Iranian Journal of Fuzzy Systems

Volume *, Number *, (****), pp. **

Original paper

Directional footrule-coefficients

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Abstract

Rank-based dependence measures such as Spearman's footrule are robust and invariant, but they often fail to capture directional or asymmetric dependence in multivariate settings. This paper introduces a new family of directional Spearman's footrule-coefficients for multivariate data, defined within the copula framework to clearly separate marginal behavior from dependence structure. We establish their main theoretical properties: vanishing under independence, attaining their extreme values under comonotonicity in a manner that depends on the direction considered; a reflection property linking a direction to its opposite via the survival copula; and a zero-sum symmetry across all directions. Nonparametric rank-based estimators are proposed and their asymptotic consistency is discussed along with an illustrative application to real data. Explicit expressions for several known families of copulas illustrate the ability of the proposed coefficients to detect directional dependence patterns undetected by classical measures.

Keywords: Copula, directional dependence, multivariate dependence, nonparametric estimators, Spearman's footrule.

2020 MSC: 62H05, 62H12, 62H20.

1 Introduction

Dependence among random variables has long been a central topic in the statistical literature, owing to its fundamental role in the modeling and analysis of multivariate stochastic systems (see for instance [19]).

Alternatively, copulas provide a powerful framework for modeling the dependence structure among random variables [1, 10, 12, 25]. By separating the marginal distributions from the joint dependence, they allow for a more nuanced characterization of relationships, capturing features such as tail dependence, asymmetries, and directional dependence that traditional correlation measures cannot. This flexibility makes copulas particularly valuable in multivariate contexts, where understanding the intricate interactions among variables is essential.

In this paper, we investigate the measurement of dependence among the components of a d-dimensional random vector ($d \geq 2$) modified along a prescribed direction $\alpha \in \{-1, 1\}^d$, within the framework of copula-based methods. More precisely, we introduce and analyze a directional extension of the multivariate Spearman's footrule coefficient. The classical bivariate version of Spearman's footrule was originally proposed by Spearman in [32] and later extended to the multivariate setting in [33] (for additional studies of this coefficient, see, e.g., [17, 28]). Recent contributions have further investigated the statistical properties and applications of Spearman's footrule. In particular, [37] establishes asymptotic representations for the classical bivariate coefficient under independence, deriving results on its asymptotic normality and the approximation of its null distribution. From a different perspective, [14] develops a dimension-reduction framework for quantifying directed dependence, showing how Spearman's footrule, applied to a suitably constructed bivariate copula, can be used to measure the dependence of a response variable on a vector of explanatory

Corresponding Author: M. Úbeda-Flores
Received: May 2026; Revised: October 2026; Accepted: November 2026.
https://doi.org/10.22111/ijfs.2026.55378.9825, https://ijfs.usb.ac.ir, Print ISSN: 1735-0654

variables. These studies highlight the continuing relevance of Spearman's footrule in the analysis of dependence, while addressing questions different from the one considered here.

Despite these advances, classical multivariate versions of Spearman's footrule are symmetric by construction: they aggregate dependence equally across all directions, treating large simultaneous values in the same way as small simultaneous values. This symmetry is a fundamental limitation when the dependence structure is inherently asymmetric—for instance, when some variables tend to take large values while others tend to take small values simultaneously, a situation that arises naturally in economics (e.g., income and deprivation), reliability engineering (e.g., failure times under opposing stress), or environmental sciences (e.g., temperature and precipitation). The existing coefficients φ_d^- and φ_d^+, introduced in [7, 8], partially address this by distinguishing lower and upper tail behavior, but they only cover the two extreme directions $\alpha = \mathbf{1} = (1, 1, \cdots, 1)$ and $\alpha = -\mathbf{1} = (-1, -1, \cdots, -1)$, leaving the $2^d - 2$ intermediate directions unexplored. The present paper fills this gap by defining a full family of directional footrule-coefficients φ_d^α for every direction $\alpha \in \{-1, 1\}^d$.

The remainder of the paper is organized as follows. Section 2 reviews the necessary background in copula theory and introduces directional dependence concepts that are essential for the development of our results. Section 3 is devoted to the definition of multidimensional directional footrule-coefficients for arbitrary directions α. We derive their main properties and examine their theoretical implications, with particular emphasis on their representation as linear combinations of lower-dimensional directional footrule-coefficients. This representation sheds light on the contribution of dependence among subsets of components to the overall dependence structure. We also present illustrative examples for well-known families of copulas. Section 4 introduces nonparametric rank-based estimators for the proposed coefficients and presents Monte Carlo simulation results, which illustrate their finite-sample performance and confirm the main theoretical properties. It also includes an application to red and white wine datasets to investigate directional dependence among physicochemical properties and their relationship with wine quality, as an illustrative example. Finally, Section 5 is devoted to conclusions.

2 Copulas and directional dependence

Copulas are a widely used tool in probability and statistics literature. Specifically, a d-dimensional copula, C, is a function $C : [0, 1]^d \longrightarrow [0, 1]$ satisfying

i) $C(\mathbf{u}) = 0$ whenever one of the components of $\mathbf{u} \in [0, 1]^d$ equals zero and $C(\mathbf{u}) = u_k$, whenever every coordinate of $\mathbf{u}$ equals 1 but u_k.

ii) For every $\mathbf{a} = (a_1, a_2, \cdots, a_d)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_d) \in [0, 1]^d$, such that $a_k \leq b_k$ for all $k = 1, \cdots, d$,

$$\sum_{\mathbf{c}} \operatorname{sgn}(\mathbf{c}) \cdot C(\mathbf{c}) \geq 0,$$

where the sum is taken over the vertices $\mathbf{c} = (c_1, c_2, \cdots, c_d)$ of $[\mathbf{a}, \mathbf{b}]$, i.e., each c_k is equal to either a_k or b_k, and $\operatorname{sgn}(\mathbf{c}) = 1$ if $c_k = a_k$ for an even number of k's, and $\operatorname{sgn}(\mathbf{c}) = -1$ otherwise.

Equivalently, a copula is a multivariate distribution function that is defined as $C(\mathbf{u}) = \mathbb{P}[U_1 \leq u_1, \cdots, U_d \leq u_d]$, where U_i is standard uniform in $[0, 1]$. The relevance of copula theory lies in Sklar's theorem [31], which states that any multivariate distribution function can be expressed in terms of a copula and its univariate marginal distribution functions. Moreover, the copula is unique when all marginal distribution functions are continuous. For a review of this result and of its different proofs, see [34].

Independence is represented by the product copula defined as $\Pi_d(\mathbf{u}) = \prod_{i=1}^d u_i$ for all $\mathbf{u} \in [0, 1]^d$, and maximal positive dependence by the upper Fréchet-Hoeffding bound $M_d(\mathbf{u}) = \min_{1 \leq i \leq d}\{u_i\}$, where subscripts in Π_d and M_d mean dimension.

Let $\widehat{C}$ be the *survival d-copula* of C [36], i.e., $\widehat{C}(\mathbf{u}) = \mathbb{P}[\mathbf{U} > \mathbf{1} - \mathbf{u}] = \overline{C}(1 - \mathbf{u})$, where $\mathbf{U}$ is a random d-vector whose components are uniform $[0, 1]$ and with associated d-copula C; and $\overline{C}(\mathbf{u}) = \mathbb{P}[\mathbf{U} > \mathbf{u}]$ is called the *survival function*.

The diagonal section of a d-copula C is a function $\delta_C : [0, 1] \to [0, 1]$, defined as $\delta_C(u) = C(u, \ldots, u)$ for all $u \in [0, 1]$.

For further information about copulas, see [10, 25].

Two of the most widely recognized concepts used to study potential dependence relationships among random variables are positive (respectively, negative) upper orthant dependence (PUOD) (respectively, NUOD) and positive (respectively, negative) lower orthant dependence (PLOD) (respectively, NLOD). Intuitively, PUOD and NUOD measure how a set of random variables $X_1, \cdots, X_d$ deviate from independence in terms of their likelihood of attaining large values simultaneously. In particular, under PUOD, random variables tend to attain large values simultaneously more

frequently than independent random variables sharing the same marginal distributions, whereas this joint behavior is less pronounced under NUOD (see, e.g., [21, 23, 25, 35] for results on positive dependence properties based on copulas). Nonetheless, dependence among random variables may be described through alternative frameworks. For example, for a given subset $J \subset \{1, 2, \cdots, d\}$, one may encounter dependence structures in which extreme (large or small) realizations of the variables $\{X_j : j \in J\}$ are consistently linked to opposite extreme realizations of the complementary variables $\{X_j : j \notin J\}$. This concept, investigated in [29] (see also [30]), motivates the following directional dependence framework, which underpins the construction of our coefficients.

The notion of directional dependence used in this paper is based on the concept of *positive dependence according to a direction* $\alpha \in \{-1, 1\}^d$, introduced in [29]. We say that $\mathbf{X}$ is *positively dependent in direction* α, denoted $\mathrm{PD}(\alpha)$, if

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i X_i > x_i)\right] \geq \prod_{i=1}^{d}\mathbb{P}[\alpha_i X_i > x_i] \qquad \text{for every } \mathbf{x} \in [-\infty, +\infty]^d.$$

The cases $\mathrm{PD}(\mathbf{1})$ and $\mathrm{PD}(-\mathbf{1})$ recover the classical notions of positive upper and lower orthant dependence, respectively. For two random vectors $\mathbf{X}$ and $\mathbf{Y}$ with respective d-copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$, we write $\mathbf{X} \leq_{\mathrm{PD}(\alpha)} \mathbf{Y}$ if

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i X_i > x_i)\right] \leq \mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i Y_i > x_i)\right] \qquad \text{for every } \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

following [3]. This order is used later in Corollary 3.11 to establish the monotonicity of the directional footrule-coefficients with respect to the dependence strength.

A key technical tool in the subsequent construction is the copula associated with a componentwise-reflected random vector. The following result, which formally appears in [13] via induction, is presented here with an alternative proof (see Appendix) that better illuminates the copula-theoretic structure underlying our framework.

Let $\mathbf{X}$ be a d-dimensional random vector with associated d-copula C and marginals uniformly distributed on $[0, 1]$. Let $J = \{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, d\}$ with $i_1 < i_2 < \cdots < i_m$, and $I = \{1, 2, \cdots, d\} \setminus J$. Let $C_J : [0, 1]^m \longrightarrow [0, 1]$ be the *J-marginal* copula of C for the random vector $(X_{i_1}, X_{i_2}, \cdots, X_{i_m})$, which is defined by setting $d - m$ arguments of C equal to 1, i.e.,

$$C_J(u_{i_1}, u_{i_2}, \cdots, u_{i_m}) = C(v_1, v_2, \cdots, v_d),$$

where $v_i = u_i$ if $i \in J$, and $v_i = 1$ otherwise. For any subset $J \subseteq \{1, 2, ..., d\}$, we define the complemented random vector

$$\mathbf{X}^{(J)} = (Y_1, \ldots, Y_d), \qquad Y_j = \begin{cases} X_j, & j \notin J, \\ 1 - X_j, & j \in J, \end{cases} \tag{2}$$

and denote by $C^{(J)}$ its associated d-copula. The number of complemented coordinates is $m = |J|$. Its associated copula can be expressed as follows:

Theorem 2.1. *For any $J \subseteq \{1, 2, \cdots, d\}$, the copula $C^{(J)}$ associated with $\mathbf{X}^{(J)}$ is given by*

$$C^{(J)}(\mathbf{u}) = \sum_{K \subseteq J} (-1)^{|K|}\, C_{I \cup K}\Big(\nu(\mathbf{u}, I, K)\Big), \tag{3}$$

for all $\mathbf{u} \in [0, 1]^d$, where, for each subset $K \subseteq J$, the mapping $\nu(\mathbf{u}, I, K)$ is defined such that it assigns u_i if $i \in I$, and $1 - u_i$ if $i \in K$.

The expression for $C^{(J)}$ obtained in Theorem 2.1 will be used directly in the next section to compute the directional footrule-coefficients.

3 Directional version of multivariate Spearman's footrule

Numerous coefficients have been proposed to capture positive or negative dependence among random variables (see, e.g., [9, 20, 24]); however, those designed to measure the directional dependence of the components of a random vector are considerably less common, particularly in the multivariate context—for dimensions 2 and 3, see, e.g., [16]. Nevertheless, they generalize the former and are capable of capturing dependence relationships that the traditional coefficients fail to

detect. Inspired by the multivariate version of Spearman's footrule for a d-dimensional random vector $\mathbf{U}$ with copula C introduced in [33] as

$$\varphi_d(C) = \frac{d+1}{d-1}\int_0^1 \left[\delta_C(u) + \delta_{\widehat{C}}(u)\right] \mathrm{d}u - \frac{2}{d-1},$$

in [7] two multivariate versions of this coefficient were described, called *downward* and *upward diagonal dependence indices*, respectively:

$$\varphi_d^-(C) = \frac{\int_0^1 \left[\delta_C(u) - \delta_\Pi(u)\right] \mathrm{d}u}{\int_0^1 \left[\delta_M(u) - \delta_\Pi(u)\right] \mathrm{d}u} = \frac{2(d+1)}{(d-1)}\int_0^1 \delta_C(u)\,\mathrm{d}u - \frac{2}{d-1},$$

and

$$\varphi_d^+(C) = \frac{\int_0^1 \left[\delta_{\widehat{C}}(u) - \delta_{\widehat{\Pi}}(u)\right] \mathrm{d}u}{\int_0^1 \left[\delta_{\widehat{M}}(u) - \delta_{\widehat{\Pi}}(u)\right] \mathrm{d}u} = \frac{2(d+1)}{(d-1)}\int_0^1 \delta_{\widehat{C}}(u)\,\mathrm{d}u - \frac{2}{d-1},$$

(see also [8]). Intuitively, the first coefficient measures the area between the diagonal section of C and the diagonal section of the independence copula, normalized by the full area between the main diagonal, which represents maximal positive dependence and the curve representing independence, and the second one measures the normalized area between the diagonal section of the survival copula $\widehat{C}$ and the curve that represents independence. It is possible to rewrite the coefficients as

$$\varphi_d^-(C) = \frac{2(d+1)}{(d-1)}\int_0^1 \mathbb{P}[U_1 \le u, \cdots, U_d \le u]\,\mathrm{d}u - \frac{2}{d-1},$$

and

$$\varphi_d^+(C) = \frac{2(d+1)}{(d-1)}\int_0^1 \mathbb{P}[U_1 > u, \cdots, U_d > u]\,\mathrm{d}u - \frac{2}{d-1}.$$

From these last expressions, we define a *directional version of Spearman's footrule* (or *directional footrule-coefficients*, for short) as follows

$$\varphi_d^{\boldsymbol{\alpha}}(C) = \frac{2(d+1)}{d-1}\int_0^1 \left(\mathbb{P}[\alpha_1 U_1 > \alpha_1 u, \cdots, \alpha_d U_d > \alpha_d u] - \prod_{i=1}^d \mathbb{P}[\alpha_i U_i > \alpha_i u]\right) \mathrm{d}u. \tag{4}$$

When $\boldsymbol{\alpha} = \mathbf{1}$, we have $\varphi_d^{\boldsymbol{\alpha}} = \varphi_d^+$; and when $\boldsymbol{\alpha} = -\mathbf{1}$, then $\varphi_d^{\boldsymbol{\alpha}} = \varphi_d^-$. Since the variables are uniformly distributed on $[0,1]$ and the event $U_i > u$ has probability $1-u$, it holds

$$\varphi_d^{\boldsymbol{\alpha}}(C) = \frac{2(d+1)}{d-1}\int_0^1 \mathbb{P}[\alpha_1 U_1 > \alpha_1 u, \cdots, \alpha_d U_d > \alpha_d u]\mathrm{d}u - \frac{2}{(d-1)\binom{d}{|J|}},$$

where $J = \{i \in \{1, \cdots, d\} : \alpha_i = 1\}$ and $I = \{1, \cdots, d\} \setminus J$.

Some of the main properties of these directional footrule-coefficients are listed in the following result.

Corollary 3.1. *Let* $\mathbf{U}$ *be a* d*-dimensional random vector with uniformly distributed marginals on* $[0,1]$ *and associated copula* C*, then*

i) $\sum_{\boldsymbol{\alpha}} \varphi_d^{\boldsymbol{\alpha}}(C) = 0$.

ii) $\varphi_d(C) = \sum_{\boldsymbol{\alpha} \neq \mathbf{1}, -\mathbf{1}} -\frac{1}{2}\varphi_d^{\boldsymbol{\alpha}}(C)$.

iii) $\varphi_d^{\boldsymbol{\alpha}}(C) = \varphi_d^{-\boldsymbol{\alpha}}(\widehat{C})$.

