MIXED VARIATIONAL INCLUSIONS INVOLVING INFINITE FAMILY OF FUZZY MAPPINGS

R. AHMAD, M. DILSHAD AND J. C. YAO

Abstract. In this paper, we introduce and study a mixed variational inclusion problem involving infinite family of fuzzy mappings. An iterative algorithm is constructed for solving a mixed variational inclusion problem involving infinite family of fuzzy mappings and the convergence of iterative sequences generated by the proposed algorithm is proved. Some illustrative examples are also given.

1. **Introduction**

The concept of fuzzy set theory was introduced by Zadeh [27] in 1965. The applications of the fuzzy set theory can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science and operational research [28]. For related topics, we refer to [22, 10].

Heilpern [6] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mapping which is fuzzy analogue of Nadler's fixed point theorem for set-valued mappings. In 1989, Chang and Zhu [3] first introduced and studied a class of variational inequalities with fuzzy mappings and extended the results of Lassonde [14], Shih and Tan [21], Takahashi [24] and Yen [26].

Since then, several classes of variational inequalities (inclusions) with fuzzy mappings were considered and studied by Chang and Huang [2], Noor [17], Huang [9], Park and Jeong [18, 19], Ding and Park [5], Ding [4], Wu and Xu [25], Kumam and Petrol [11], Lee et al. [15] and Lan [12], etc..

In this paper, we introduce and study a mixed variational inclusion problem involving infinite family of fuzzy mappings in real uniformly smooth Banach spaces, which includes as special cases, many known classes of variational inequalities with fuzzy mappings.

In the next section, we recall some basic definitions and formulate the main problem, i.e. a mixed variational inclusion problem involving infinite family of fuzzy mappings. Some examples and some special cases are given.

In section 3, we construct an iterative algorithm for solving a mixed variational inclusion problem involving infinite family of fuzzy mappings. Section 4 is devoted

Received: October 2010; Revised: April 2011 and October 2011; Accepted: December 2011

Key words and phrases: Mixed variational inclusions, Infinite family, Fuzzy mappings, Algorithm.

This work is supported by Department of Science and Technology, Government of India under grant no. SR/S4/MS: 577/09.

to the proof of an existence and convergence result for a mixed variational inclusion problem involving infinite family of fuzzy mappings. In the last section, conclusion of the paper is given.

2. **Preliminaries**

Let E be a real Banach space with norm $\|\cdot\|$, E^* is the topological dual of E, d is the metric induced by the norm $\|\cdot\|$, $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E and $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$\mathcal{D}(A,B)=\max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(A,y)\},$$

where $d(x,B)=\inf_{y\in B}\ d(x,y)$ and $d(A,y)=\inf_{x\in A}\ d(x,y)$. We also assume that $\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^* and $\mathcal{J}:E\to 2^{E^*}$ is the normalized duality mapping defined by

$$\mathcal{J}(x)=\{f\in E^*:\langle x,f\rangle=\|x\|\|f\|,\ \text{and}\ \|f\|=\|x\|\ \},\ \text{for all}\ x\in E.$$

The function $\gamma_E:[0,\infty)\to[0,\infty)$ defined by

$$\gamma_E(t)\ =\ \sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1:\ \|x\|\le 1,\ \|y\|\le t\right\},$$

is called the modulus of smoothness of E. We remark that a Banach space E is uniformly smooth if $\lim\limits_{t\to 0}\frac{\gamma_E(t)}{t}=0$.

Definition 2.1. A mapping $g:E\to E$ is said to be

(i) δ-strongly accretive, if for any $x,y\in E$, $j(x-y)\in\mathcal{J}(x-y)$, there exists a constant $\delta>0$ such that

$$\langle g(x)-g(y),\ j(x-y)\rangle\ge\delta\|x-y\|^2;$$

(ii) Lipschitz continuous, if for any $x,y\in E$, there exists a constant $\lambda_g>0$ such that

$$\|g(x)-g(y)\|\le\lambda_g\|x-y\|;$$

(iii) α-relaxed Lipschitz continuous, if for any $x,y\in E$, $j(x-y)\in\mathcal{J}(x-y)$, there exists a constant $\alpha>0$ such that

$$\langle g(x)-g(y),\ j(x-y)\rangle\le-\alpha\|x-y\|^2.$$

Definition 2.2. A mapping $A:E\to E$ is said to be

(i) r-strongly η-accretive, if for any $x,y\in E$, $j(\eta(x,y))\in\mathcal{J}(\eta(x,y))$, there exists a constant $r>0$ such that

$$\langle A(x)-A(y),\ j(\eta(x,y))\rangle\ge r\|x-y\|^2;$$

(ii) strictly η-accretive, if A is η-accretive and the equality holds if and only if $x=y$.

