.

Example 4.9. (i) Let $P = (\{0_P, a, b, c, d, 1_P\}, \trianglelefteq)$ be a complete trellis and $Q = (\{0_Q, e, h, k, l, m, n, r, 1_Q\}, \trianglelefteq)$ be a finite chain, whose Hasse diagrams are given in Fig. 3(a) and Fig. 3(b), respectively. Let $f_2 : P \to Q$ be given in Table 5, $Q_1 = \{0_Q, h\}$ and $Q_2 = \{l\}$. Then $K_1 = \{0_Q, h\} \subseteq Q_1$ and $K_2 \setminus K_1 = \{l\} \subseteq Q_2$. Since f_2 is an injective function, we can know $f_2^\wedge = f_2^\vee = f_2^*$. We take f_2^* in Table 5 and the binary function F_2 on Q such that

$$F_2(x,y) = \begin{cases} h, & (x,y) \in \{(k,k),(r,k),(k,r),(n,k),(k,n),(m,k),(k,m),(l,k),(k,l)\}, \\ l, & (x,y) \in \{(m,m),(n,m),(m,n),(r,m),(m,r)\}, \\ min(x,y), & \text{otherwise.} \end{cases}$$

It is easy to verify that F_2 is a t-norm on Q and the conditions (i)-(ii) are satisfied in Theorem 4.7. Then Table 6 shows the construction of a t-norm using F_2, f_2 and f_2^* by Theorem 4.7.

(ii) Let $P = (\{0_P, a, b, c, d, 1_P\}, \trianglelefteq)$ be a complete trellis with the Hasse diagram given by Fig. 3(a). In the following, we employ a multiplicative generator to construct a t-norm on P. We take $f_3 : P \longrightarrow [0,1]$, $[0,1] \setminus \mathrm{Ran}(f_3) \subseteq Q_1$ and the quasi-inverse $f_3^{(-1)} : [0,1] \longrightarrow P$ of f_3 in Table 7. It can be verified that f_3 and $f_3^{(-1)}$ satisfy conditions (i) and (ii) in Remark 4.8(2). Thus, the t-norm T_3 constructed via the multiplicative generator f_3 is given by Table 8.

From Proposition 3.9 and Theorem 4.7, we have the following corollary.

Corollary 4.10. *Let P and Q be complete trellises, $f : P \to Q$ be an order-embedding, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^* : Q \to P$ be a quasi-inverse of f, $K_1 \subseteq Q_1$, $f(1_P) = 1_Q$ and $F : Q^2 \to Q$ be a t-norm. If $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1$ for any $x, y \in P$, then the binary operator $T_{f^*} : P^2 \to P$ defined by*

$$T_{f^*}(x,y) = f^*(F(f(x), f(y))),$$

is a t-norm on P.

Table 6: A t-norm constructed by Theorem 4.7 using f_2, f_2^* and F_2.

T_2	0_P	a	b	c	d	1_P
0_P	0_P	0_P	0_P	0_P	0_P	0_P
a	0_P	0_P	0_P	0_P	0_P	a
b	0_P	0_P	b	b	0_P	b
c	0_P	0_P	b	c	0_P	c
d	0_P	0_P	0_P	0_P	0_P	d
1_P	0_P	a	b	c	d	1_P

Table 7: The function f_3 of Example 4.9.

f_3	0_P	a	b	c	d	1_P
	0	0.1	0.15	$\sqrt{0.15}$	0.14	1

$f_3^{(-1)}$	$[0, 0.1[$	$[0.1, 0.14[$	0.14	$]0.14, 0.15[$	0.15	$]0.15, \sqrt{0.15}[$	$\sqrt{0.15}$	$]\sqrt{0.15}, 1]$
	0_P	a	d	1_P	b	1_P	c	1_P

Table 8: A t-norm constructed by a multiplicative generator in Example 4.9.