Proof. We prove part i). By (4), we have

$$\sum_{\boldsymbol{\alpha}} \varphi_d^{\boldsymbol{\alpha}}(C) = \frac{2(d+1)}{d-1}\int_0^1 \left(\sum_{\boldsymbol{\alpha}} \mathbb{P}\left[\bigcap_{i=1}^d (\alpha_i U_i > \alpha_i u)\right] - \sum_{\boldsymbol{\alpha}} \prod_{i=1}^d \mathbb{P}[\alpha_i U_i > \alpha_i u]\right) \mathrm{d}u.$$

For the first inner sum, note that the 2^d events $\bigcap_{i=1}^d (\alpha_i U_i > \alpha_i u)$, indexed by $\boldsymbol{\alpha} \in \{-1, 1\}^d$, are pairwise almost surely disjoint —they can only overlap on $\{U_i = u\}$ for some i, which has probability zero since each U_i is continuous—, and their union equals $\bigcap_{i=1}^d (U_i \neq u)$, which also has probability one. Therefore,

$$\sum_{\boldsymbol{\alpha}} \mathbb{P}\left[\bigcap_{i=1}^d (\alpha_i U_i > \alpha_i u)\right] = \mathbb{P}\left[\bigcap_{i=1}^d (U_i \neq u)\right] = 1.$$

For the second inner sum, since the factors depend only on the sign of each α_i independently,

$$\sum_{\alpha}\prod_{i=1}^{d}\mathbb{P}[\alpha_i U_i > \alpha_i u] = \prod_{i=1}^{d}\big(\mathbb{P}[U_i > u] + \mathbb{P}[U_i < u]\big) = \prod_{i=1}^{d} 1 = 1.$$

Therefore, the integrand is $1 - 1 = 0$ for (Lebesgue-almost) every u, and $\sum_\alpha \varphi_d^\alpha(C) = 0$.

Now we prove part ii). From part i),

$$\varphi_d^{\mathbf{1}}(C) + \varphi_d^{-\mathbf{1}}(C) + \sum_{\alpha\neq\pm\mathbf{1}} \varphi_d^\alpha(C) = 0,$$

so

$$\frac{\varphi_d^{\mathbf{1}}(C) + \varphi_d^{-\mathbf{1}}(C)}{2} = \sum_{\alpha\neq\pm\mathbf{1}} \left(-\frac{1}{2}\right)\varphi_d^\alpha(C).$$

By definition, $\varphi_d^{\mathbf{1}} = \varphi_d^+$ and $\varphi_d^{-\mathbf{1}} = \varphi_d^-$, and it is shown in [8] that $\varphi_d(C) = (\varphi_d^+(C) + \varphi_d^-(C))/2$. Combining these two facts gives the desired identity.

Finally, to prove part iii), let $\mathbf{V} = \mathbf{1} - \mathbf{U} = (1-U_1, \cdots, 1-U_d)$. Since $\mathbf{U}$ has copula C and each $U_i \sim \text{Uniform}(0,1)$, the random vector $\mathbf{V}$ also has uniform marginals and its associated copula is $\widehat{C}$. By definition,

$$\varphi_d^{-\alpha}(\widehat{C}) = \frac{2(d+1)}{d-1}\int_0^1 \left(\mathbb{P}\left[\bigcap_{i=1}^{d}(-\alpha_i V_i > -\alpha_i u)\right] - \prod_{i=1}^{d}\mathbb{P}[-\alpha_i V_i > -\alpha_i u]\right) \mathrm{d}u.$$

Since $V_i = 1 - U_i$, we have $-\alpha_i V_i = -\alpha_i(1-U_i) = \alpha_i U_i - \alpha_i$, so the event $(-\alpha_i V_i > -\alpha_i u)$ is equivalent to $(\alpha_i U_i > \alpha_i(1-u))$. Therefore,

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(-\alpha_i V_i > -\alpha_i u)\right] = \mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i U_i > \alpha_i(1-u))\right],$$

and

$$\prod_{i=1}^{d}\mathbb{P}[-\alpha_i V_i > -\alpha_i u] = \prod_{i=1}^{d}\mathbb{P}[\alpha_i U_i > \alpha_i(1-u)].$$

Substituting and applying the change of variables $u \mapsto 1-u$ —which maps $[0,1]$ to itself and has unit Jacobian—:

$$\begin{aligned}
\varphi_d^{-\alpha}(\widehat{C}) &= \frac{2(d+1)}{d-1}\int_0^1 \left(\mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i U_i > \alpha_i(1-u))\right] - \prod_{i=1}^{d}\mathbb{P}[\alpha_i U_i > \alpha_i(1-u)]\right)\mathrm{d}u \\
&= \frac{2(d+1)}{d-1}\int_0^1 \left(\mathbb{P}\left[\bigcap_{i=1}^{d}\{\alpha_i U_i > \alpha_i u'\}\right] - \prod_{i=1}^{d}\mathbb{P}[\alpha_i U_i > \alpha_i u']\right)\mathrm{d}u' \\
&= \varphi_d^\alpha(C),
\end{aligned}$$

where $u' = 1 - u$. This completes the proof. □

In [26], it was conjectured that multidimensional directional ρ-coefficients—another type of multidimensional measure of directionality—can be expressed as a linear combination of lower-dimensional directional coefficients. That conjecture was proved in [4]. Now we show that the new directional coefficients can also be expressed in terms of lower-dimensional coefficients.

Theorem 3.2. *Let $\mathbf{X} = (X_1, \cdots, X_d)$ be a d-dimensional random vector whose marginals are uniform on $[0,1]$, let C be its associated d-copula, and let $\alpha \in \mathbb{R}^d$ be such that $\alpha_i \in \{-1, 1\}$ for all $i = 1, \cdots, d$. Let $I \subseteq \{1, \cdots, d\}$ be the set of indices such that $\alpha_i = -1$ if $i \in I$, and let $J = \{1, \cdots, d\} \setminus I$ be the set of indices such that $\alpha_i = 1$ if $i \in J$. Assume $|I| \geq 1$ and $|J| \geq 1$ (i.e., $\alpha \neq \pm\mathbf{1}$). Then*

$$\varphi_d^\alpha(C) = \frac{2(d+1)}{d-1}\cdot \sum_{\substack{S\subseteq J \\ |I|+|S|\geq 2}} (-1)^{|S|}\frac{|I|+|S|-1}{2(|I|+|S|+1)}\,\varphi^-_{X_{I\cup S}}, \tag{5}$$

where $\mathbf{X}_K$ denotes the random subvector $(X_i)_{i\in K}$ for any $K \subseteq \{1, \cdots, d\}$, and $\varphi^-_{X_K}$ denotes the coefficient $\varphi^-_{|K|}$ applied to $\mathbf{X}_K$.

Proof. By using ([2, Lemma 3.4 and Remark 3.5]), it holds

$$\varphi_d^{\alpha}(C) = \frac{2(d+1)}{d-1}\int_0^1 \mathbb{P}[\alpha_1 U_1 > \alpha_1 u, \cdots, \alpha_d U_d > \alpha_d u]\mathrm{d}u - \frac{2}{(d-1)\binom{d}{|J|}}$$
$$= \frac{2(d+1)}{d-1}\int_0^1 \left(\sum_{k=0}^{|J|}(-1)^k \sum_{\substack{S\subseteq J\\ |S|=k}} C(\mathbf{u}_{I\cup S}, \mathbf{1})\right)\mathrm{d}u - \frac{2}{(d-1)\binom{d}{|J|}},$$

where the arguments with indices outside $I \cup S$ are fixed at 1. After some elementary calculations, we obtain

$$\int_0^1 C(\mathbf{u}_{I\cup S}, \mathbf{1})\mathrm{d}u = \frac{|I|+|S|-1}{2(|I|+|S|+1)}\left(\varphi^-_{\mathbf{X}_{I\cup S}} + \frac{2}{|I|+|S|-1}\right) = \frac{|I|+|S|-1}{2(|I|+|S|+1)}\varphi^-_{\mathbf{X}_{I\cup S}} + \frac{1}{|I|+|S|+1},$$

since

$$\varphi^-_{\mathbf{X}_{I\cup S}}(C) = \frac{2(|I|+|S|+1)}{|I|+|S|-1}\int_0^1 C(\mathbf{u}_{I\cup S}, \mathbf{1})\,\mathrm{d}u - \frac{2}{|I|+|S|-1}.$$

So, using this equality we have

$$\varphi_d^{\alpha}(C) = \frac{2(d+1)}{d-1}\left(\sum_{k=0}^{|J|}(-1)^k \sum_{\substack{S\subseteq J\\ |S|=k}}\left\{\frac{|I|+|S|-1}{2(|I|+|S|+1)}\varphi^-_{\mathbf{X}_{I\cup S}} + \frac{1}{|I|+|S|+1}\right\}\right) - \frac{2}{(d-1)\binom{d}{|J|}}$$
$$= \frac{2(d+1)}{d-1}\sum_{S\subseteq J}(-1)^{|S|}\frac{|I|+|S|-1}{2(|I|+|S|+1)}\varphi^-_{\mathbf{X}_{I\cup S}}.$$

The terms that do not involve the φ^- coefficients vanish because

$$\sum_{S\subseteq J}(-1)^{|S|}\frac{1}{|I|+|S|+1} = \sum_{k=0}^{|J|}(-1)^k\binom{|J|}{k}\frac{1}{|I|+k+1} = \frac{|I|!|J|!}{(d+1)!},$$

(see, e.g., [18]), and

$$\frac{2(d+1)}{d-1}\cdot\frac{|I|!|J|!}{(d+1)!} = \frac{2}{(d-1)\binom{d}{|J|}}.$$

Therefore, the desired formula is obtained. □

Remark 3.3. *The restriction $|I|+|S|\geq 2$ in the sum (5) ensures that every term $\varphi^-_{X_{I\cup S}}$ is well defined, since the coefficient φ_d^- requires dimension $d\geq 2$. The only case excluded by this restriction is $|I|=1$ and $S=\emptyset$, i.e., when I consists of a single index and S is empty. In this case, the corresponding term in the sum would be null, and therefore it vanishes identically regardless of the value of $\varphi^-_{X_I}$. Consequently, excluding this term from the sum does not alter the value of $\varphi_d^{\alpha}(C)$, and the restriction $|I|+|S|\geq 2$ is merely a matter of notational precision rather than a substantive constraint.*

Remark 3.4. *The case $J=\emptyset$, which is excluded from the statement of the theorem, is trivial; whereas in the case $I=\emptyset$ it is also possible to compute the expression of $\varphi_d^+(C)$ in terms of $\varphi^-_{\mathbf{X}_S}$. It suffices to note that $\frac{2(|I|+|S|+1)}{|I|+|S|-1}\int_0^1 C(\mathbf{u}_{I\cup S},\mathbf{1})\,\mathrm{d}u - \frac{2}{|I|+|S|-1} = 0$ for $|I|=|S|=0$, since $C(\mathbf{u}_\emptyset,\mathbf{1})=1$. Therefore, it does not affect the overall computation of the expression. For the cases where $|I|=0$ and $|S|=1$, the corresponding terms in the sum vanish regardless of the value of $\varphi^-_{\mathbf{X}_S}$, which likewise does not alter the value of $\varphi_d^{\alpha}(C)$.*

Remark 3.5. *With the aid of Theorem 3.2, we can rewrite the probabilities in (4) using expectations. First note*

$$\varphi_d^{\alpha}(C) = \frac{2(d+1)}{d-1}\int_0^1\left(\mathbb{E}\left[\prod_{i=1}^d \mathbb{1}(\alpha_i U_i > \alpha_i u)\right] - \prod_{i=1}^d \mathbb{E}[\mathbb{1}(\alpha_i U_i > \alpha_i u)]\right)\mathrm{d}u,$$

where $\mathbb{1}(A)$ *denotes the indicator function of event* A*. Here, the expectation* $\mathbb{E}$ *is taken with respect to the joint distribution of the random vector* $\mathbf{U}$ *under the* d*-copula* C*. Note that, since the integrand is bounded by 1, by the Fubini-Tonelli theorem we have*

$$\int_0^1 \mathbb{E}\left[\prod_{i=1}^{d} \mathbb{1}(\alpha_i U_i > \alpha_i u)\right] \mathrm{d}u = \mathbb{E}\left[\int_0^1 \prod_{i=1}^{d} \mathbb{1}(\alpha_i U_i > \alpha_i u)\mathrm{d}u\right],$$

and $\prod_{i=1}^{d} \mathbb{1}(\alpha_i U_i > \alpha_i u) = \mathbb{1}(u < \min_{i\in J} U_i, u > \max_{i\in I} U_i)$*; therefore,*

$$\int_0^1 \prod_{i=1}^{d} \mathbb{1}(\alpha_i U_i > \alpha_i u)\mathrm{d}u = \max\left(\min_{i\in J} U_i - \max_{i\in I} U_i, 0\right),$$

since $\prod_{i=1}^{d} \mathbb{1}(\alpha_i U_i > \alpha_i u) = 1$ *if, and only if,* $\max_{i\in I} U_i < u < \min_{i\in J} U_i$*. On the other hand, from the proof of Theorem 3.2, we have*

$$\int_0^1 \prod_{i=1}^{d} \mathbb{E}\left[\mathbb{1}(\alpha_i U_i > \alpha_i u)\right] \mathrm{d}u = \frac{|I|!|J|!}{(d+1)!}.$$

Thus, the formula can be written as:

$$\varphi_d^{\alpha}(C) = \frac{2(d+1)}{d-1}\left(\mathbb{E}\left[\left(\min_{i\in J} U_i - \max_{i\in I} U_i\right)_+\right] - \frac{|I|!|J|!}{(d+1)!}\right), \tag{6}$$

where $(A)_+$ *denotes* $\max(A, 0)$.

Table 1 shows the sixteen directional footrule-coefficients for $d = 4$ in terms of lower dimensional footrule-coefficients.

Direction α	**Coefficient** $\varphi_4^{\alpha}(C)$
$(1,1,1,1)$	$\frac{5}{9}(\varphi^-_{\mathbf{x}_{12}} + \varphi^-_{\mathbf{x}_{13}} + \varphi^-_{\mathbf{x}_{14}} + \varphi^-_{\mathbf{x}_{23}} + \varphi^-_{\mathbf{x}_{24}} + \varphi^-_{\mathbf{x}_{34}}) - \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{124}} + \varphi^-_{\mathbf{x}_{134}} + \varphi^-_{\mathbf{x}_{234}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(-1,1,1,1)$	$-\frac{5}{9}(\varphi^-_{\mathbf{x}_{12}} + \varphi^-_{\mathbf{x}_{13}} + \varphi^-_{\mathbf{x}_{14}}) + \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{124}} + \varphi^-_{\mathbf{x}_{134}}) - \varphi^-_{\mathbf{x}_{1234}}$
$(1,-1,1,1)$	$-\frac{5}{9}(\varphi^-_{\mathbf{x}_{12}} + \varphi^-_{\mathbf{x}_{23}} + \varphi^-_{\mathbf{x}_{24}}) + \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{124}} + \varphi^-_{\mathbf{x}_{234}}) - \varphi^-_{\mathbf{x}_{1234}}$
$(1,1,-1,1)$	$-\frac{5}{9}(\varphi^-_{\mathbf{x}_{13}} + \varphi^-_{\mathbf{x}_{23}} + \varphi^-_{\mathbf{x}_{34}}) + \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{134}} + \varphi^-_{\mathbf{x}_{234}}) - \varphi^-_{\mathbf{x}_{1234}}$
$(1,1,1,-1)$	$-\frac{5}{9}(\varphi^-_{\mathbf{x}_{14}} + \varphi^-_{\mathbf{x}_{24}} + \varphi^-_{\mathbf{x}_{34}}) + \frac{5}{6}(\varphi^-_{\mathbf{x}_{124}} + \varphi^-_{\mathbf{x}_{134}} + \varphi^-_{\mathbf{x}_{234}}) - \varphi^-_{\mathbf{x}_{1234}}$
$(-1,-1,1,1)$	$\frac{5}{9}\varphi^-_{\mathbf{x}_{12}} - \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{124}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(-1,1,-1,1)$	$\frac{5}{9}\varphi^-_{\mathbf{x}_{13}} - \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{134}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(-1,1,1,-1)$	$\frac{5}{9}\varphi^-_{\mathbf{x}_{14}} - \frac{5}{6}(\varphi^-_{\mathbf{x}_{124}} + \varphi^-_{\mathbf{x}_{134}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(1,-1,-1,1)$	$\frac{5}{9}\varphi^-_{\mathbf{x}_{23}} - \frac{5}{6}(\varphi^-_{\mathbf{x}_{123}} + \varphi^-_{\mathbf{x}_{234}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(1,-1,1,-1)$	$\frac{5}{9}\varphi^-_{\mathbf{x}_{24}} - \frac{5}{6}(\varphi^-_{\mathbf{x}_{124}} + \varphi^-_{\mathbf{x}_{234}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(1,1,-1,-1)$	$\frac{5}{9}\varphi^-_{\mathbf{x}_{34}} - \frac{5}{6}(\varphi^-_{\mathbf{x}_{134}} + \varphi^-_{\mathbf{x}_{234}}) + \varphi^-_{\mathbf{x}_{1234}}$
$(-1,-1,-1,1)$	$\frac{5}{6}\varphi^-_{\mathbf{x}_{123}} - \varphi^-_{\mathbf{x}_{1234}}$
$(-1,-1,1,-1)$	$\frac{5}{6}\varphi^-_{\mathbf{x}_{124}} - \varphi^-_{\mathbf{x}_{1234}}$
$(-1,1,-1,-1)$	$\frac{5}{6}\varphi^-_{\mathbf{x}_{134}} - \varphi^-_{\mathbf{x}_{1234}}$
$(1,-1,-1,-1)$	$\frac{5}{6}\varphi^-_{\mathbf{x}_{234}} - \varphi^-_{\mathbf{x}_{1234}}$
$(-1,-1,-1,-1)$	$\varphi^-_{\mathbf{x}_{1234}}$

Table 1: The sixteen directional footrule-coefficients for $d = 4$ expressed in terms of lower dimensional footrule-coefficients.