Definition 2.3. A set-valued mapping $M : E \to 2^E$ is said to be m-relaxed η-accretive, if for any $x, y \in E, u \in M(x), v \in M(y)$, there exists a constant $m > 0$ such that

$$\langle u - v,\ j(\eta(x, y))\rangle \geq -m\|x - y\|^2.$$

Definition 2.4. Let $A : E \to E,\ \eta : E \times E \to E$ be the single-valued mappings. Then the set-valued mapping $M : E \to 2^E$ is said to be (A, η)-accretive, if

(i) M is m-relaxed η-accretive;

(ii) $(A + \rho M)(E) = E$, for every $\rho > 0$.

Definition 2.5. A mapping $N : E^\infty = E \times E \times E \ldots \to E$ is said Lipschitz continuous in the i^{th} argument with constants β_i, if

$$\|N(\cdots, x_i, \cdots) - N(\cdots, y_i, \cdots)\| \leq \beta_i\|x_i - y_i\|, \text{ for all } x_i, y_i \in E, i = 1, 2, \ldots\ .$$

A mapping F from a set E to the collection $\mathcal{F}(E) = \{B : E \to [0, 1],\ \text{ a function }\}$ of fuzzy sets over E is called a fuzzy mapping, which means that for each $x \in E$ a fuzzy set $F(x)$, denoted by F_x, is a function from E to $[0, 1]$. For each $y \in E$, $F_x(y)$ denotes the membership-grade of y in F_x.

A fuzzy mapping $F : E \to \mathcal{F}(E)$ is said to be closed if for each $x \in E$, the function $y \to F_x(y)$ is upper semi-continuous, that is, for any given net $\{y_\alpha\} \subset E$, satisfying $y_\alpha \to y_0 \in E$, we have

$$\limsup_\alpha F_x(y_\alpha) \leq F_x(y_0).$$

For $B \in \mathcal{F}(E)$ and $\lambda \in [0, 1]$, the set $(B)_\lambda = \{x \in E : B(x) \geq \lambda\}$ is called a λ-cut set of B. Let $F : E \to \mathcal{F}(E)$ be a closed fuzzy mapping satisfying the following condition:

Condition $(*)$: If there exists a function $a : E \to [0, 1]$ such that for each $x \in E$, the set $(F_x)_{a(x)} = \{y \in E : F_x(y) \geq a(x)\}$ is a nonempty bounded subset of E.

Remark 2.6. Let E be a normed vector space. If F is a closed fuzzy mapping satisfying the condition $(*)$, then for each $x \in E$, the set $(F_x)_{a(x)}$ belongs to the collection $CB(E)$ of all nonempty closed and bounded subsets of E. In fact, let $\{y_\alpha\} \subset (F_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in E$, then $(F_x)(y_\alpha) \geq a(x)$ for each α. Since F is closed, we have

$$F_x(y_0) \geq \lim_\alpha F_x(y_\alpha) \geq a(x),$$

which implies that $y_0 \in (F_x)_{a(x)}$ and so $(F_x)_{a(x)} \in CB(E)$.

Let $T_i : E \to \mathcal{F}(E),\ i = 1, 2, \ldots$ be an infinite family of closed fuzzy mappings satisfying condition $(*)$. Then there exist functions $a_i : E \to [0, 1],\ i = 1, 2, \ldots$ such that for each $u \in E$, we have $(T_1u)_{a_1(u)},\ (T_2u)_{a_2(u)}, \ldots \in CB(E)$. Therefore, we can define set-valued mappings $\tilde{T}_i : E \to CB(E),\ i = 1, 2, \ldots$ by $\tilde{T}_1(u) = (T_{1u})_{a_1(u)},\ \tilde{T}_2(u) = (T_{2u})_{a_2(u)}, \ldots$ for all $u \in E$. In the sequel, $\tilde{T}_1,\ \tilde{T}_2,\ \tilde{T}_3, \ldots$ are called the set-valued mappings induced by the fuzzy mappings $T_1\ , T_2,\ T_3, \ldots$ respectively.