T_3	0_P	a	b	c	d	1_P
0_P	0_P	0_P	0_P	0_P	0_P	0_P
a	0_P	0_P	0_P	0_P	0_P	a
b	0_P	0_P	0_P	0_P	0_P	b
c	0_P	0_P	0_P	b	0_P	c
d	0_P	0_P	0_P	0_P	0_P	d
1_P	0_P	a	b	c	d	1_P

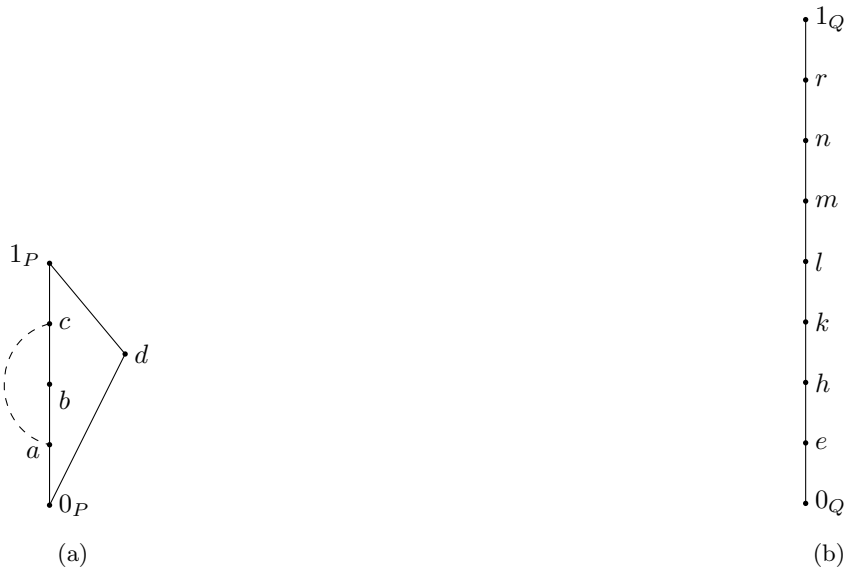


Fig. 3. Hasse diagrams of a complete trellis (left) and a finite chain (right) in Example 4.9.

When P and Q are both complete lattices in Corollary 4.10 and $Q_2 = \emptyset$, the conclusion generalizes Theorem 4.5 in [5] in two aspects. On the one hand, Theorem 4.5 in [5] requires that f be an injective complete inf-homomorphism, while Corollary 4.10 requires that f be an order embedding. In fact, on a complete lattice, if an unary function is an injective complete inf-homomorphism, then it can be shown to be an order-embedding. The converse, however, does not hold. On the other hand, Theorem 4.5 in [5] requires that $F(f(x), f(y)) \in \mathrm{Ran}(f) \cup [\,0_Q, f(0_P)[\cup \mathcal{A}$ for any $x, y \in P$. Clearly, $[\,0_Q, f(0_P)[\cup \mathcal{A} \subseteq K_1$. Next, we now turn to a discussion of the Archimedean property for the t-norms generated by the aforementioned method.

Proposition 4.11. *Let P and Q be complete trellises, $f : P \to Q$ be an injective increasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^* : Q \to P$ be a quasi-inverse of f, $K_1 \subseteq Q_1$ and $f(1_P) = 1_Q$. If F is an Archimedean t-norm on Q, $\downarrow K_1 \setminus \{f(0_P)\} = K_1$ and for any $x, y \in P$*

$$F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1,$$

then T_{f^*} defined by (2) is an Archimedean t-norm on P.

Proof. (i) We prove that for any $n \in \mathbb{N}$ and $r \in P$, $f(r)_F^{(n)} \in \mathrm{Ran}(f) \cup K_1$ using mathematical induction. If $n = 2$, then

$$f(r)_F^{(2)} = F(f(r), f(r)) \in \mathrm{Ran}(f) \cup K_1.$$

Suppose that $f(r)_F^{(n)} \in \mathrm{Ran}(f) \cup K_1$ when $n = k$. Let $n = k + 1$. Then

$$f(r)_F^{(k+1)} = F(f(r)_F^{(k)}, f(r)).$$

Since F is a t-norm, we can get $f(r)_F^{(k+1)} \trianglelefteq f(r)_F^{(k)}$. If $f(r)_F^{(k)} \in \mathrm{Ran}(f)$, then $f(r)_F^{(k+1)} \in \mathrm{Ran}(f) \cup K_1$. If $f(r)_F^{(k)} \in K_1$, then $f(r)_F^{(k+1)} \in \{f(0_P)\} \cup K_1$ by $\downarrow K_1 \setminus \{f(0_P)\} = K_1$. In summary, $f(r)_F^{(n)} \in \mathrm{Ran}(f) \cup K_1$.