Next we provide several examples of directional footrule-coefficients for well-known families of copulas. The behavior under independence is captured in Example 3.6, and under extreme dependence is established in Example 3.7.

Example 3.6. For the independence copula Π_d we have $\varphi_d^{\alpha}(\Pi_d) = 0$ since, in this case,

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i X_i > x_i)\right] = \prod_{i=1}^{d} \mathbb{P}[\alpha_i X_i > x_i].$$

Example 3.7. For the comonotonicity copula M_d, the random variables satisfy $X_1 = X_2 = \cdots = X_d = X$ almost surely, where $X \sim \text{Uniform}(0,1)$. Let $k = |I|$ denote the number of components of α equal to -1, so that $|J| = d - k$. We study three cases.

Case 1: $k = 0$ (i.e., $\alpha = \mathbf{1}$). The joint probability is

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(X > u)\right] = \mathbb{P}[X > u] = 1 - u,$$

and the product of marginal probabilities is $(1-u)^d$. Therefore,

$$\int_0^1 \left[(1-u) - (1-u)^d\right] \mathrm{d}u = \frac{1}{2} - \frac{1}{d+1} = \frac{d-1}{2(d+1)},$$

and hence

$$\varphi_d^{\mathbf{1}}(M_d) = \frac{2(d+1)}{d-1} \cdot \frac{d-1}{2(d+1)} = 1.$$

Case 2: $k = d$ (i.e., $\alpha = -\mathbf{1}$). The joint probability is

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(X < u)\right] = \mathbb{P}[X < u] = u,$$

and the product of marginal probabilities is u^d. Therefore,

$$\int_0^1 \left[u - u^d\right] \mathrm{d}u = \frac{1}{2} - \frac{1}{d+1} = \frac{d-1}{2(d+1)},$$

whence

$$\varphi_d^{-\mathbf{1}}(M_d) = \frac{2(d+1)}{d-1} \cdot \frac{d-1}{2(d+1)} = 1.$$

Case 3: $0 < k < d$. Since $X_1 = \cdots = X_d = X \sim \text{Uniform}(0,1)$ under M_d, the event $\bigcap_{i=1}^{d}(\alpha_i X > \alpha_i u)$ requires simultaneously $X > u$ (for all $i \in J$) and $X < u$ (for all $i \in I$). These two conditions are mutually exclusive for almost every $u \in (0,1)$, so

$$\mathbb{P}\left[\bigcap_{i=1}^{d}(\alpha_i X > \alpha_i u)\right] = 0 \quad \text{for all } u \in (0,1).$$

Therefore,

$$\int_0^1 \left(0 - \prod_{i=1}^{d} \mathbb{P}[\alpha_i X > \alpha_i u]\right) \mathrm{d}u = -\int_0^1 u^k (1-u)^{d-k} \, \mathrm{d}u = -\frac{k!\,(d-k)!}{(d+1)!},$$

where we have used $\mathbb{P}[X < u] = u$ for $i \in I$ and $\mathbb{P}[X > u] = 1 - u$ for $i \in J$. Therefore

$$\varphi_d^{\alpha}(M_d) = \frac{2(d+1)}{d-1} \cdot \left(-\frac{k!\,(d-k)!}{(d+1)!}\right) = -\frac{2}{(d-1)\binom{d}{k}}.$$

Note that this result holds for all $0 < k < d$, regardless of whether k is even or odd, since the joint probability vanishes in both cases for the same reason.

In summary,

$$\varphi_d^{\alpha}(M_d) = \begin{cases} 1, & \text{if } k = 0 \text{ or } k = d, \\ -\dfrac{2}{(d-1)\binom{d}{k}}, & \text{if } 0 < k < d, \end{cases}$$

where $k = |I|$ is the number of components of α equal to -1.

Example 3.8. Let us consider the FGM family of d-copulas given by

$$C_\lambda^{FGM}(\mathbf{u}) = \prod_{i=1}^{d} u_i \left(1 + \lambda \prod_{i=1}^{d}(1-u_i)\right), \quad \lambda \in [-1,1],$$

for all $\mathbf{u} \in [0,1]^d$ (see [10, 25]). On the other hand, the expression of the d-copula associated with the reflected vector $\mathbf{X}^{(J)}$ is obtained from Theorem 2.1 as

$$(C_\lambda^{FGM})^{(J)}(\mathbf{u}) = \prod_{i=1}^{d} u_i \left(1 + (-1)^{|J|}\lambda \prod_{i=1}^{d}(1-u_i)\right),$$

for all $\mathbf{u} \in [0,1]^d$, where J is the set of coordinates of α equal to 1. Therefore,

$$\begin{aligned}
\varphi_d^\alpha(C_\lambda^{FGM}) &= \frac{2(d+1)}{d-1}\int_0^1 (-1)^{|J|}\lambda \prod_{i=1}^{d} \mathbb{P}[\alpha_i U_i > \alpha_i u](1 - \mathbb{P}[\alpha_i U_i > \alpha_i u])\,\mathrm{d}u \\
&= \frac{2(d+1)}{d-1}(-1)^{|J|}\lambda \int_0^1 u^{|I|}(1-u)^{|I|}u^{|J|}(1-u)^{|J|}\,\mathrm{d}u \\
&= \frac{2(d+1)}{d-1}(-1)^{|J|}\lambda \int_0^1 u^{d}(1-u)^{d}\,\mathrm{d}u \\
&= \frac{2(d+1)}{d-1}(-1)^{|J|}\lambda \frac{(d!)^2}{(2d+1)!} \\
&= \frac{2\lambda(-1)^{|J|}(d+1)(d!)^2}{(d-1)(2d+1)!}.
\end{aligned}$$

So, for an even number of coordinates equal to 1 of α, the coefficient takes the value

$$\varphi_d^\alpha(C_\lambda^{FGM}) = \frac{2\lambda(d+1)(d!)^2}{(d-1)(2d+1)!},$$

and for an odd number

$$\varphi_d^\alpha(C_\lambda^{FGM}) = -\frac{2\lambda(d+1)(d!)^2}{(d-1)(2d+1)!}.$$

Example 3.8 shows that $\varphi_d^\alpha(C_\lambda^{FGM})$ takes the same absolute value for all directions but alternates in sign depending on $|J|$. In contrast, the classical footrule $\varphi_d(C_\lambda^{FGM})$ is a single scalar which equals the average $(\varphi_d^{\mathbf{1}} + \varphi_d^{-\mathbf{1}})/2$ of the two extreme directional coefficients (see the proof of Corollary 3.1), thereby masking the directional structure. For example, for $d = 3$ we have $\varphi_3^{\mathbf{1}}(C_\lambda^{FGM}) = -\lambda/35$ and $\varphi_3^{-\mathbf{1}}(C_\lambda^{FGM}) = \lambda/35$, so that $\varphi_3(C_\lambda^{FGM}) = 0$, whereas all the directional coefficients equal $\pm\lambda/35 \neq 0$ for $\lambda \neq 0$. More generally, $\varphi_d(C_\lambda^{FGM}) = 0$ for every odd d, illustrating that the classical measure can be uninformative when directional asymmetries cancel out.

Example 3.9. Let C_θ^{CL} be a Clayton d-copula, defined as

$$C_\theta^{CL}(\mathbf{u}) = \left(\sum_{i=1}^{d} u_i^{-\theta} - d + 1\right)^{-1/\theta}, \tag{7}$$

for all $\mathbf{u} \in [0,1]^d$ and $\theta > 0$. Due to the exchangeability of the copula, it is not important which of the variables is transformed by changing the sign; only the number of signs changed matters. The general expression for the directional footrule-coefficients for C_θ^{CL}, using the inclusion–exclusion representation in the proof of Theorem 3.2, is

$$\varphi_d^\alpha(C_\theta^{CL}) = \frac{2(d+1)}{d-1}\sum_{k=0}^{|J|}(-1)^k\binom{|J|}{k}\int_0^1 ((|I|+k)u^{-\theta} + 1 - |I| - k)^{-1/\theta}\,\mathrm{d}u - \frac{2}{(d-1)\binom{d}{|J|}}.$$

Note that in the extreme case $|J| = 0$ (i.e., $\alpha = -\mathbf{1}$) the formula reduces to a single term ($k = 0$), whereas for $|J| = d$ (i.e., $\alpha = \mathbf{1}$) all the lower-dimensional marginals are involved; in both cases, one can verify that

$$\lim_{\theta\to\infty} \varphi_d^{\pm\mathbf{1}}(C_\theta^{\mathrm{CL}}) = 1,$$

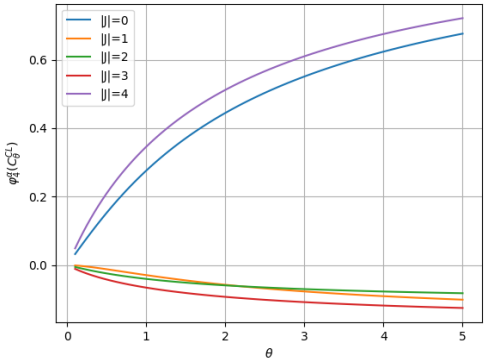


Figure 1: Values of $\varphi_4^{\alpha}(C_\theta^{CL})$ for Clayton 4-copulas as functions of the parameter θ, for all possible values of $|J|$. The curves for $|J| = 0$ and $|J| = 4$ (corresponding to $\alpha = -\mathbf{1}$ and $\alpha = \mathbf{1}$, respectively) exhibit positive dependence that increases with θ, while intermediate values of $|J|$ display negative directional dependence.

consistently with Example 3.7, since the Clayton copula converges to the d-copula M_d as $\theta \to \infty$.

Figure 1 shows the values of $\varphi_4^{\alpha}(C_\theta^{\mathrm{CL}})$ as a function of θ for all possible values of $|J|$, illustrating how the extreme directions $|J| = 0$ and $|J| = 4$ yield positive and increasing dependence, while the intermediate directions display negative directional dependence.

Example 3.10. Let C_θ^{CA} be a multivariate Cuadras-Augé d-copula whose expression is given by

$$C_\theta^{CA}(\mathbf{u}) = (\Pi(\mathbf{u}))^{1-\theta}(M(\mathbf{u}))^{\theta}, \tag{8}$$

for all $\mathbf{u} \in [0,1]^d$ and $\theta \in [0,1]$ (see [10, 25]). These d-copulas satisfy that $C_0^{CA} = \Pi$ and $C_1^{CA} = M$. Moreover, every lower j-dimensional marginal of this d-copula is a Cuadras-Augé j-copula. Since this copula is exchangeable, it does not matter which components we keep equal to 1, but their number. The general expression for these coefficients for C_θ^{CA} is, using (5),

$$\varphi_d^{\alpha}\left(C_\theta^{CA}\right) = \frac{2(d+1)}{d-1} \sum_{\substack{k=0 \\ |I|+k\geq 2}}^{|J|} (-1)^k \binom{|J|}{k} \frac{\theta(|I|+k-1)}{(|I|+k+1)^2 - \theta[(|I|+k)^2-1]}. \tag{9}$$

By using (9), we obtain the value of $\varphi_4^{\alpha}(C_\theta^{CA})$ for all the possible numbers of components of α equal to 1

i) $|J| = 0$

$$\varphi_4^{-}(C_\theta^{CA}) = \frac{2\theta}{5-3\theta}.$$

ii) $|J| = 1$

$$\varphi_4^{\alpha}(C_\theta^{CA}) = \frac{10}{3}\left(\frac{\theta}{8-4\theta} - \frac{3\theta}{25-15\theta}\right).$$

iii) $|J| = 2$

$$\varphi_4^{\alpha}(C_\theta^{CA}) = \frac{10}{3}\left(\frac{\theta}{9-3\theta} - \frac{\theta}{4-2\theta} + \frac{3\theta}{25-15\theta}\right).$$

iv) $|J| = 3$

$$\varphi_4^{\alpha}(C_\theta^{CA}) = \frac{10}{3}\left(-\frac{\theta}{3-\theta} + \frac{3\theta}{8-4\theta} - \frac{3\theta}{25-15\theta}\right).$$

v) $|J| = 4$

$$\varphi_4^{+}(C_\theta^{CA}) = \frac{10}{3}\left[\frac{2\theta}{3-\theta} - \frac{\theta}{2-\theta} + \frac{3\theta}{25-15\theta}\right].$$

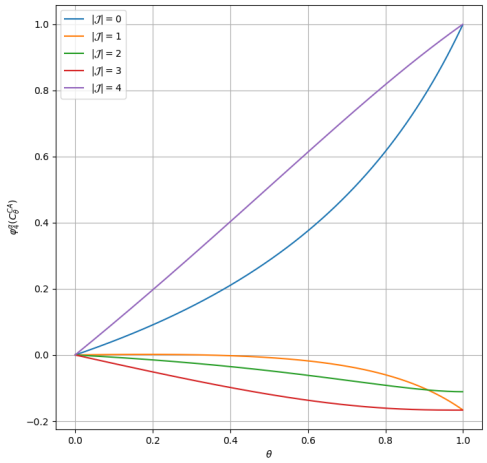


Figure 2: Values of φ_4^α for Cuadras-Augé 4-copulas as a function of θ, for all possible values of α.

In Figure 2 we observe that, as anticipated, increasing values of the parameter θ drive the copula away from independence and toward maximal dependence in the cases $\alpha = \mathbf{1}$ and $\alpha = -\mathbf{1}$. For the mixed directions ($0 < |J| < 4$), the coefficients are negative for $|J| \in \{2, 3\}$ and every $\theta \in (0, 1]$, the strongest negative dependence being attained for $|J| = 3$. For $|J| = 1$, the coefficient is positive, although smaller than 0.002, for $\theta < 1/3$, and negative for $\theta > 1/3$, approaching $-1/6$ as $\theta \to 1$ (see Example 3.7).

The following result can be trivially extracted from (1) and the expression of the directional footrule-coefficients given by (4).

Corollary 3.11. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* d*-dimensional random vectors of continuous random variables uniformly distributed on* $[0, 1]$*, whose respective joint distribution functions are the* d*-copulas* $C_{\mathbf{X}}$ *and* $C_{\mathbf{Y}}$*. Let* $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ *such that* $|\alpha_i| = 1$ *for all* $i = 1, \cdots, d$*. If* $\mathbf{X} \leq_{\mathrm{PD}(\alpha)} \mathbf{Y}$*, then* $\varphi_d^\alpha(C_{\mathbf{X}}) \leq \varphi_d^\alpha(C_{\mathbf{Y}})$*.*

As a direct implication of this ordering property, the coefficient φ_d^α is monotone with respect to the parameter λ: it is increasing in λ when $|J|$ is even, and decreasing in λ when $|J|$ is odd, consistently with the expressions obtained in Example 3.8. In all cases, $\varphi_d^\alpha(C_{\lambda_1}^{FGM}) \leq \varphi_d^\alpha(C_{\lambda_2}^{FGM})$ if and only if, $\lambda_1 \leq \lambda_2$ and $|J|$ is even, or $\lambda_1 \geq \lambda_2$ and $|J|$ is odd.