Let $N : E^\infty = E \times E \times E \ldots \to E,\ g : E \to E$ and $\eta : E \times E \to E$ be three mappings such that $g \not\equiv 0$. Let $T_i : E \to \mathcal{F}(E)$ be the fuzzy mappings and

$a_i : E \to [0,1]$ be the given functions, $i = 1, 2, \dots$. Let $M : E \times E \to 2^E$ be a set-valued mapping such that for each $u \in E, M(\cdot, u)$ is an (A, η)-accretive mapping in the first argument with $g(u) \in \text{ dom } M(\cdot, u)$. We consider the following problem:

Find $u, x_1, x_2, \dots \in E$, such that

$$\begin{aligned} &(T_{1u})(x_1) \geq a_1(u),\ (T_{2u})(x_2) \geq a_2(u), \dots \text{ and} \\ &0 \in N(x_1, x_2, \cdots) + M(g(u), u). \end{aligned} \tag{1}$$

Problem (1) is called mixed variational inclusion problem involving infinite family of fuzzy mappings.

In support of problem (1), we give the following examples.

Example 2.7. Let $E = [0,1]$ and suppose

(i) the closed fuzzy mappings $T_i : E \to \mathcal{F}(E),\ i = 1, 2, \dots$ are defined by

$$T_{iu}(x_i) = \begin{cases} \dfrac{u + x_i}{2}, & \text{if } u \in [0,\ \frac{1}{2}),\ x_i \in [0,1]; \\ ((1 - \dfrac{1}{i})u + \frac{1}{i})x_i, & \text{if } u \in [\frac{1}{2},\ 1],\ x_i \in [0,1]. \end{cases}$$

(ii) the functions $a_i : E \to E,\ i = 1, 2, \dots$ are defined by

$$a_i(u) = \begin{cases} \dfrac{u}{2}, \text{ if } u \in [0,\ \frac{1}{2}); \\ 0, \text{ if } u \in [\frac{1}{2},\ 1]. \end{cases}$$

(iii) for all $x_i \in [0,1]$, the mapping $N : E^\infty : E \times E \times E \dots \to E$, is defined by

$$N(x_1, x_2, \cdots) = \begin{cases} 1, \text{ if } x_1 > x_2 > x_3 > \cdots; \\ \dfrac{1}{2}, \text{ if } x_1 < x_2 < x_3 < \cdots; \\ 0, \text{ if } x_1 = x_2 = x_3 = \cdots; \\ \dfrac{1}{3}, \text{ otherwise.} \end{cases}$$

(iv) the mappings $g, A : E \to E$ are defined by

$$(a) \quad g(u) = u^2, \text{ for all } u \in [0,1],\ g \not\equiv 0;$$

and

$$(b) \quad A(u) = 0, \text{ for all } u \in [0,1].$$

(v) the mapping $\eta : E \times E \to E$ is defined by

$$\eta(x,y) = \begin{cases} x-y, & \text{if } x > y, \\ y-x, & \text{if } y > x, \\ 0, & \text{if } x = y, \end{cases} \text{, for all } x, y \in [0,1].$$

(vi) the set-valued mapping $M : E \times E \to 2^E$ is defined by

$$M(g(u), u) = \{g(u)\} \cup \{u\}, \text{ for all } u \in E,$$

such that $g(u) \in \text{dom}M(\cdot, u)$. Then it is easy to check that M is an (A, η)-accretive in the first argument for $\rho = 1$.

In view of assumptions (i)-(vi), it is clear that all the conditions of problem (1) are satisfied.

The following problem plays an important role in fuzzy games, which can be obtain from problem (1).

Example 2.8. Consider a 'group of players' identified with an interval Ω of the straight line. The 'fuzzy coalitions' of players are identified by the measurable functions from Ω to $[0, 1]$.