(ii) We prove that for any $n \in \mathbb{N}$ and $r \in P$, $r_{T_{f^*}}^{(n)} = f^*(f(r)_F^{(n)})$ using mathematical induction. If $n = 2$, then

$$r_{T_{f^*}}^{(2)} = f^*(F(f(r), f(r))) = f^*(f(r)_F^{(2)}).$$

Suppose that $r_{T_{f^*}}^{(n)} = f^*(f(r)_F^{(n)})$ when $n = k$. Let $n = k + 1$. Then

$$r_{T_{f^*}}^{(k+1)} = T_{f^*}(r_{T_{f^*}}^{(k)}, r) = f^*(F(f(r), f(f^*(f(r)_F^{(k)})))).$$

If $f(r)_F^{(k)} \in \mathrm{Ran}(f)$, then

$$r_{T_{f^*}}^{(k+1)} = f^*(F(f(r), f(r)_F^{(k)})) = f^*(f(r)_F^{(k+1)}).$$

If $f(r)_F^{(k)} \in K_1$, then $f(r)_F^{(k+1)} \in \{f(0_P)\} \cup K_1$. Thus,

$$r_{T_{f^*}}^{(k+1)} = f^*(F(f(r), f(0_P))) = 0_P = f^*(f(r)_F^{(k+1)}).$$

In summary, $r_{T_{f^*}}^{(n)} = f^*(f(r)_F^{(n)})$.

Next we prove that T is Archimedean. Assume that T_{f^*} is not Archimedean. Then there exist $s, t \in P$ with $s \neq 1_P$, $t \neq 0_P$ and $t \trianglelefteq s_{T_{f^*}}^{(n)}$ for all $n \in \mathbb{N}$. $s \neq 1_P$ and $t \neq 0_P$, then $f(s) \neq 1_Q$ and $f(t) \neq 0_Q$ by f is injective. Since F is Archimedean, we can obtain there exists $n_1 \in \mathbb{N}$ such that $f(s)_F^{(n_1)} \lhd f(t)$ or $f(s)_F^{(n_1)} \parallel f(t)$. If $f(s)_F^{(n_1)} \in \mathrm{Ran}(f)$ and $f(s)_F^{(n_1)} \lhd f(t)$, then

$$s_{T_{f^*}}^{(n_1)} = f^*(f(s)_F^{(n_1)}) \lhd f^*(f(t)) = t,$$

or

$$s_{T_{f^*}}^{(n_1)} = f^*(f(s)_F^{(n_1)}) \parallel f^*(f(t)) = t,$$

by f is increasing. If $f(s)_F^{(n_1)} \in \mathrm{Ran}(f)$ and $f(s)_F^{(n_1)} \parallel f(t)$, then

$$s_{T_{f^*}}^{(n_1)} = f^*(f(s)_F^{(n_1)}) \parallel f^*(f(t)) = t,$$

by f is increasing. If $f(s)_F^{(n_1)} \in K_1$, then

$$s_{T_{f^*}}^{(n_1)} = f^*(f(s)_F^{(n_1)}) = 0_P \lhd t.$$

To sum up, $s_{T_{f^*}}^{(n_1)} \lhd t$ or $s_{T_{f^*}}^{(n_1)} \parallel t$. But this is a contradiction. Thus, T_{f^*} is Archimedean. □

Remark 4.12. (i) In Proposition 4.11, if Q is a complete lattice, then $\downarrow K_1 \setminus \{f(0_P)\} = K_1$ is naturally valid. Proposition 4.11 generalizes the conclusion of Proposition 4.12 in [5] in two aspects. On the one hand, Proposition 4.11 only requires that f be an injective increasing function, rather than injective complete inf-homomorphism. On the other hand, $[0_Q, f(0_P)[\cup \mathcal{A} \subseteq K_1$.