4 Nonparametric estimators

Consider a d-dimensional random sample of size n, $\{X_{1j}, \cdots, X_{dj}\}_{j=1}^n$, of a d-dimensional random vector $(X_1, \cdots, X_d)$ with associated d-copula C. For each $i \in \{1, \cdots, d\}$ and $j \in \{1, \cdots, n\}$, let R_{ij} denote the rank of X_{ij} among $X_{i1}, \cdots, X_{in}$, i.e., $R_{ij} = \sum_{k=1}^n \mathbb{1}(X_{ik} \leq X_{ij})$, and let $\overline{R}_{ij} = n + 1 - R_{ij}$ denote the complementary rank of X_{ij}. The pseudo-observations are defined as $\widehat{U}_{ij} = R_{ij}/(n+1)$. From (6), and using the pseudo-observations $\widehat{U}_{ij}$, we define the nonparametric rank-based estimator (for related rank-based estimators of multivariate versions of Spearman's rho and footrule, see [27, 28]) for $\alpha \neq \pm\mathbf{1}$ (i.e., when both I and J are non-empty) as

$$\widetilde{\varphi}_{n,d}^\alpha = \frac{2(d+1)}{(d-1)(n+1)} \left\{ \frac{1}{n} \sum_{j=1}^n \left(\min_{i \in J} R_{ij} - \max_{i \in I} R_{ij} \right)_+ - \frac{(n+1)\,|I|!\,|J|!}{(d+1)!} \right\}. \tag{10}$$

The cases $\alpha = \mathbf{1}$ and $\alpha = -\mathbf{1}$, for which $\max_{i \in I} R_{ij}$ and $\min_{i \in J} R_{ij}$ are respectively undefined, are handled separately. Note that when $\alpha = \mathbf{1}$ and $\alpha = -\mathbf{1}$, it holds that

$$\widetilde{\varphi}_{n,d}^+ = \frac{2(d+1)}{(d-1)(n+1)} \left(\frac{1}{n} \sum_{j=1}^n \min_{1 \leq i \leq d} R_{ij} - \frac{n+1}{d+1} \right),$$

and

$$\widetilde{\varphi}^{-}_{n,d} = \frac{2(d+1)}{(d-1)(n+1)} \left(\frac{1}{n} \sum_{j=1}^{n} \min_{1\le i\le d} \overline{R}_{ij} - \frac{n+1}{d+1} \right),$$

since, with the conventions $\min_{i\in\emptyset} R_{ij} = n+1$ and $\max_{i\in\emptyset} R_{ij} = 0$, when $J = \emptyset$ the term $\min_{i\in J} R_{ij} - \max_{i\in I} R_{ij}$ equals $n+1-\max_{1\le i\le d} R_{ij} = \min_{1\le i\le d} \overline{R}_{ij}$. The expressions obtained for the extreme cases of the directional footrule estimators are consistent and coherent with the previously derived expressions presented in [8]. If ties are present, as may happen in practice when the data are recorded with limited precision, each R_{ij} is replaced by the corresponding midrank, i.e., the average of the ranks that the tied observations would otherwise receive. Unlike other conventions, such as assigning the maximum rank to tied observations, midranks preserve the identity $\widetilde{\varphi}^{\alpha}_{n,d}(\mathbf{X}) = \widetilde{\varphi}^{-\alpha}_{n,d}(-\mathbf{X})$, where the right-hand side denotes the estimator computed from the sample $\{-X_{1j}, \cdots, -X_{dj}\}_{j=1}^{n}$; this identity is the empirical counterpart of Corollary 3.1 iii).

Before proving the decomposition of $\widetilde{\varphi}^{\alpha}_{n,d}$ in terms of lower-dimensional estimators, we establish a useful integral representation. Observe that, for each fixed $j \in \{1, \cdots, n\}$,

$$\int_0^1 \mathbb{1}\left(\max_{i\in I} \widehat{U}_{ij} \le u \le \min_{i\in J} \widehat{U}_{ij} \right) \mathrm{d}u = \left(\min_{i\in J} \widehat{U}_{ij} - \max_{i\in I} \widehat{U}_{ij} \right)_+ = \frac{1}{n+1} \left(\min_{i\in J} R_{ij} - \max_{i\in I} R_{ij} \right)_+,$$

since the integrand equals 1 on the interval $[\max_{i\in I} \widehat{U}_{ij}, \min_{i\in J} \widehat{U}_{ij}]$ and 0 elsewhere, with the convention that the length is zero whenever $\min_{i\in J} \widehat{U}_{ij} < \max_{i\in I} \widehat{U}_{ij}$. Summing over $j = 1, \cdots, n$, dividing by n, and substituting into the definition of $\widetilde{\varphi}^{\alpha}_{n,d}$, we obtain

$$\widetilde{\varphi}^{\alpha}_{n,d} = \frac{2(d+1)}{(d-1)n} \sum_{j=1}^{n} \int_0^1 \mathbb{1}\left(\max_{i\in I} \widehat{U}_{ij} \le u \le \min_{i\in J} \widehat{U}_{ij} \right) \mathrm{d}u - \frac{2}{(d-1)\binom{d}{|J|}}. \tag{11}$$

It can be shown that the proposed estimators admit a decomposition as a linear combination of the lower-dimensional $\widetilde{\varphi}$-estimators, following a structure analogous to that established in Theorem 3.2.

Theorem 4.1. *Let $\{X_{1j}, \cdots, X_{dj}\}_{j=1}^{n}$ be a d-dimensional random sample of size n of a d-dimensional random vector $(X_1, \cdots, X_d)$ with associated d-copula C. Let $\alpha \in \mathbb{R}^d$ be such that $\alpha_i \in \{-1, 1\}$ for all $i = 1, \cdots, d$. Let $I \subseteq \{1, 2, \cdots, d\}$ such that $\alpha_i = -1$ if $i \in I$ and $\alpha_i = 1$ if $i \in J = \{1, 2, \cdots, d\} \setminus I$, and assume $|I| \ge 1$ and $|J| \ge 1$ (i.e., $\alpha \ne \pm\mathbf{1}$). Then*

$$\widetilde{\varphi}^{\alpha}_{n,d} = \frac{2(d+1)}{d-1} \sum_{\substack{K\subseteq J \\ |I|+|K|\ge 2}} (-1)^{|K|} \frac{|I|+|K|-1}{2(|I|+|K|+1)} \widetilde{\varphi}^{-}_{n,I\cup K}, \tag{12}$$

where, for any $L \subseteq \{1, \cdots, d\}$ with $|L| \ge 2$, $\widetilde{\varphi}^{-}_{n,L}$ denotes the estimator $\widetilde{\varphi}^{-}_{n,|L|}$ computed from the subsample $\{(X_{ij})_{i\in L}\}_{j=1}^{n}$.

Proof. We start from the integral representation (11). For each fixed $j \in \{1, \cdots, n\}$, we write

$$\mathbb{1}\left(\max_{i\in I} \widehat{U}_{ij} \le u \le \min_{i\in J} \widehat{U}_{ij} \right) = \mathbb{1}\left(\max_{i\in I} \widehat{U}_{ij} \le u \right) \cdot \prod_{i\in J} \mathbb{1}\left(u \le \widehat{U}_{ij} \right).$$

Applying the identity

$$\prod_{i\in J} \mathbb{1}(u \le \widehat{U}_{ij}) = \prod_{i\in J} \big(1 - \mathbb{1}(\widehat{U}_{ij} < u)\big) = \sum_{K\subseteq J} (-1)^{|K|} \prod_{i\in K} \mathbb{1}(\widehat{U}_{ij} < u),$$

we obtain

$$\begin{aligned} \mathbb{1}\left(\max_{i\in I} \widehat{U}_{ij} \le u \le \min_{i\in J} \widehat{U}_{ij} \right) &= \sum_{K\subseteq J} (-1)^{|K|} \mathbb{1}\left(\max_{i\in I} \widehat{U}_{ij} \le u \right) \cdot \mathbb{1}\left(\max_{i\in K} \widehat{U}_{ij} < u \right) \\ &= \sum_{K\subseteq J} (-1)^{|K|} \mathbb{1}\left(\max_{i\in I\cup K} \widehat{U}_{ij} < u \right), \end{aligned}$$

where the last equality uses the fact that both conditions together are equivalent to $\max_{i\in I\cup K}\widehat{U}_{ij} < u$ up to a set of Lebesgue measure zero. Integrating over $u \in [0,1]$ for each fixed j:

$$\int_0^1 \mathbb{1}\left(\max_{i\in I\cup K}\widehat{U}_{ij} < u\right) \mathrm{d}u = 1 - \max_{i\in I\cup K}\widehat{U}_{ij} = \min_{i\in I\cup K}\overline{\widehat{U}}_{ij},$$

where $\overline{\widehat{U}}_{ij} = \overline{R}_{ij}/(n+1)$. Hence,

$$\int_0^1 \mathbb{1}\left(\max_{i\in I}\widehat{U}_{ij} \le u \le \min_{i\in J}\widehat{U}_{ij}\right) \mathrm{d}u = \sum_{K\subseteq J}(-1)^{|K|}\min_{i\in I\cup K}\overline{\widehat{U}}_{ij}.$$

Summing over $j = 1, \cdots, n$, dividing by n, and substituting into (11):

$$\widetilde{\varphi}^{\alpha}_{n,d} = \frac{2(d+1)}{d-1}\sum_{K\subseteq J}(-1)^{|K|}\left(\frac{1}{n}\sum_{j=1}^{n}\min_{i\in I\cup K}\overline{\widehat{U}}_{ij}\right) - \frac{2}{(d-1)\binom{d}{|J|}}.$$

Using the identity

$$\frac{1}{n}\sum_{j=1}^{n}\min_{i\in I\cup K}\overline{\widehat{U}}_{ij} = \frac{|I|+|K|-1}{2(|I|+|K|+1)}\,\widetilde{\varphi}^{-}_{n,I\cup K} + \frac{1}{|I|+|K|+1},$$

which follows directly from the definition of $\widetilde{\varphi}^{-}_{n,I\cup K}$, and substituting:

$$\widetilde{\varphi}^{\alpha}_{n,d} = \frac{2(d+1)}{d-1}\sum_{K\subseteq J}(-1)^{|K|}\left(\frac{|I|+|K|-1}{2(|I|+|K|+1)}\,\widetilde{\varphi}^{-}_{n,I\cup K} + \frac{1}{|I|+|K|+1}\right) - \frac{2}{(d-1)\binom{d}{|J|}}.$$

The terms not involving the estimators satisfy (see, e.g., [18])

$$\sum_{K\subseteq J}(-1)^{|K|}\frac{1}{|I|+|K|+1} = \frac{|I|!\,|J|!}{(d+1)!},$$

and therefore

$$\frac{2(d+1)}{d-1}\cdot\frac{|I|!\,|J|!}{(d+1)!} = \frac{2}{(d-1)\binom{d}{|J|}},$$

which exactly cancels the constant term. When $|I| = 1$ and $K = \emptyset$, the factor $\frac{|I|+|K|-1}{2(|I|+|K|+1)} = 0$, so those terms vanish identically. We thus obtain (12), as desired. □

We now illustrate the good performance of these estimators through a practical example in the 3-dimensional setting. Table 2 presents the 3-dimensional directional footrule-coefficients of the Clayton 3-copula with $\theta = 5$ for all possible directions $\alpha \in \{-1,1\}^3$, computed exactly via numerical integration and subsequently approximated using our proposed estimators, based on 1000 independent samples, each one of size 500. To further illustrate the behavior of the proposed

Table 2: Results obtained in Monte Carlo simulations to calculate the value of $\widetilde{\varphi}^{\alpha}_{n,d}$ in every direction $\alpha \in \{-1,1\}^3$ based on a total of 1000 samples of size 500 generated for the Clayton 3-copula for $\theta = 5$.

α	φ^{α}_{3}	$\widetilde{\varphi}^{\alpha}_{n,3}$
$(-1,-1,-1)$	0.68975	0.68924
$(-1,1,1)$	−0.24935	−0.24848
$(1,-1,1)$	−0.24935	−0.24889
$(1,1,-1)$	−0.24935	−0.24874
$(1,-1,-1)$	−0.22020	−0.22004
$(-1,-1,1)$	−0.22020	−0.22030
$(-1,1,-1)$	−0.22020	−0.22045
$(1,1,1)$	0.71890	0.71768

estimators, Figure 3 presents their values as functions of parameter θ for three-dimensional Clayton family copulas. The figure highlights how the estimators capture the variation in directional dependence across different values of θ, providing a visual confirmation of their finite-sample performance and their alignment with the theoretical directional footrule-coefficients.

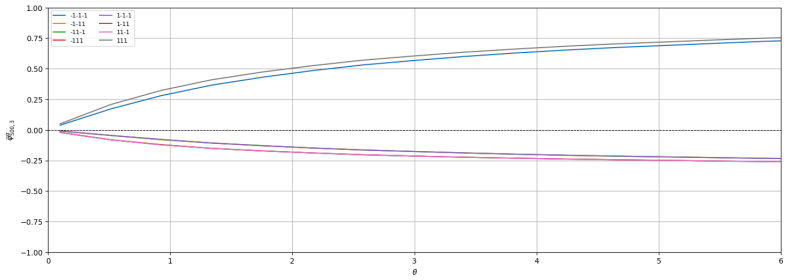


Figure 3: Values of the different estimators of the directional footrule-coefficients as functions of the parameter θ for 3-dimensional Clayton family copulas.

4.1 Empirical processes and some properties of these estimators

We now establish the asymptotic properties of the proposed directional footrule estimators. Building on the theory of empirical copula processes, we derive conditions under which these estimators are consistent and asymptotically normal. This theoretical foundation enables statistical inference in practical applications and validates the use of these estimators for directional dependence analysis.

4.1.1 Empirical process

In [16], an empirical process was introduced to estimate a generalization of a 3-copula which can be easily extended to the d-dimensional case. As the authors did, we begin by considering an index set β such that $\beta \subseteq \{1, \cdots, d\}$. We define $x_\beta = (x_i)_{i\in\beta}$ an arbitrary value of $\mathbf{X}_\beta = (X_i)_{i\in\beta}$. Let $|\beta|$ denote the cardinality of β and $\alpha \in \mathbb{R}^d$ a direction. Consider the function

$$H_{\beta,\alpha}(x_\beta) = \mathbb{P}\left[\alpha_i X_i \leq \alpha_i x_i, \text{ for all } i \in \beta\right], \tag{13}$$

i.e.,

$$H_{\beta,\alpha}(x_\beta) = \mathbb{P}\left[\bigcap_{\substack{i\in\beta\\ \alpha_i=1}} \{X_i \leq x_i\}, \bigcap_{\substack{i\in\beta\\ \alpha_i=-1}} \{X_i \geq x_i\}\right].$$

Let u_β be defined analogously to x_β, let F_i be the marginal cumulative distribution function of X_i, and let $U_i = F_i(X_i)$. Then

$$C_{\beta,\alpha}(u_\beta) = H_{\beta,\alpha}\big((F_i^{-1}(u_i))_{i\in\beta}\big) = \mathbb{P}\left[\alpha_i U_i \leq \alpha_i u_i, \text{ for all } i \in \beta\right], \tag{14}$$

which reduces to the d-copula C when $\beta = \{1, \cdots, d\}$ and $\alpha = \mathbf{1}$. From this expression, we can introduce, as in [16], the empirical process to estimate it

$$C_{\beta,\alpha,n}(u_\beta) = \frac{1}{n+1}\sum_{j=1}^{n}\prod_{i\in\beta} \mathbb{1}\left(\alpha_i \frac{R_{ij}}{n+1} \leq \alpha_i u_i\right). \tag{15}$$

When $\beta = \{1, \cdots, d\}$ and $\alpha = \mathbf{1}$, Expression (15) is the empirical estimator of the d-copula.