We associate each player with its action $f(\omega, \cdot)$ where $f : \Omega \times L \to \mathbb{R}^n$, L is a nonempty subset of $\mathbb{R}^n$, and each fuzzy coalition $c(\omega)$ with its action $\int_\Omega f(\omega, x)c(\omega)d(\omega)$.

The problem of fuzzy game is to find $x \in E$ and $c \in L^\infty(\Omega, [0, 1])$ such that

$$\int_\Omega f(\omega, x)c(\omega)d\omega = 0.$$

The above problem can be obtain from problem (1). If we take $E = \mathbb{R}^n$ and define $N(\cdot, \cdots) = N(\cdot, \cdot) : \Omega \times L \to E$ by

$$N(\omega, x) = \int_\Omega f(\omega, x)c(\omega)d(\omega),$$

and all the other functions involved in the formulation of problem (1) are equal to zero. For more details we refer to the book 'Optima and Equilibria' by Aubin [1].

Some special cases of problem (1)

(i) If $N(\cdot, \cdots) = N(\cdot, \cdot)$; $T_1 = T$, $T_2 = F$; $T_3 = T_4 = \ldots = 0$; $a_1, a_2 : E \to [0, 1]$, $a_3 = a_4 = \ldots = 0$; $\eta(u, v) = u - v$, for all $u, v \in E$ and $M(\cdot, \cdot) = M(\cdot)$ be an m-accretive mapping. Then the problem (2.1) reduces to the problem of finding $u, w, q \in E$ such that

$$\begin{aligned} &T_u(w) \geq a_1(u), \ F_u(q) \geq a_2(u) \ \text{ and} \\ &0 \in N(w, q) + M(g(u)). \end{aligned} \tag{2}$$

Problem (2) is introduced and studied by Siddiqi et al.[23].

(ii) If $T_i : E \to CB(E), i = 1, 2, \ldots$ are classical set-valued mappings, we can define fuzzy mappings $T_i : E \to \mathcal{F}(E)$ by

$$u \to \chi_{T_i(u)}, \ i = 1, 2, \ldots,$$

where $\chi_{T_i(u)}$ are characteristic functions of $T_i(u), i = 1, 2, \ldots$.

Taking $a_i(u) = 1, i = 1, 2, \dots$ for all $u \in E$, $\eta(u, v) = u - v$ and $M : E \times E \to 2^E$ to be an m-accretive mapping. The problem (1) is equivalent to the following problem considered by W.Y-Heng [7].

Find $u \in E, x_i \in T_i(u), i = 1, 2, \dots$ such that

$$0 \in N(x_1, x_2, \cdots) + M(g(u), u). \tag{3}$$

For suitable choice of mappings involved in the formulation of the problem (1), a number of known classes of variational inequalities (inclusions) studied in [5, 8], can be obtained from the problem (1).

3. **Iterative Algorithm**

In this section, we construct an iterative algorithm for solving problem (1).

Definition 3.1. Let $A : E \to E$ be the strictly η-accretive mapping and $M : E \times E \to 2^E$ be an (A, η)-accretive mapping in the first argument. The resolvent operator $J^{\rho,A}_{\eta,M(\cdot,u)} : E \to E$ is defined by

$$J^{\rho,A}_{\eta,M(\cdot,u)}(v) = (A + \rho M(\cdot, u))^{-1}(v), \text{ for all } u, v \in E.$$

Proposition 3.2. [13] *Let E be a real Banach space and $\eta : E \times E \to E$ be τ-Lipschitz continuous mapping. Let $A : E \to E$ be r-strongly η-accretive mapping and $M : E \times E \to 2^E$ be an (A, η)-accretive mapping in the first argument. Then the resolvent operator $J^{\rho,A}_{\eta,M(\cdot,u)} : E \to E$ is $\frac{\tau}{r-\rho m}$-Lipschitz continuous, i.e.,*

$$\|J^{\rho,A}_{\eta,M(\cdot,u)}(x) - J^{\rho,A}_{\eta,M(\cdot,u)}(y)\| \le \frac{\tau}{r - \rho m}\|x - y\|, \quad \text{for all } x, y \in E,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

Proposition 3.3. [20] *Let E be a uniformly smooth Banach space and $J : E \to 2^{E^*}$ be a normalized duality mapping. Then, for any $x, y \in E$, $j(x + y) \in \mathcal{J}(x + y)$,*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle$,

(ii) $\langle x - y,\ j(x) - j(y)\rangle \le 2D^2_{\gamma_E}(4\|x - y\|/D)$, *where* $D = \sqrt{\|x\|^2 + \|y\|^2/2}$.