(ii) If a t-norm T on a complete trellis P has a multiplicative generator $f : P \longrightarrow [0,1]$, then T is Archimedean.

Proposition 4.13. *Let P be a complete trellis, T be a t-norm on P with a multiplicative generator $f : P \longrightarrow [0,1]$ and $f(x) \cdot f(y) \in \mathrm{Ran}(f) \cup [0, f(0_P)]$ for any $x, y \in P$. Then the following conclusions hold:*

(i) *$f(0_P) = 0$ if and only if T is strictly monotone;*

(ii) *$f(0_P) > 0$ if and only if every $a \in P \setminus \{0_P, 1_P\}$ is a nilpotent element of T.*

Proof. (i) Let $f(0_P) = 0$. Then $f(x) \cdot f(y) \in \mathrm{Ran}(f)$ for any $x, y \in P$. If $0_P \lhd s_1 \trianglelefteq s_2$ and $w_1 \lhd w_2$ for any $s_1, s_2, w_1, w_2 \in P$, then $f(s_1) \cdot f(w_1), f(s_2) \cdot f(w_2) \in \mathrm{Ran}(f)$ and $f(s_1) \cdot f(w_1) \lhd f(s_2) \cdot f(w_2)$. Thus,

$$T(s_1, w_1) = f^{(-1)}(f(s_1) \cdot f(w_1)) \lhd f^{(-1)}(f(s_2) \cdot f(w_2)) = T(s_2, w_2).$$

Let T is strictly monotone. Assumes that $f(0_P) > 0$, then for any $a \in P \setminus \{0_P, 1_P\}$, $f(a) > 0$. Since the multiplication operation on $[0,1]$ is Archimedean, we can obtain $f(a)^n < f(0_P)$ for some $n \in \mathbb{N}$. Hence, $a_T^{(n)} = f^{(-1)}(f(a)^n) = 0_P$, which implies that a is a nilpotent element of T. But this is a contradiction. Thus, $f(0_P) = 0$.

(ii) According to the proof of Proposition 4.13(i), if $f(0_P) > 0$, then every $a \in P \setminus \{0_P, 1_P\}$ is a nilpotent element of T. If every $a \in P \setminus \{0_P, 1_P\}$ is a nilpotent element of T, then T is not strictly monotone, and thus, $f(0_P) > 0$. □

4.2 Constructing a t-norm using a decreasing function and its quasi-inverse

Below we will list the methods of constructing t-norms using decreasing functions and their pseudo-inverses. This part is analogous to the method based on increasing functions and their quasi-inverses. For this reason, we state only the results and omit the proofs.

Theorem 4.14. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be a non-constant decreasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1^{'} \subseteq Q_1$, $K_2^{'} \setminus K_1^{'} \subseteq Q_2$, $f^* : Q \to P$ be a quasi-inverse of f and $F : Q^2 \to Q$ be a commutative and associative function such that for any $x, y, z, t \in P \setminus \{1_P\}$ the following conditions are satisfied:*

(i) *$x \trianglelefteq y$ implies $f^*(F(f(x), f(z))) \trianglelefteq y$,*

(ii) *$F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1^{'} \cup K_2^{'}$,*

(iii) *$x \trianglelefteq y$, $z \trianglelefteq t$ and $F(f(x), f(z)) \in \mathrm{Ran}(f)$ implies $f^*(F(f(x), f(z))) \trianglelefteq f^*(F(f(y), f(t)))$.*

Then the binary operator $T_{f^*} : P^2 \to P$ defined by

$$T_{f^*}(x, y) = \begin{cases} f^*(F(f(x), f(y))), & x, y \in P \setminus \{1_P\}, \\ x \wedge y, & \text{otherwise} \end{cases}$$

is a t-norm on P.

The following two corollaries are obtained by combining Theorem 4.14 with items (iii) and (v) of Proposition 3.13, respectively.