To determine the weak convergence —denoted by "$\xrightarrow{w}$"— of the empirical process

$$\sqrt{n}\{C_{\beta,\alpha,n}(u_\beta) - C_{\beta,\alpha}(u_\beta)\}, \qquad u_\beta \in [0,1]^{|\beta|},$$

it suffices to proceed as in [16], where the following two conditions are considered. For that, first recall that a $C_{\beta,\alpha}$-tight centered Gaussian process on $[0,1]^{|\beta|}$ is a zero-mean Gaussian process $\mathbb{B}_{C_{\beta,\alpha}}$ indexed by $u_\beta \in [0,1]^{|\beta|}$ with covariance structure determined by $C_{\beta,\alpha}$: $\mathrm{Cov}(\mathbb{B}_{C_{\beta,\alpha}}(u_\beta), \mathbb{B}_{C_{\beta,\alpha}}(v_\beta)) = C_{\beta,\alpha}(w_\beta) - C_{\beta,\alpha}(u_\beta)C_{\beta,\alpha}(v_\beta)$, where $(w_\beta)_j = \min(u_j, v_j)$ if $\alpha_j = 1$ and $(w_\beta)_j = \max(u_j, v_j)$ if $\alpha_j = -1$ —tightness here is understood in the sense of weak convergence in $\ell^\infty([0,1]^{|\beta|})$: see [11, 15].

Condition 4.2. For each $i \in \beta$, the i-th first-order partial derivative $\dot{C}_{i,\beta,\alpha}$ exists and is continuous on the set $\{u_\beta \in [0,1]^{|\beta|} : 0 < u_i < 1\}$.

Condition 4.3. $\mathbb{B}_{C_{\beta,\alpha}}(u_\beta)$ is a $C_{\beta,\alpha}$-tight centered Gaussian process on $[0,1]^{|\beta|}$. $u_\beta^{(i)}$ is a vector such that its j-th component, for $j \in \beta$, is:

$$(u_\beta^{(i)})_j = \begin{cases} 1 & \text{if } j \neq i \text{ when } \alpha_j = 1, \\ 0 & \text{if } j \neq i \text{ when } \alpha_j = -1, \\ u_i & \text{if } j = i. \end{cases}$$

The covariance function is $\mathrm{Cov}(\mathbb{B}_{C_{\beta,\alpha}}(u_\beta), \mathbb{B}_{C_{\beta,\alpha}}(v_\beta)) = C_{\beta,\alpha}(w_\beta) - C_{\beta,\alpha}(u_\beta)C_{\beta,\alpha}(v_\beta)$, where the j-th component $(w_\beta)_j$ is:

$$(w_\beta)_j = \begin{cases} \min(u_j, v_j) & \text{if } \alpha_j = 1, \\ \max(u_j, v_j) & \text{if } \alpha_j = -1, \end{cases}$$

for all $j \in \beta$.

From Conditions 4.2 and 4.3, the following result is obtained, which is valid for any dimension d and, therefore, guarantees the weak convergence of our process.

Theorem 4.4. *Let $H_{\beta,\alpha}$ be a $|\beta|$-dimensional distribution function, given by (13) with continuous marginal distributions F_i, $i \in \beta$ and with $C_{\beta,\alpha}$ given by (14), where $\beta \subseteq \{1,...,d\}$ and $\alpha \in \mathbb{R}^d$, $\alpha_i \in \{-1,1\}$. Under Condition 4.2 on the function $C_{\beta,\alpha}$, when $n \longrightarrow \infty$*

$$\sqrt{n}\,\{C_{\beta,\alpha,n}(u_\beta) - C_{\beta,\alpha}(u_\beta)\} \xrightarrow{w} \mathbb{G}_{C_{\beta,\alpha}}(u_\beta).$$

Weak convergence takes place in $\ell^\infty([0,1]^{|\beta|})$ and $\mathbb{G}_{C_{\beta,\alpha}}(u_\beta) = \mathbb{B}_{C_{\beta,\alpha}}(u_\beta) - \sum_{i\in\beta} \dot{C}_{i,\beta,\alpha}(u_\beta)\mathbb{B}_{C_{\beta,\alpha}}(u_\beta^{(i)})$, where $\mathbb{B}_{C_{\beta,\alpha}}$ is a $C_{\beta,\alpha}$-tight centered Gaussian process on $[0,1]^{|\beta|}$ and $\mathbb{G}_{C_{\beta,\alpha}}$ follows Condition 4.3.

The proof of Theorem 4.4 follows by adapting the argument of [16, Theorem 3.1] to the general d-dimensional directional setting. Specifically, since $C_{\beta,\alpha,n}$ is a generalized empirical copula process —as defined in (15)—, the weak convergence in $\ell^\infty([0,1]^{|\beta|})$ is established by verifying that

$$\sqrt{n}\{C_{\beta,\alpha,n}(u_\beta) - C_{\beta,\alpha}(u_\beta)\},$$

satisfies the conditions of [11, Theorem 1] and [15]. Condition 4.2 ensures the existence of the Hadamard derivative of the functional $C \mapsto C_{\beta,\alpha}$, and Condition 4.3 guarantees the tightness of the limiting Gaussian process $\mathbb{G}_{C_{\beta,\alpha}}$ in $\ell^\infty([0,1]^{|\beta|})$ following the framework of [11, Theorem 6].

4.1.2 Properties of the estimators

To establish the asymptotic normality of our estimators, we will use the following adaptation to integrals along the main diagonal of $[0,1]^{|\beta|}$, of a result in [16] (see also [11, Theorem 6] and [15]); recall from (11) that our estimators are integrals of this type.

Theorem 4.5. *Under the assumptions of Theorem 4.4, let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers satisfying $\sqrt{n}(a_n - a_0) = O(n^{-1/2})$ and $\sqrt{n}(b_n - b_0) = O(n^{-1/2})$, respectively, where a_0 and b_0 are constant values. For every bounded function f on $[0,1]^{|\beta|}$, let*

$$T_n(f) = a_n \int_0^1 f(u\mathbf{1}_\beta)\,\mathrm{d}u + b_n, \qquad n \geq 0,$$

where $u\mathbf{1}_\beta = (u, \cdots, u) \in [0,1]^{|\beta|}$. Then, when $n \longrightarrow \infty$,

$$\sqrt{n}\{T_n(C_{\beta,\alpha,n}) - T_0(C_{\beta,\alpha})\} \xrightarrow{w} Z_{C_{\beta,\alpha}} \sim N(0, \sigma^2_{C_{\beta,\alpha}}),$$

where

$$Z_{C_{\beta,\alpha}} := a_0 \int_0^1 \mathbb{G}_{C_{\beta,\alpha}}(u\mathbf{1}_\beta)\,\mathrm{d}u \quad \text{and} \quad \sigma^2_{C_{\beta,\alpha}} = a_0^2 \int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_{C_{\beta,\alpha}}(u\mathbf{1}_\beta)\,\mathbb{G}_{C_{\beta,\alpha}}(v\mathbf{1}_\beta)\right]\mathrm{d}u\,\mathrm{d}v.$$

Proof. We have

$$\sqrt{n}\{T_n(C_{\beta,\alpha,n}) - T_0(C_{\beta,\alpha})\} = a_n \int_0^1 \sqrt{n}\,\{C_{\beta,\alpha,n}(u\mathbf{1}_\beta) - C_{\beta,\alpha}(u\mathbf{1}_\beta)\}\,\mathrm{d}u + \sqrt{n}(a_n - a_0)\int_0^1 C_{\beta,\alpha}(u\mathbf{1}_\beta)\,\mathrm{d}u + \sqrt{n}(b_n - b_0).$$

The last two terms converge to zero by assumption. Since the linear map $f \mapsto \int_0^1 f(u\mathbf{1}_\beta)\,\mathrm{d}u$ is continuous on $\ell^\infty([0,1]^{|\beta|})$, Theorem 4.4, the continuous mapping theorem and Slutsky's lemma (note that $a_n \to a_0$) imply that the first term converges weakly to $Z_{C_{\beta,\alpha}}$. Finally, $Z_{C_{\beta,\alpha}}$ is a centered normal random variable with variance $\sigma^2_{C_{\beta,\alpha}}$, since it is a continuous linear functional of the centered Gaussian process $\mathbb{G}_{C_{\beta,\alpha}}$. □

As a consequence, we have the following result.

Corollary 4.6. *Let $\beta = \{1, 2, \cdots, d\}$ and let $\alpha \in \mathbb{R}^d$ be such that $\alpha_i \in \{-1, 1\}$ for all $i = 1, \cdots, d$. Assume that the marginal distribution functions are continuous and that Condition 4.2 holds for $C_{\beta,-\alpha}$. Then, when $n \longrightarrow \infty$,*

$$\sqrt{n}\{\widetilde{\varphi}^\alpha_{n,d} - \varphi^\alpha_d\} \xrightarrow{w} Z_{C_{\beta,-\alpha}} \sim N(0, \sigma^2_{C_{\beta,-\alpha}}).$$

Proof. Since the marginals are continuous, the events $\bigcap_{i=1}^d(\alpha_i U_i > \alpha_i u)$ and $\bigcap_{i=1}^d(-\alpha_i U_i \leq -\alpha_i u)$ coincide up to a null set. Hence, by (4) and (14), we have $\varphi^\alpha_d = T_0(C_{\beta,-\alpha})$ with

$$a_0 = \frac{2(d+1)}{d-1} \quad \text{and} \quad b_0 = -\frac{2}{(d-1)\binom{d}{|J|}}.$$

Similarly, by (15) and (11), which also holds for $\alpha = \pm\mathbf{1}$ with the conventions $\max_{i\in\emptyset}\widehat{U}_{ij} = 0$ and $\min_{i\in\emptyset}\widehat{U}_{ij} = 1$, we have $\widetilde{\varphi}^\alpha_{n,d} = T_n(C_{\beta,-\alpha,n})$ with $b_n = b_0$ for all n and

$$a_n = \frac{2(d+1)(n+1)}{(d-1)n}.$$

The condition $\sqrt{n}(a_n - a_0) = O(n^{-1/2})$ holds, since

$$\sqrt{n}(a_n - a_0) = \sqrt{n}\left(\frac{2(d+1)(n+1)}{(d-1)n} - \frac{2(d+1)}{d-1}\right) = \sqrt{n}\left(\frac{2(d+1)}{(d-1)n}\right) = O(n^{-1/2}).$$

The result now follows from Theorem 4.5. □

Remark 4.7. *We want to stress that the limiting process in Corollary 4.6 is indexed by $-\alpha$ rather than α because, as shown in its proof, φ^α_d is a functional of $C_{\beta,-\alpha}$: the event $\bigcap_{i=1}^d(\alpha_i U_i > \alpha_i u)$ coincides, up to a null set, with $\bigcap_{i=1}^d(-\alpha_i U_i \leq -\alpha_i u)$. Accordingly, the estimator $\widetilde{\varphi}^\alpha_{n,d}$ is a functional of the empirical process $C_{\beta,-\alpha,n}$.*

By Corollary 4.6, $\sqrt{n}\,(\widetilde{\varphi}^\alpha_{n,d} - \varphi^\alpha_d)$ converges weakly to a centered normal distribution. Consequently, $\widetilde{\varphi}^\alpha_{n,d} - \varphi^\alpha_d = O_p(n^{-1/2})$, so that $\widetilde{\varphi}^\alpha_{n,d} \xrightarrow{p} \varphi^\alpha_d$, i.e., the estimators $\widetilde{\varphi}^\alpha_{n,d}$ are consistent and asymptotically unbiased.

With the results obtained above, the conjecture posed in [8], involving the extreme footrule-estimators $\widetilde{\varphi}^+_{n,d}$ and $\widetilde{\varphi}^-_{n,d}$, is now resolved. Specifically, it is established that the proposed estimators are consistent and asymptotically normally distributed, provided that Condition 4.2 is satisfied. This confirmation not only validates the theoretical properties anticipated in the earlier work, but also provides a rigorous foundation for the application of these estimators in practical settings, including multivariate dependence modeling and directional coefficient estimation.

4.2 Simulations

To evaluate the empirical behavior of the proposed estimators, Monte Carlo simulations for d-copulas of the parametric Clayton and Cuadras-Augé families have been carried out. The expressions of these copulas are given by (7) and (8), respectively. The Clayton family satisfies Condition 4.2, since its first-order partial derivatives are continuous on the interior of the unit hypercube, and hence Corollary 4.6 applies to it. In contrast, the Cuadras–Augé copula has a singular component concentrated on the main diagonal of $[0,1]^d$, so that its first-order partial derivatives are discontinuous at the points where two or more coordinates attain the minimum, and Condition 4.2 does not hold. Hence, the simulation results for this family illustrate the behavior of the estimators beyond the scope of Corollary 4.6.

Two dimensions, $d \in \{4, 5\}$, and four sample sizes, $n \in \{20, 50, 100, 500\}$, have been considered. A Monte Carlo study was conducted to evaluate the behavior of the proposed estimators under different sampling scenarios. For each

combination of direction and parameter value, 1000 independent samples of size n were generated. Based on each sample, the directional estimators $\widetilde{\varphi}^{\alpha}_{n,d}$ were computed, and their empirical averages were subsequently obtained. These averages are reported in the corresponding tables, together with reference values of the associated directional coefficients φ^{α}_{d}, obtained through numerical integration.

The simulation results are organized according to the dimension, direction, and copula parameter. In particular, Table 3 summarizes the estimated values of $\varphi_4^{(-1,1,1,-1)}$ for the 4-dimensional Clayton copula, considering sample sizes $n \in \{20, 50, 100, 500\}$ and parameter values $\theta \in \{0.4, 0.6, 1, 2, 5\}$. Table 4 presents the estimation results for the coefficient $\varphi_5^{(1,-1,1,-1,1)}$ associated with the five-dimensional Clayton copula, again for the same range of sample sizes and parameter values. In all cases, the numerical approximations of the true coefficients serve as benchmarks for assessing the accuracy of the proposed estimators.

Table 3: Results obtained in Monte Carlo simulations to calculate the value of $\widetilde{\varphi}_{n,4}^{(-1,1,1,-1)}$ based on a total of 1000 samples of size n generated for the Clayton 4-copula for different values of the parameter θ.

Sample size			$n = 20$	$n = 50$	$n = 100$	$n = 500$
θ	φ_4^{α}	$\widetilde{\varphi}_{n,4}^{\alpha}$	mean	mean	mean	mean
0.4	-0.02042	$\widetilde{\varphi}_{n,4}^{(-1,1,1,-1)}$	-0.02660	-0.02147	-0.02198	-0.02036
0.6	-0.02829	$\widetilde{\varphi}_{n,4}^{(-1,1,1,-1)}$	-0.03366	-0.02891	-0.02960	-0.02816
1	-0.04053	$\widetilde{\varphi}_{n,4}^{(-1,1,1,-1)}$	-0.04402	-0.04090	-0.04141	-0.04029
2	-0.05946	$\widetilde{\varphi}_{n,4}^{(-1,1,1,-1)}$	-0.06104	-0.05920	-0.06000	-0.05915
5	-0.08234	$\widetilde{\varphi}_{n,4}^{(-1,1,1,-1)}$	-0.08166	-0.08144	-0.08218	-0.08213

Table 4: Results obtained in Monte Carlo simulations in order to calculate the value of $\widetilde{\varphi}_{n,5}^{(1,-1,1,-1,1)}$ based on a total of 1000 samples of size n generated for the Clayton 5-copula for different values of the parameter θ.

Sample size			$n = 20$	$n = 50$	$n = 100$	$n = 500$
θ	φ_5^{α}	$\widetilde{\varphi}_{n,5}^{\alpha}$	mean	mean	mean	mean
0.4	-0.01105	$\widetilde{\varphi}_{n,5}^{(1,-1,1,-1,1)}$	-0.01144	-0.01246	-0.01064	-0.01125
0.6	-0.01483	$\widetilde{\varphi}_{n,5}^{(1,-1,1,-1,1)}$	-0.01607	-0.01475	-0.01491	-0.01475
1	-0.02042	$\widetilde{\varphi}_{n,5}^{(1,-1,1,-1,1)}$	-0.02135	-0.02085	-0.02135	-0.02055
2	-0.02864	$\widetilde{\varphi}_{n,5}^{(1,-1,1,-1,1)}$	-0.02897	-0.02892	-0.02863	-0.02865
5	-0.03821	$\widetilde{\varphi}_{n,5}^{(1,-1,1,-1,1)}$	-0.03756	-0.03789	-0.03749	-0.03817

A careful inspection of the numerical results reveals that the proposed estimators closely track the corresponding theoretical directional footrule-coefficients, with their accuracy improving as the sample size increases. Across all configurations examined, the estimated coefficients take negative values for the directions under consideration. Nevertheless, for the Clayton d-copula, extreme directions lead to positive coefficient values, which increase toward unity as the dependence parameter θ grows, in agreement with the expected dependence behavior of the model.