Lemma 3.4. $(u, x_1, x_2, \cdots)$ *is a solution of problem* (1) *if and only if it is a solution of the following equation:*

$$g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho N(x_1, x_2, \cdots)]. \tag{4}$$

Proof. The conclusion can be drawn directly from the definition of the resolvent operator $J^{\rho,A}_{\eta,M(\cdot,u)}$. □

Using Lemma 3.2 and Nadler's theorem [16], we construct an iterative algorithm for solving problem (1).

Iterative Algorithm 3.5. Let $T_i : E \to \mathcal{F}(E),\ i = 1, 2, \dots$ be an infinite family of closed fuzzy mappings satisfying the condition $(*)$ and let $\tilde{T}_i : E \to CB(E),\ i = 1, 2, \dots$ be an infinite family of set-valued mappings induced by the fuzzy mappings $T_i, i = 1, 2, \dots$, respectively. Let $N = E^\infty = E \times E \times E \dots \to E$, $\eta : E \times E \to E$,

$g, A : E \to E$ be the mappings such that $g \not\equiv 0$ and $M : E \times E \to 2^E$ be a set-valued mapping such that for each $u \in E, M(\cdot, u)$ is an (A, η)-accretive mapping in the first argument such that $g(u) \in \mathrm{dom}M(\cdot, u)$.

Let $Q : E \to 2^E$ be a set-valued mapping such that for each $u \in E, Q(u) \subseteq g(E)$, where Q is defined by

$$Q(u) = \bigcup_{x_1 \in \tilde{T}_1(u)} \bigcup_{x_2 \in \tilde{T}_2(u)} \bigcup_{x_3 \in \tilde{T}_3(u)} \cdots \bigcup \cdots [A(g(u)) - \rho N(x_1, x_2, \cdots)]. \tag{5}$$

For given $u_0 \in E, x_1^0 \in \tilde{T}_1(u_0), x_2^0 \in \tilde{T}_2(u_0), \ldots$ and let

$$t_0 = J^{\rho,A}_{\eta,M(\cdot,u_0)}[A(g(u_0)) - \rho N(x_1^0, x_2^0, \cdots)] \subseteq Q(u_0) \subseteq g(E).$$

Hence, there exists $u_1 \in E$ such that $t_0 = g(u_1)$. Further, since $\tilde{T}_i(u_i) \in CB(E), i = 1, 2, \ldots$ by Nadler's [16], there exist $x_1^1 \in \tilde{T}_1(u_1), x_2^1 \in \tilde{T}_2(u_1), \ldots$ such that

$$\|x_1^0 - x_1^1\| \leq \tilde{D}(\tilde{T}_1(u_0), \tilde{T}_1(u_1));$$
$$\|x_2^0 - x_2^1\| \leq \tilde{D}(\tilde{T}_2(u_0), \tilde{T}_2(u_1));$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Let $t_1 = J^{\rho,A}_{\eta,M(\cdot,u_1)}[A(g(u_1)) - \rho N(x_1^1, x_2^1, \cdots)] \subseteq Q(u_1) \subseteq g(E)$. Hence there exists $u_2 \in E$ such that $t_1 = g(u_2)$. Continuing the above process inductively, we can define iterative sequences $\{u_n\}, \{x_i^n\}, i = 1, 2, \ldots$ satisfying

$$g(u_{n+1}) = J^{\rho,A}_{\eta,M(\cdot,u_n)}[A(g(u_n)) - \rho N(x_1^n, x_2^n, \cdots)], \tag{6}$$

and

$$x_i^n \in \tilde{T}_i(u_n),\ \|x_i^n - x_i^{n-1}\| \leq \tilde{D}(\tilde{T}(u_n), \tilde{T}(u_{n-1})),$$

for all $i = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$.