Corollary 4.15. *Let P and Q be complete trellises, $f : P \to Q$ be a complete inf-sup-homomorphism, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1^{'} \subseteq Q_1$, $f^{\wedge} : Q \to P$ be a quasi-inverse of f and $F : Q^2 \to Q$ be a commutative, increasing and associative function such that for any $x, y, z \in P \setminus \{1_P\}$ the following conditions are satisfied:*

(i) *$F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1^{'}$,*

(ii) *$x \trianglelefteq y$ implies $f^{\wedge}(F(f(x), f(z))) \trianglelefteq y$.*

Then the binary operator $T_{f^\wedge} : P^2 \to P$ defined by

$$T_{f^\wedge}(x,y) = \begin{cases} f^\wedge(F(f(x),f(y))), & x,y \in P \setminus \{1_P\}, \\ x \wedge y, & \text{otherwise} \end{cases}$$

is a t-norm on P.

Corollary 4.16. *Let P and Q be complete trellises, $f : P \to Q$ be a complete sup-inf-homomorphism, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $K_1^{'} \subseteq Q_1$, $f^\vee : Q \to P$ be a quasi-inverse of f, $f^\vee(f(0_P)) = 0_P$ and $F : Q^2 \to Q$ be a commutative, increasing and associative function such that for any $x,y,z \in P \setminus \{1_P\}$ the following conditions are satisfied:*

(i) $F(f(x),f(y)) \in \mathrm{Ran}(f) \cup K_1^{'}$,

(ii) $x \trianglelefteq y$ *implies* $f^\vee(F(f(x),f(z))) \trianglelefteq y$.

Then the binary operator $T_{f^\vee} : P^2 \to P$ defined by

$$T_{f^\vee}(x,y) = \begin{cases} f^\vee(F(f(x),f(y))), & x,y \in P \setminus \{1_P\}, \\ x \wedge y, & \text{otherwise} \end{cases}$$

is a t-norm on P.

Theorem 4.17. *Let P be a complete trellis, Q be a psoset, $f : P \to Q$ be an injective decreasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^* : Q \to P$ be a quasi-inverse of f, $K_1^{'} \subseteq Q_1$, $K_2^{'} \setminus K_1^{'} \subseteq Q_2$ and $F : Q^2 \to Q$ be a commutative and associative function with neutral element $f(1_P)$ such that for any $x,y,z,t \in P$ the following conditions are satisfied:*

(i) $F(f(x),f(y)) \in \mathrm{Ran}(f) \cup K_1^{'} \cup K_2^{'}$,

(ii) $x \trianglelefteq y$, $z \trianglelefteq t$ *and* $F(f(x),f(z)) \in \mathrm{Ran}(f)$ *implies* $f^*(F(f(x),f(z))) \trianglelefteq f^*(F(f(y),f(t)))$.

Then the binary operator $T_{f^*} : P^2 \to P$ defined by

$$T_{f^*}(x,y) = f^*(F(f(x),f(y))),$$

is a t-norm on P.

Remark 4.18. (i) Distinguishingly, if $P = [0,1]$, $Q = [0,\infty]$, $F = +$ (Since for all $x \in [0,\infty]$, $x + \infty = \infty$, the restriction $+\big|_{[0,\infty]^2}$ can be regarded as a t-conorm on $[0,\infty]$), $f(1) = 0$ and $f(0^+) = f(0)$ in Theorem 4.17, then we can obtain that f is an additive generator of T_{f^*} on $[0,1]$ and Theorem 4.15 corresponds to the method of constructing a t-norm on $[0,1]$ using an additive generator in [16].

(ii) Based on Theorem 4.17, we propose the additive generators theorem of t-norms on complete trellises. Let P be a complete trellis, $f : P \to [0,\infty]$ be an injective decreasing function, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^{(-1)} : [0,\infty] \to P$ be a quasi-inverse of f, $K_1^{'} \subseteq Q_1$ and $f(1_P) = 0$, such that for any $x,y,z,t,p,q \in P$ the following conditions are satisfied:

(i) $f(x) + f(y) \in Ran(f) \cup K_1^{'}$,

(ii) $x \trianglelefteq y$, $z \trianglelefteq t$, $f(p) = f(x) + f(z)$ and $f(q) = f(y) + f(t)$ implies $p \trianglelefteq q$.