Table 5 displays the performance of the estimators in all directions $\alpha \in \{-1, 1\}^4$ for 4-copulas from the Cuadras–Augé family with parameter values $\theta \in \{0.4, 0.8\}$, together with the theoretical values obtained from (9). The results are based on Monte Carlo simulations conducted on 1000 samples of different sizes, $n \in \{20, 50, 100, 500\}$, with the table reporting the outcomes for the case $n = 500$. The samples were generated by means of the stochastic representation $U_i = \max\{V_i^{1/(1-\theta)}, W^{1/\theta}\}$, $i = 1, \cdots, 4$, where $V_1, \cdots, V_4, W$ are independent random variables uniformly distributed on $[0, 1]$; it is easy to check that the distribution function of $(U_1, \cdots, U_4)$ is C_{θ}^{CA}.

The estimators correctly capture the sign and the relative magnitude of the coefficients in all directions. In the mixed directions with $|J| \in \{1, 2\}$, the estimates are very close to the theoretical values, whereas in the extreme directions $\alpha = \pm\mathbf{1}$ and, to a lesser extent, in the directions with $|J| = 3$, they are biased towards zero. This bias decreases as the sample size increases, but only at rate $n^{-1/2}$, i.e., at the same rate as the standard deviation of the estimators: for instance, for $\theta = 0.4$, the mean of $\widetilde{\varphi}_{n,4}^{(1,1,1,1)}$ equals 0.36514, 0.38481, 0.39390 and 0.39806 for $n = 100$, 500, 2000 and 8000, respectively (based on 1000 samples for $n \leq 2000$ and 300 samples for $n = 8000$), whereas $\varphi_4^{(1,1,1,1)} = 0.40283$. This behavior is due to the singular component of the Cuadras–Augé copula on the main diagonal, for which Condition 4.2

Table 5: Theoretical values φ_4^α and means of the estimates $\widetilde{\varphi}_{n,4}^\alpha$ in every direction $\alpha \in \{-1,1\}^4$, based on a total of 1000 samples of size 500 generated from the Cuadras-Augé 4-copula for $\theta = 0.4$ and $\theta = 0.8$.

	$\theta = 0.4$		$\theta = 0.8$	
α	φ_4^α	$\widetilde{\varphi}_{n,4}^\alpha$	φ_4^α	$\widetilde{\varphi}_{n,4}^\alpha$
$(-1,-1,-1,-1)$	0.21053	0.20445	0.61538	0.60172
$(1,-1,-1,-1)$	-0.00219	-0.00339	-0.05983	-0.05945
$(-1,1,-1,-1)$	-0.00219	-0.00233	-0.05983	-0.05955
$(1,1,-1,-1)$	-0.03520	-0.03418	-0.09169	-0.08942
$(-1,-1,1,-1)$	-0.00219	-0.00302	-0.05983	-0.05943
$(1,-1,1,-1)$	-0.03520	-0.03423	-0.09169	-0.08956
$(-1,1,1,-1)$	-0.03520	-0.03461	-0.09169	-0.08942
$(1,1,1,-1)$	-0.09835	-0.09269	-0.16084	-0.15489
$(-1,-1,-1,1)$	-0.00219	-0.00234	-0.05983	-0.05958
$(1,-1,-1,1)$	-0.03520	-0.03432	-0.09169	-0.08937
$(-1,1,-1,1)$	-0.03520	-0.03484	-0.09169	-0.08957
$(1,1,-1,1)$	-0.09835	-0.09304	-0.16084	-0.15478
$(-1,-1,1,1)$	-0.03520	-0.03420	-0.09169	-0.08957
$(1,-1,1,1)$	-0.09835	-0.09295	-0.16084	-0.15476
$(-1,1,1,1)$	-0.09835	-0.09311	-0.16084	-0.15460
$(1,1,1,1)$	0.40283	0.38481	0.81740	0.79224

does not hold. Nevertheless, as illustrated in Figures 4 and 5, the estimators converge toward the corresponding theoretical values as the sample size increases.

These properties make the proposed directional footrule-estimators particularly suitable for practical applications involving multivariate dependence modeling, such as risk management, reliability analysis, and financial or environmental data. In such contexts, Cuadras–Augé copulas provide a flexible framework for modeling radially asymmetric dependence structures, and the reliability of the estimators ensures their applicability within a broad class of mathematical models where directional dependence plays a relevant role.

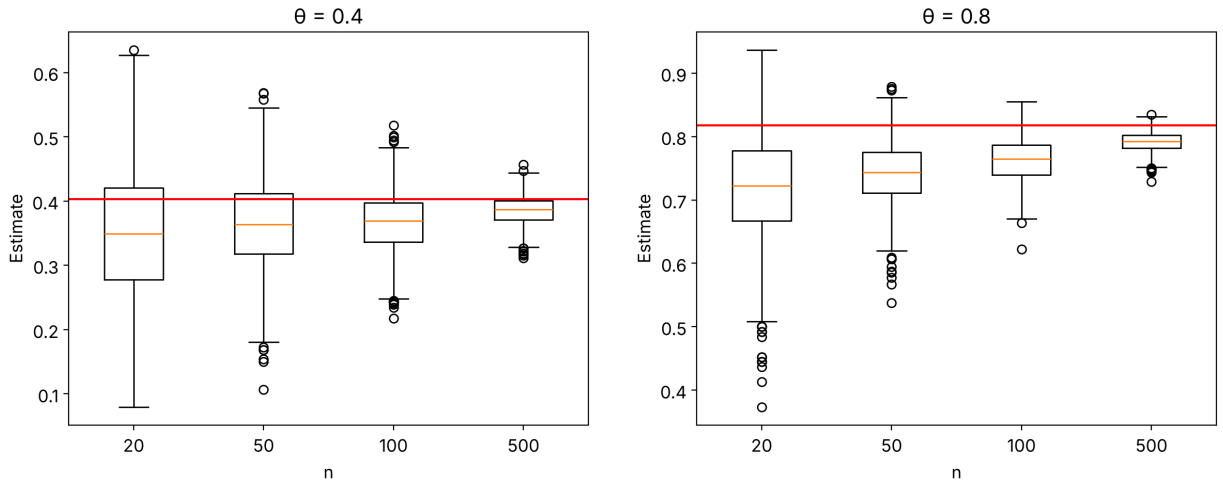


Figure 4: Empirical distribution of $\widetilde{\varphi}_{n,4}^{(1,1,1,1)}$ for the Cuadras-Augé 4-copula with parameter values $\theta = \{0.4, 0.8\}$ and sample sizes $n = \{20, 50, 100, 500\}$. The horizontal red line shows the value of $\varphi_4^{(1,1,1,1)}$.

Finally, to conclude this analysis, Figure 6 presents a comparative plot of the estimated directional footrule-coefficients for both the Clayton and Cuadras–Augé copulas. In the case of the four-dimensional Clayton copula, the directions $\mathbf{1}$ and $-\mathbf{1}$ exhibit positive coefficients. Simulations have made the exchangeability of the copula evident, as reflected by the equality of the coefficients depending on the number of coordinates equal to -1.

In contrast, for the four-dimensional Cuadras–Augé copula the coefficients reveal a clear radial asymmetry: for

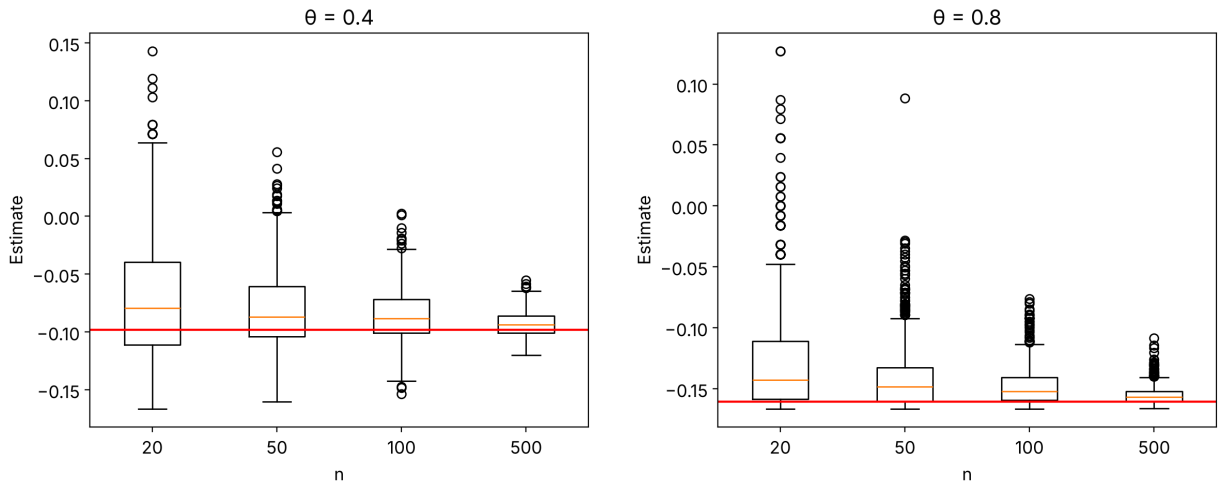


Figure 5: Empirical distribution of $\widetilde{\varphi}_{n,4}^{(1,-1,1,1)}$ for the Cuadras-Augé 4-copula with parameter values $\theta = \{0.4, 0.8\}$ and sample sizes $n = \{20, 50, 100, 500\}$. The horizontal red line shows the value of $\varphi_4^{(1,-1,1,1)}$.

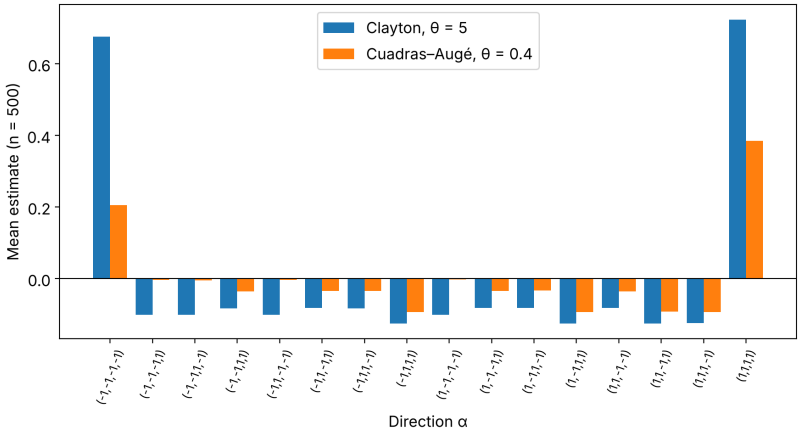


Figure 6: Comparative study of the estimated directional footrule-coefficients for two 4-dimensional copulas corresponding to the parametric families of Clayton and Cuadras-Augé. For the Clayton copula we have considered a parameter value $\theta = 5$, and for the Cuadras–Augé copula, $\theta = 0.4$.

$\theta = 0.4$, the theoretical values are $\varphi_4^{(1,1,1,1)} = 0.40283$ and $\varphi_4^{(-1,-1,-1,-1)} = 0.21053$ (see Table 5), so that simultaneously large values are more strongly associated than simultaneously small ones. Mixed directions yield negative coefficients whose magnitude increases with the number of coordinates equal to 1 (namely, -0.00219, -0.03520 and -0.09835 for $|J| = 1, 2, 3$, respectively). These results illustrate the singular-plus-continuous nature of the Cuadras–Augé copula and highlight the ability of the proposed estimators to capture subtle differences in dependence structure across copula families.

To conclude, we present a final example aimed at illustrating the relevance of the proposed coefficients in detecting and quantifying directional dependence in asymmetric settings. In particular, we consider an asymmetric extension of the Clayton family introduced in [22], where different components exhibit varying levels of dependence. This example highlights the ability of the directional footrule coefficients to reveal heterogeneous dependence patterns across directions.

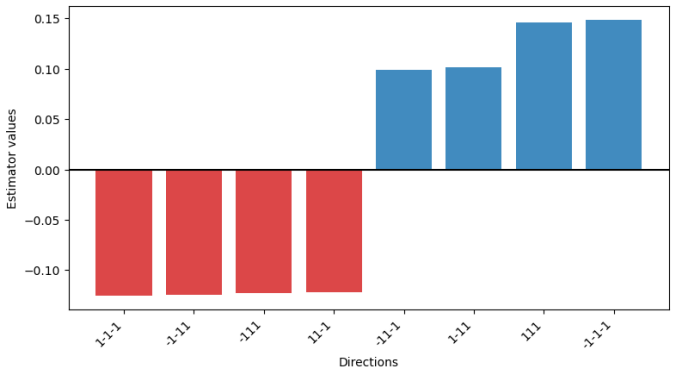


Figure 7: Values of the directional footrule estimators for the different directions in the asymmetric Clayton 3-copula defined in Example 4.8.

Example 4.8. To illustrate the potential of the proposed coefficients to capture directional dependence, we consider an asymmetric extension of the Clayton family of 3-dimensional copulas given by

$$C(\mathbf{u}) = \prod_{i=1}^{3} u_i^{1-\theta_i} \left(1 + \sum_{i=1}^{3} (u_i^{-\gamma\theta_i} - 1)\right)^{-\frac{1}{\gamma}},$$

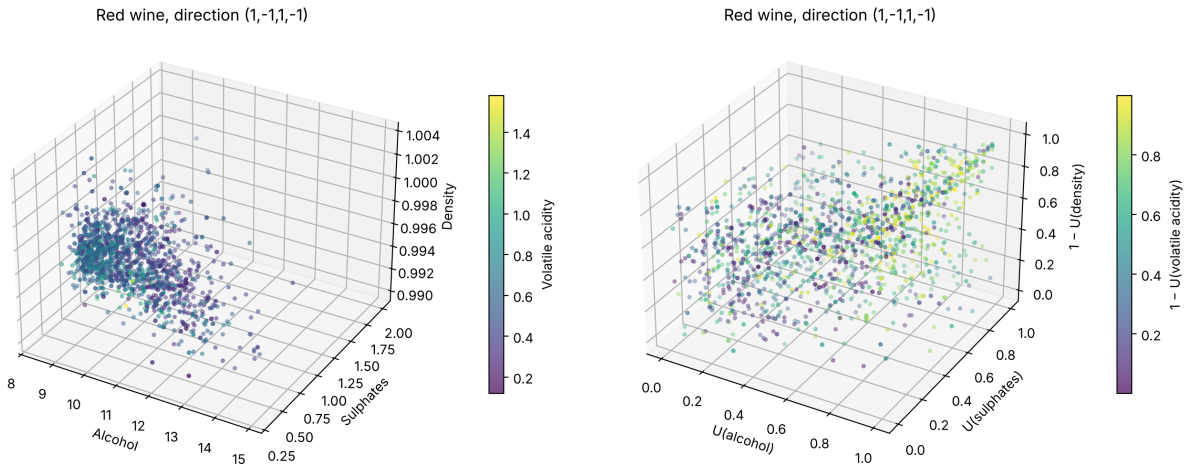
with parameters $\gamma = 100$ and $\boldsymbol{\theta} = (0.95759, 0.05, 0.98926)$. Figure 7 displays the values of the directional footrule estimators for the different directions $\alpha \in \{-1, 1\}^3$. The results clearly reveal a marked heterogeneity across directions, reflecting the asymmetric nature of the dependence structure induced by the parameter vector θ. In particular, the sign of each coefficient is determined by the relative orientation of the first and third components, which have the largest parameters ($\theta_1 = 0.95759$ and $\theta_3 = 0.98926$) and are therefore strongly concordant: the theoretical values of the four coefficients with $\alpha_1 = \alpha_3$ lie between 0.287 and 0.333, whereas those of the four coefficients with $\alpha_1 \neq \alpha_3$ are approximately -0.31. By contrast, the orientation of the second component, which is almost independent of the other two since $\theta_2 = 0.05$, has only a minor effect; for instance, $\varphi_3^{(1,1,1)} = 0.33259$, whereas $\varphi_3^{(1,-1,1)} = 0.28676$. This highlights the ability of the proposed coefficients to disentangle and quantify directional effects that remain hidden using classical, non-directional measures.

4.3 Real-data application: directional dependence among physicochemical wine components

To illustrate the practical usefulness of the proposed directional Spearman footrule coefficients, we consider the red and white wine datasets from the UCI Machine Learning Repository ([6]), comprising 1,599 and 4,898 observations, respectively. We focus on four physicochemical variables: alcohol, volatile acidity, sulphates, and density, in this order when defining the direction vector $\alpha \in \{-1, 1\}^4$. Since these variables are recorded with limited precision, the data contain many ties; for instance, alcohol takes only 65 distinct values among the 1,599 red wines. Ties are handled by midranks, as described in Section 4, so that the estimates satisfy the empirical counterpart of Corollary 3.1 iii). Confidence intervals are percentile bootstrap intervals based on 2000 resamples of the observations.

For red wines, the largest estimated coefficient is attained in the direction $\alpha = (1, -1, 1, -1)$, with an estimate of 0.09622 and a 95% bootstrap confidence interval of $[0.07838, 0.11283]$. This direction combines high ranks in alcohol and sulphates with low ranks in volatile acidity and density. Figure 8 shows the original observations along with the directionally oriented pseudo-observations. The opposite direction, $(-1, 1, -1, 1)$, yields the second largest estimate, 0.07678. The estimated difference, defined as the coefficient in the opposite direction minus that in the maximizing direction, is -0.01944, with a 95% bootstrap confidence interval of $[-0.03731, -0.00112]$. This interval excludes zero, although narrowly, providing some evidence of an asymmetry between the coefficients in these two directions. Note that, by Corollary 3.1 iii), $\varphi_4^{-\alpha}(C) = \varphi_4^{\alpha}(\widehat{C})$; hence, if C were radially symmetric (i.e., $C = \widehat{C}$), the coefficients in opposite directions would coincide, so that a difference significantly different from zero indicates a departure from radial symmetry.

Other directions also yield positive estimates. In particular, the coefficients for $(-1, 1, -1, -1)$ and $(1, -1, 1, 1)$ are 0.04707 and 0.03602, respectively. By contrast, the estimates for $(-1, -1, -1, 1)$ and $(1, 1, 1, -1)$ are close to zero, and their confidence intervals include zero. The most negative estimate is obtained for $(1, -1, -1, 1)$, at -0.07828, with a



(a) Original observations for red wine.

(b) Directionally oriented pseudo-observations.

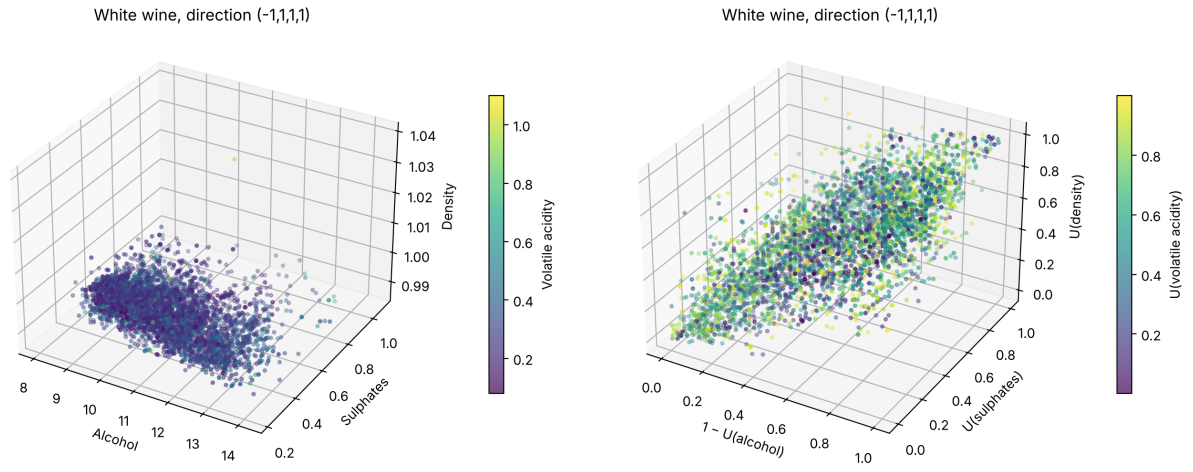
Figure 8: Three-dimensional scatterplots for the red wine dataset in the direction $(1, -1, 1, -1)$. The plots display alcohol, sulphates, and density on the three axes, with volatile acidity represented by the colour scale. The left panel uses the original observations, whereas the right panel uses directionally oriented pseudo-observations.

95% confidence interval of $[-0.08501, -0.07079]$. Overall, the estimates reveal substantial variation across directions, illustrating how the proposed coefficients distinguish different directional patterns of dependence. Table 6 reports the estimates for all directions, while Table 8 summarizes the comparisons between opposite directions. Figure 10a provides a graphical summary.

Table 6: Directional Spearman footrule coefficients for the red wine dataset ($n = 1599$), with bootstrap standard errors and 95% confidence intervals based on 2000 bootstrap samples. The variables are ordered as alcohol, volatile acidity, sulphates, and density. Midranks handle ties

Direction α	Estimate	Bootstrap SE	Lower 95% CI	Upper 95% CI
$(1, -1, 1, -1)$	0.09622	0.00897	0.07838	0.11283
$(-1, 1, -1, 1)$	0.07678	0.00811	0.06063	0.09244
$(-1, 1, -1, -1)$	0.04707	0.00718	0.03280	0.06116
$(1, -1, 1, 1)$	0.03602	0.00712	0.02267	0.05023
$(-1, -1, 1, 1)$	0.03138	0.00734	0.01739	0.04629
$(-1, 1, 1, 1)$	0.03041	0.00677	0.01708	0.04350
$(1, 1, -1, -1)$	0.02973	0.00789	0.01431	0.04525
$(1, -1, -1, -1)$	0.02448	0.00814	0.00875	0.04076
$(-1, -1, -1, 1)$	0.00307	0.00650	-0.00914	0.01550
$(1, 1, 1, -1)$	-0.00878	0.00693	-0.02230	0.00497
$(1, 1, 1, 1)$	-0.04202	0.00689	-0.05553	-0.02846
$(-1, -1, -1, -1)$	-0.04549	0.00709	-0.05902	-0.03066
$(1, 1, -1, 1)$	-0.05736	0.00491	-0.06612	-0.04748
$(-1, -1, 1, -1)$	-0.06740	0.00456	-0.07606	-0.05832
$(-1, 1, 1, -1)$	-0.07583	0.00392	-0.08299	-0.06809
$(1, -1, -1, 1)$	-0.07828	0.00359	-0.08501	-0.07079

The white-wine dataset exhibits a different pattern. Its largest estimated coefficient is attained in the direction $(-1, 1, 1, 1)$, with an estimate of 0.10712 and a 95% bootstrap confidence interval of $[0.09844, 0.11668]$. This direction combines low alcohol ranks with high ranks in volatile acidity, sulphates, and density. Figure 9 shows the original observations along with the directionally oriented pseudo-observations. The second-largest estimate corresponds to $(1, 1, -1, -1)$, with a coefficient of 0.09406 and a confidence interval of $[0.08400, 0.10369]$. Both intervals lie entirely



(a) Original observations for white wine.

(b) Directionally oriented pseudo-observations.

Figure 9: Three-dimensional scatterplots for the white wine dataset in the direction $(-1, 1, 1, 1)$. The plots display alcohol, sulphates, and density on the three axes, with volatile acidity represented by the colour scale. The left panel uses the original observations, whereas the right panel uses directionally oriented pseudo-observations.

above zero. In Table 7 we report the estimates for all directions, while Table 9 summarizes the comparisons between opposite directions. Figure 10b provides a graphical summary.

Unlike the red-wine results, all sixteen bootstrap confidence intervals for the white-wine directional coefficients exclude zero. Nevertheless, the estimates vary substantially, ranging from 0.10712 for $(-1, 1, 1, 1)$ to -0.07900 for $(-1, 1, 1, -1)$. The latter direction combines low alcohol ranks and high ranks in volatile acidity and sulphates with low density ranks. This variation reinforces the importance of examining the full family of directional coefficients rather than relying on a single summary value.

Comparisons between opposite directions further highlight these differences. For white wines, the difference between the coefficients in directions $(-1, -1, 1, 1)$ and $(1, 1, -1, -1)$ is -0.05731, with a 95% bootstrap confidence interval of $[-0.06713, -0.04686]$. Conversely, the coefficient in direction $(-1, 1, 1, 1)$ exceeds that in its opposite direction, $(1, -1, -1, -1)$, by 0.05896, with a confidence interval of $[0.04902, 0.06866]$. These results illustrate how comparisons between opposite directions can reveal asymmetries that are not apparent from examining individual coefficients in isolation. In fact, the confidence intervals of all eight differences between opposite directions exclude zero for white wines (Table 9). In contrast, for red wines this only happens for the pair discussed above (Table 8). Hence, the dependence structure of the four white-wine variables appears to be radially asymmetric, whereas for red wines radial symmetry cannot be ruled out for seven of the eight pairs of opposite directions.

The maximizing directions also differ between the two datasets: $(1, -1, 1, -1)$ for red wines and $(-1, 1, 1, 1)$ for white wines. Moreover, two directional estimates for red wines have confidence intervals that include zero, whereas all estimated coefficients for white wines have confidence intervals excluding zero; this is partly due to the larger sample size of the white-wine dataset, which yields narrower intervals. These findings suggest differences in the directional dependence patterns of the selected physicochemical variables across the two datasets. However, the separate confidence intervals do not constitute a formal test of differences between red and white wines.

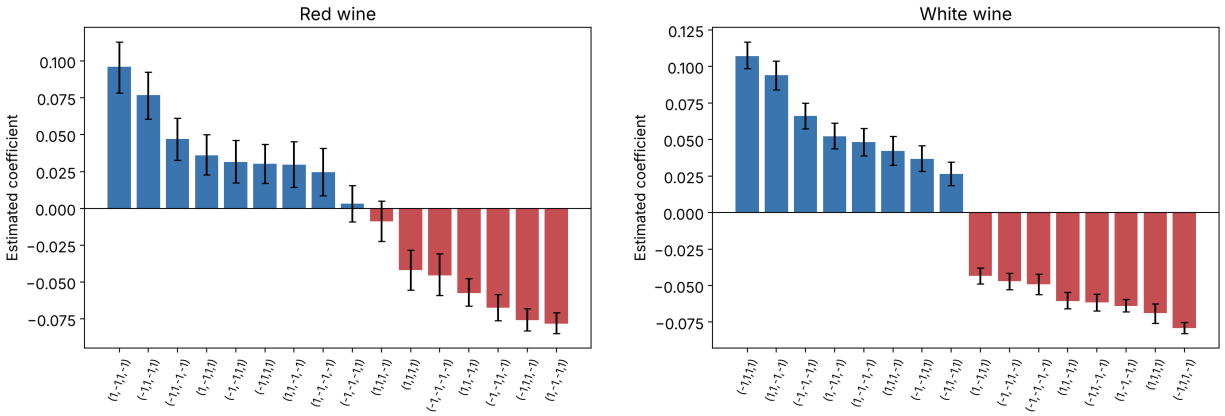
Overall, this first analysis illustrates the ability of directional Spearman footrule coefficients to characterize multivariate dependence across different orientations. The comparisons between opposite directions provide additional information about asymmetry, while the bootstrap procedure quantifies sampling uncertainty. The analysis therefore establishes a descriptive basis for the subsequent investigation of how these dependence patterns relate to sensory quality.

The preceding analysis characterized the dependence structure among four physicochemical properties of red and white wines without incorporating the sensory quality score. We now extend the analysis to investigate the relationship between these properties and wine quality. Specifically, we examine the marginal associations between quality and individual physicochemical variables and then assess whether the dominant directional dependence patterns among the four variables differ across quality groups.

This analysis complements the work of [5], which focused on predicting wine quality from physicochemical measure-

Table 7: Directional Spearman footrule coefficients for the white wine dataset ($n = 4898$), with bootstrap standard errors and 95% confidence intervals based on 2000 bootstrap samples. The variables are ordered as in Table 6.

Direction α	Estimate	Bootstrap SE	Lower 95% CI	Upper 95% CI
$(-1,1,1,1)$	0.10712	0.00454	0.09844	0.11668
$(1,1,-1,-1)$	0.09406	0.00511	0.08400	0.10369
$(-1,-1,-1,1)$	0.06619	0.00446	0.05734	0.07492
$(1,-1,1,-1)$	0.05223	0.00461	0.04361	0.06139
$(1,-1,-1,-1)$	0.04816	0.00481	0.03893	0.05779
$(1,1,1,-1)$	0.04209	0.00500	0.03238	0.05208
$(-1,-1,1,1)$	0.03675	0.00445	0.02823	0.04575
$(-1,1,-1,1)$	0.02649	0.00418	0.01844	0.03465
$(1,-1,1,1)$	-0.04336	0.00276	-0.04881	-0.03803
$(-1,-1,1,-1)$	-0.04699	0.00288	-0.05278	-0.04161
$(-1,-1,-1,-1)$	-0.04916	0.00354	-0.05611	-0.04199
$(1,1,-1,1)$	-0.06051	0.00292	-0.06590	-0.05455
$(-1,1,-1,-1)$	-0.06140	0.00291	-0.06726	-0.05571
$(1,-1,-1,1)$	-0.06383	0.00228	-0.06799	-0.05932
$(1,1,1,1)$	-0.06885	0.00344	-0.07569	-0.06246
$(-1,1,1,-1)$	-0.07900	0.00200	-0.08291	-0.07518



(a) Red wine.

(b) White wine.

Figure 10: Estimated directional Spearman footrule coefficients for the physicochemical properties of red and white wines across all directions $\alpha \in \{-1,1\}^4$.

Table 8: Bootstrap comparisons between opposite directions for the red wine dataset ($n = 1599$). The difference is the coefficient in direction α minus the coefficient in direction $-\alpha$.

Direction α	Direction $-\alpha$	Difference	Bootstrap SE	95% CI
$(-1,-1,-1,-1)$	$(1,1,1,1)$	-0.00346	0.00875	$[-0.01964, 0.01379]$
$(-1,-1,-1,1)$	$(1,1,1,-1)$	0.01185	0.00816	$[-0.00497, 0.02717]$
$(-1,-1,1,-1)$	$(1,1,-1,1)$	-0.01005	0.00664	$[-0.02344, 0.00238]$
$(-1,-1,1,1)$	$(1,1,-1,-1)$	0.00166	0.00857	$[-0.01533, 0.01860]$
$(-1,1,-1,-1)$	$(1,-1,1,1)$	0.01105	0.00826	$[-0.00547, 0.02701]$
$(-1,1,-1,1)$	$(1,-1,1,-1)$	-0.01944	0.00914	$[-0.03731, -0.00112]$
$(-1,1,1,-1)$	$(1,-1,-1,1)$	0.00246	0.00463	$[-0.00632, 0.01164]$
$(-1,1,1,1)$	$(1,-1,-1,-1)$	0.00593	0.00872	$[-0.01123, 0.02373]$

Table 9: Bootstrap comparisons between opposite directions for the white wine dataset ($n = 4898$). The difference is the coefficient in direction α minus the coefficient in direction $-\alpha$.

Direction α	Direction $-\alpha$	Difference	Bootstrap SE	95% CI
$(-1,-1,-1,-1)$	$(1,1,1,1)$	0.01969	0.00503	$[0.00976, 0.03018]$
$(-1,-1,-1,1)$	$(1,1,1,-1)$	0.02410	0.00521	$[0.01370, 0.03416]$
$(-1,-1,1,-1)$	$(1,1,-1,1)$	0.01352	0.00398	$[0.00540, 0.02092]$
$(-1,-1,1,1)$	$(1,1,-1,-1)$	-0.05731	0.00522	$[-0.06713, -0.04686]$
$(-1,1,-1,-1)$	$(1,-1,1,1)$	-0.01804	0.00403	$[-0.02589, -0.00992]$
$(-1,1,-1,1)$	$(1,-1,1,-1)$	-0.02574	0.00486	$[-0.03530, -0.01643]$
$(-1,1,1,-1)$	$(1,-1,-1,1)$	-0.01517	0.00283	$[-0.02084, -0.00977]$
$(-1,1,1,1)$	$(1,-1,-1,-1)$	0.05896	0.00501	$[0.04902, 0.06866]$

ments using data-mining techniques. Rather than evaluating predictive performance, we aim is to provide a descriptive characterization of the associations with quality and of the multivariate directional dependence patterns observed across quality groups. Red and white wines are analyzed separately.