Then the following binary operator $T : P^2 \to P$ defined by

$$T(x,y) = f^{(-1)}(f(x) + f(y)),$$

is a t-norm on P and we call f an additive generator of T. Under these conditions, $K_1^{'} =]f(0_P),\infty] \bigcup \{x \in [0,\infty] \mid \forall z \in P \setminus \{0_P\},\ f(z) < x < f(0_P)\}$.

(iii) Based on the additive generator of the t-norm complete trellises given in (2), if $Q_2 = \emptyset$ and P is a bounded lattice such that f^* exists, then the additive generator of the t-norm on bounded lattices can be obtained. This conclusion is a generalization of Theorem 3.5 in [22], since in this case $]f(0_P),\infty] \subseteq K_1^{'}$.

From Proposition 3.13 and Theorem 4.14, we have the following corollary.

Corollary 4.19. *Let P and Q be complete trellises, $f : P \to Q$ be an order-reversing embedding, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^* : Q \to P$ be a quasi-inverse of f, $K_1^{'} \subseteq Q_1$, $f(1_P) = 0_Q$ and $F : Q^2 \to Q$ be a t-conorm. If $F(f(x), f(y)) \in \mathrm{Ran}(f) \bigcup K_1^{'}$ for any $x, y \in P$, then the binary operator $T_{f^*} : P^2 \to P$ defined by*

$$T_{f^*}(x, y) = f^*(F(f(x), f(y))),$$

is a t-norm on P.

Proposition 4.20. *Let P and Q be complete trellises, $f : P \to Q$ be an injective decreasing, $Q \setminus \mathrm{Ran}(f) = Q_1 \sqcup Q_2$, $f^* : Q \to P$ be a quasi-inverse of f, $K_1^{'} \subseteq Q_1$ and $f(1_P) = 0_Q$. If F is an Archimedean t-conorm on Q, $\uparrow K_1^{'} \setminus \{f(0_P)\} = K_1^{'}$ and for any $x, y \in P$*

$$F(f(x), f(y)) \in \mathrm{Ran}(f) \cup K_1^{'},$$

then T_{f^*} defined by (2) is an Archimedean t-norm on P.

In Proposition 4.20, if Q is a complete lattice, then $\uparrow K_1^{'} \setminus \{f(0_P)\} = K_1^{'}$ is naturally valid. If a t-norm T on a complete trellis P has an additive generator $f : P \longrightarrow [0, 1]$, then T is Archimedean.

Proposition 4.21. *Let P be a complete trellis, T be a t-norm on P with an additive generator $f : P \longrightarrow [\,0, \infty\,]$ and $f(x) + f(y) \in \mathrm{Ran}(f) \cup [\,f(0_P), \infty\,]$ for any $x, y \in P$. Then the following conclusions hold:*

(i) *$f(0_P) = \infty$ if and only if T is strictly monotone;*

(ii) *$f(0_P) < \infty$ if and only if every $a \in P \setminus \{0_P, 1_P\}$ is a nilpotent element of T.*

5 Conclusions

In the present article, we have developed several methods for constructing t-norms on complete trellises using monotone functions and their quasi-inverses. These approaches are built upon the introduction of quasi-inverse and pseudo-inverse concepts for monotone functions on complete trellises. These concepts not only generalize the corresponding notions for real intervals, but also unify the two existing definitions of pseudo-inverse on complete lattices. The corresponding results for t-conorms can be derived dually. Several problems remain open for further investigation: for instance, can alternative forms be proposed for defining quasi-inverses and pseudo-inverses? How can additive and multiplicative generators be utilized to characterize t-norms and t-conorms on complete trellises? These questions are the subject of our future investigation.

Declarations

Data availability: No data.
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Declaration of Generative AI and AI-Assisted Technologies:
The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript.

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