We first calculate Spearman's rank correlation coefficients between quality and alcohol, volatile acidity, sulphates, and density. Table 10 reports the estimates, 95% confidence intervals, and corresponding p-values.

Alcohol displays the strongest positive marginal association with quality in both datasets, with Spearman coefficients of 0.47853 for red wines and 0.44037 for white wines. Volatile acidity is negatively associated with quality, with a stronger association for red wines (-0.38065) than for white wines (-0.19656). The relationship between sulphates and quality differs more markedly: the correlation is positive and substantially stronger for red wines (0.37706), whereas it is weak for white wines (0.03332), despite being statistically significant at the 5% level. Finally, density is negatively associated with quality in both datasets, with a stronger association for white wines (-0.34835) than for red wines (-0.17707). These results indicate that the marginal association patterns differ across wine types, although formal tests are needed to establish whether the corresponding correlations differ significantly. Figure 11 examines the relationships between wine quality and selected chemical properties.

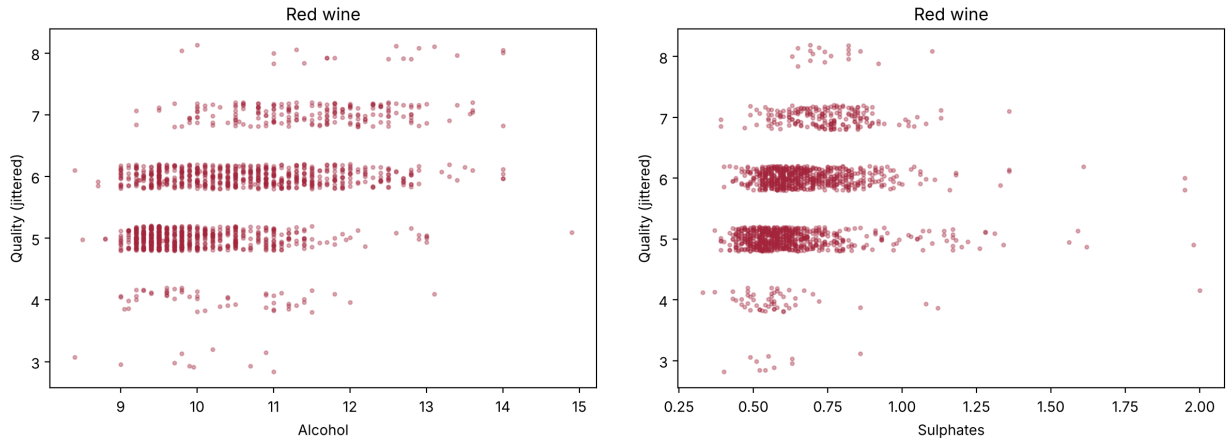
Table 10: Spearman correlations between physicochemical properties and wine quality. The sample sizes are $n = 1599$ for red wines and $n = 4898$ for white wines. Confidence intervals are based on the bootstrap procedure, whereas p-values are asymptotic.

	Red wines			White wines		
Variable	ρ_S	95% CI	p-value	ρ_S	95% CI	p-value
Alcohol	0.47853	[0.43890, 0.51729]	2.7268×10^{-92}	0.44037	[0.41636, 0.46532]	1.6592×10^{-231}
Volatile acidity	-0.38065	[-0.42303, -0.33676]	2.7349×10^{-56}	-0.19656	[-0.22447, -0.17002]	7.4172×10^{-44}
Sulphates	0.37706	[0.33310, 0.41695]	3.4777×10^{-55}	0.03332	[0.00341, 0.06269]	0.01971
Density	-0.17707	[-0.22563, -0.13075]	9.9181×10^{-13}	-0.34835	[-0.37365, -0.32175]	9.4971×10^{-140}

Next, we examine the directional footrule coefficients among the four physicochemical variables within each quality group. For each group, we identify the direction $\alpha \in \{-1, 1\}^4$ that yields the largest estimated coefficient. Table 11 summarizes the maximizing directions, estimates, and 95% confidence intervals.

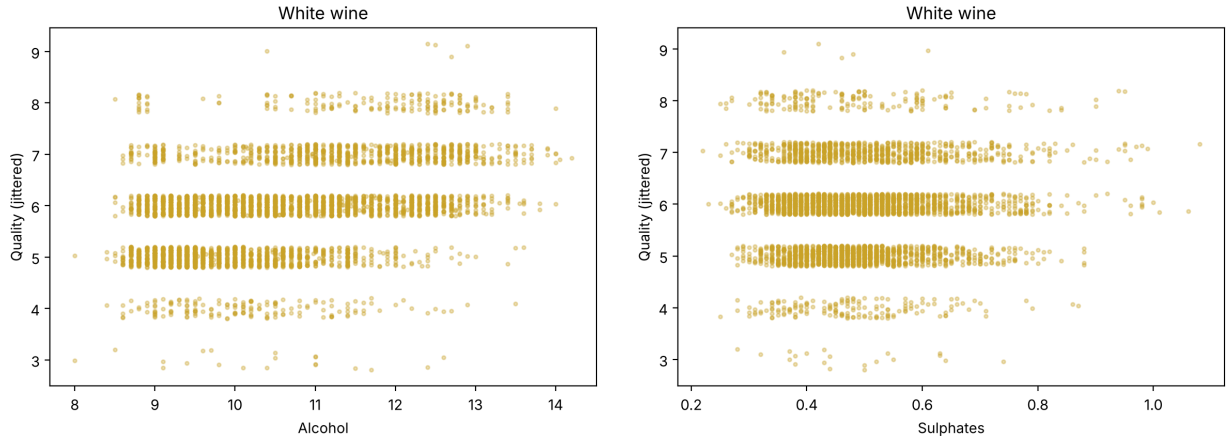
For red wines, the maximizing direction in the full sample is $(1, -1, 1, -1)$. Within the quality groups, however, $(-1, -1, 1, 1)$ is selected for quality groups 3–4 (where it ties with $(1, 1, -1, -1)$), 6, and 7–8, whereas $(1, 1, -1, -1)$ is selected for quality 5. Thus, the direction attaining the maximum estimate varies across the groups, although these differences do not by themselves establish statistically significant changes in the underlying dependence structure. Interestingly, the signs of the full-sample maximizing direction $(1, -1, 1, -1)$ coincide with those of the Spearman correlations of the four variables with quality (Table 10), and the coefficient in this direction, which equals 0.09622 in the full sample, ranges only between 0.00997 and 0.04018 within the quality groups. This suggests that a substantial part of the directional dependence detected in the full red-wine sample is induced by the heterogeneity across quality levels.

For white wines, the direction $(-1, 1, 1, 1)$ is selected for the full sample and for quality groups 3–4 and 5. In contrast, $(1, 1, -1, -1)$ is selected for quality groups 6 and 7–9. The corresponding maximum estimate increases from 0.10944 for quality 6 to 0.15233 for quality 7–9. Moreover, the coefficient in the direction $(-1, 1, 1, 1)$, which dominates in the full sample, remains close to 0.10 in quality groups 3–4, 5 and 6, but drops to 0.01098 in quality group 7–9. This



(a) Alcohol content in red wine.

(b) Sulphates in red wine.



(c) Alcohol content in white wine.

(d) Sulphates in white wine.

Figure 11: Scatterplots of wine quality against alcohol content and sulphate levels for red and white wines. Vertical jitter is added to the quality scores to improve the visualisation of overlapping observations.

Table 11: Maximum estimated directional footrule coefficients by wine type and quality group.

Wine type	Quality group	n	Direction α	$\widetilde{\varphi}^{\alpha}_{n,4}$	95% CI
Red	All	1599	$(1,-1,1,-1)$	0.09622	[0.07838, 0.11283]
Red	3–4	63	$(-1,-1,1,1)$ *	0.10549	[0.02571, 0.18196]
Red	5	681	$(1,1,-1,-1)$	0.07925	[0.05397, 0.10403]
Red	6	638	$(-1,-1,1,1)$	0.08436	[0.05961, 0.11030]
Red	7–8	217	$(-1,-1,1,1)$	0.10736	[0.05483, 0.15718]
White	All	4898	$(-1,1,1,1)$	0.10712	[0.09844, 0.11668]
White	3–4	183	$(-1,1,1,1)$	0.09661	[0.04761, 0.14965]
White	5	1457	$(-1,1,1,1)$	0.11064	[0.09385, 0.12765]
White	6	2198	$(1,1,-1,-1)$	0.10944	[0.09369, 0.12529]
White	7–9	1060	$(1,1,-1,-1)$	0.15233	[0.13144, 0.17542]

Notes. The table reports the direction attaining the largest estimated coefficient in each group. Since the direction is selected by maximizing the estimates, the confidence intervals should be interpreted cautiously because they do not account explicitly for this selection. The quality scores are grouped as follows: scores 3–4 represent the lowest-quality wines, score 5 low-to-intermediate quality, score 6 intermediate quality, and scores 7–9 the highest-quality wines (no red wine received a score of 9, so the corresponding group only contains scores 7 and 8). *In this group, the direction $(1,1,-1,-1)$ attains the same estimate.

pattern suggests that the strongest estimated directional dependence is more pronounced in the highest-quality group considered. However, it does not establish a monotonic relationship between wine quality and directional dependence.

These group-specific results should be interpreted with some caution. In particular, the reported confidence intervals do not explicitly account for selecting the largest estimate among sixteen directions. Consequently, formal comparisons between quality groups would require an inferential procedure that accounts for this selection, for example, a suitable bootstrap or permutation procedure.

Taken together, the results provide two complementary perspectives on the relationship between physicochemical properties and wine quality. Spearman's coefficients quantify the marginal association between each variable and quality, whereas the directional footrule coefficients characterize joint dependence patterns among the physicochemical variables. The changes in maximizing direction across some quality groups suggest that the joint dependence structure may vary with quality in ways that cannot be described by marginal correlations alone.

These findings support the use of directional footrule coefficients as a descriptive complement to conventional association measures in wine-data analysis. Nevertheless, the present results do not establish causal relationships or demonstrate an improvement in predictive performance. Assessing whether directional dependence information improves wine-quality prediction would require comparing models with and without such information using held-out observations or an appropriate cross-validation procedure.

5 Conclusions

In this paper, we have introduced a new family of directional multivariate footrule-coefficients within the copula framework, aimed at capturing direction-specific dependence structures in multivariate data. By extending the classical Spearman's footrule to a directional setting, the proposed coefficients capture dependence patterns undetectable by traditional measures.

We have established the main theoretical properties of the proposed coefficients, showing their consistency with existing multivariate versions in the extreme directional cases and highlighting their structural coherence. In particular, the representation of the directional coefficients as linear combinations of lower-dimensional directional footrule-coefficients offers new insight into the contribution of partial dependence among subsets of components to the overall dependence structure.

To facilitate practical implementation, we have proposed nonparametric rank-based estimators for the directional coefficients and studied their finite-sample behavior through Monte Carlo simulations. The numerical results demonstrate that the proposed estimators provide accurate approximations of the theoretical coefficients and exhibit improved performance as the sample size increases. Explicit expressions and illustrative examples for well-known copula families, including the Farlie–Gumbel–Morgenstern, Clayton and Cuadras–Augé copulas, further emphasize the interpretability and flexibility of the proposed approach. An application to red and white wine datasets further illustrates the practical usefulness of the proposed coefficients for identifying directional dependence patterns among physicochemical properties and examining how these patterns vary across wine quality levels.

Overall, the directional footrule-coefficients introduced in this work enrich the toolbox of rank-based dependence measures and offer a versatile framework for analyzing complex multivariate dependence structures. Possible directions for future research include the derivation of explicit expressions for the asymptotic variances, which would allow inference without resorting to the bootstrap; the development of formal tests for directional dependence and radial asymmetry, including procedures that account for the selection of the maximizing direction; the study of the estimators for copulas with singular components, such as the Cuadras–Augé family, for which Condition 4.2 fails; and the extension of the proposed methodology to broader classes of dependence models.

Declarations

Data availability: The data employed in this study are publicly accessible through the UCI Machine Learning Repository: Wine Quality [Dataset] (see [6]: https://doi.org/10.24432/C56S3T).

Compliance with Ethical Standards:

- The authors declare that they have no conflict of interest.
- This paper does not contain any studies with human participants or animals performed by any of the authors.

Declaration of Generative AI and AI-Assisted Technologies:

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript.

Acknowledgement

The authors acknowledge the comments of a reviewer. The first author is partially supported by the CDTIME of the University of Almería (Spain).

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Appendix

Proof of Theorem 2.1. Let $\mathbf{X}^{(J)} = (Y_1, \cdots, Y_d)$ be the random vector whose components are given by (2). For $j \notin J$ the inequality $Y_j \le u_j$ is equivalent to $X_j \le u_j$, and for $j \in J$ it is equivalent to $X_j \ge 1 - u_j$. Thus, the d-copula $C^{(J)}$ associated with $\mathbf{X}^{(J)}$ is given by

$$C^{(J)}(\mathbf{u}) = \mathbb{P}\left[\mathbf{X}^{(J)} \le \mathbf{u}\right] = \mathbb{P}\left[\bigcap_{j\notin J}\{X_j \le u_j\}, \bigcap_{j\in J}\{X_j \ge 1 - u_j\}\right]$$

for all $\mathbf{u} \in [0,1]^d$. Define

$$A = \bigcap_{j\notin J}\{X_j \le u_j\}.$$

Since for $j \in J$ the event $\{X_j \ge 1 - u_j\}$ coincides, up to a null set, with the complement of $\{X_j \le 1 - u_j\}$, we have

$$\begin{aligned}\left(\bigcap_{j\notin J}\{X_j \le u_j\}, \bigcap_{j\in J}\{X_j \ge 1 - u_j\}\right) &= \left\{\omega \in \Omega :\, \omega \in A \text{ and } \omega \notin \bigcup_{j\in J}\{X_j \le 1 - u_j\}\right\} \\ &= A \setminus \left(A \cap \bigcup_{j\in J}\{X_j \le 1 - u_j\}\right).\end{aligned}$$

Thus,

$$C^{(J)}(\mathbf{u}) = \mathbb{P}[A] - \mathbb{P}\left[A \cap \bigcup_{j\in J}\{X_j \le 1 - u_j\}\right]. \tag{16}$$

By the inclusion–exclusion principle, we can express the probability of the union as

$$\mathbb{P}\left[A \cap \bigcup_{j\in J}\{X_j \le 1 - u_j\}\right] = \sum_{\emptyset\neq K\subseteq J} (-1)^{|K|-1}\, \mathbb{P}\left[A \cap \bigcap_{j\in K}\{X_j \le 1 - u_j\}\right].$$

Since C is the copula of $\mathbf{X}$, for any subset $L \subseteq \{1, 2, \cdots, d\}$ and any numbers $\{t_j\}_{j\in L} \subseteq [0,1]$ we have

$$\mathbb{P}\left[\bigcap_{j\in L}\{X_j \le t_j\}\right] = C_L\Big((t_j)_{j\in L}\Big).$$

Then we have

$$\mathbb{P}(A) = \mathbb{P}\left[\bigcap_{j\notin J}\{X_j \le u_j\}\right] = C_I\Big(\nu(\mathbf{u}, I, \emptyset)\Big),$$

and for any nonempty $K \subseteq J$,

$$\mathbb{P}\left[A \cap \bigcap_{j\in K}\{X_j \le 1 - u_j\}\right] = C_{I\cup K}\Big(\nu(\mathbf{u}, I, K)\Big).$$

Substituting the above expressions into (16), we obtain

$$\begin{aligned}C^{(J)}(\mathbf{u}) &= C_I\Big(\nu(\mathbf{u}, I, \emptyset)\Big) - \sum_{\emptyset\neq K\subseteq J} (-1)^{|K|-1}\, C_{I\cup K}\Big(\nu(\mathbf{u}, I, K)\Big) \\ &= C_I\Big(\nu(\mathbf{u}, I, \emptyset)\Big) - \sum_{1\le k\le m} C_{I\cup\{i_k\}}\Big(\nu(\mathbf{u}, I, \{i_k\})\Big) \\ &\quad + \sum_{1\le k<l\le m} C_{I\cup\{i_k, i_l\}}\Big(\nu(\mathbf{u}, I, \{i_k, i_l\})\Big) + \cdots + (-1)^m\, C\Big(\nu(\mathbf{u}, I, J)\Big),\end{aligned}$$

where we have enumerated the elements of J as $\{i_1, i_2, \cdots, i_m\}$. Therefore, Expression (3) easily follows. □