4. **Existence of Solutions and Convergence of Iterative Sequences**

Theorem 4.1. *Let E be a uniformly smooth Banach space with modulus of smoothness $\gamma_E(t) \leq Ct^2$, for $C > 0$. Let $T_i : E \to \mathcal{F}(E), i = 1, 2, \ldots$ be an infinite family of closed fuzzy mappings satisfying condition $(*)$ and let $\tilde{T}_i : E \to CB(E)$ be the set-valued mappings induced by the fuzzy mappings $T_i, i = 1, 2, \ldots$ such that $\tilde{T}_i$ are $\tilde{D}$-Lipschitz continuous mappings in all the arguments with constants $\lambda_{\tilde{T}_i}, i = 1, 2, \ldots$. Let $N : E^\infty = E \times E \times E \ldots \to E$ be a Lipschitz continuous mapping in all the arguments with constants $\beta_i, i = 1, 2, \ldots$ and $\eta : E \to E$ be τ-Lipschitz continuous mapping. Let $g : E \to E$ is α-relaxed Lipschitz and λ_g-Lipschitz continuous with $g \not\equiv 0$ and let $(g - I)$ is δ-strongly accretive, where I is the identity mapping. Let $A : E \to E$ be r-strongly η-accretive and λ_A-Lipschitz continuous mapping. Let $M : E \times E \to 2^E$ be set-valued mapping such that for each $u \in E, M(\cdot, u)$ is an (A, η)-accretive mapping in the first argument with $g(u) \in \mathrm{dom}M(\cdot, u)$ and let for each $u \in E$, $Q(u) \subseteq g(E)$, where Q is defined by (5). Suppose that there exist $\rho > 0$ and $\mu > 0$ such that for each $x, y, z \in E$,*

$$\|J^{\rho,A}_{\eta,M(\cdot,x)}(z) - J^{\rho,A}_{\eta,M(\cdot,y)}(z)\| \leq \mu\|x - y\|, \tag{7}$$

and

$$\lambda_A(1+\sqrt{1-2\alpha+64C\lambda_{g^2}})+\rho\sum_{i=1}^{\infty}\beta_i\lambda_{\tilde{T}_i}<\frac{(\sqrt{1+2\delta}-\mu)(r-\rho m)}{\tau},$$

$$\sqrt{1+2\delta}>\mu,\ \tfrac{r}{\rho}>m. \tag{8}$$

Then the iterative sequences $\{u_n\},\{x_1^n\},\{x_2^n\},\dots$ generated by Algorithm 3.5 converge strongly to $u,x_1,x_2,\dots.$ respectively and $(u,x_1,x_2,\cdots)$ is a solution of problem (1).

Proof. Since $(g-I)$ is δ-strongly accretive mapping, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &= \|g(u_{n+1})-g(u_n)+u_{n+1}-u_n-(g(u_{n+1})-g(u_n))\|^2\\ &\le \|g(u_{n+1})-g(u_n)\|^2-2\langle(g-I)(u_{n+1})-(g-I)(u_n),\ j(u_{n+1}-u_n)\rangle\\ &\le \|g(u_{n+1})-g(u_n)\|^2-2\delta\|u_{n+1}-u_n\|^2,\end{aligned}$$

which implies that

$$\|u_{n+1}+u_n\|\le\frac{1}{\sqrt{1+2\delta}}\|g(u_{n+1})-g(u_n)\|. \tag{9}$$

By Proposition 3.1, Algorithm 3.1 and (4.1), we have

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\| &= \|J^{\rho,A}_{\eta,M(.,u_n)}[A(g(u_n))-\rho N(x_1^n,x_2^n,\cdots)]\\ &\quad - J^{\rho,A}_{\eta,M(.,u_{n-1})}[A(g(u_{n-1}))-\rho N(x_1^{n-1},x_2^{n-1},\cdots)]\|\\ &= \|J^{\rho,A}_{\eta,M(\cdot,u_n)}[A(g(u_n))-\rho N(x_1^n,x_2^n,\cdots)]\\ &\quad - J^{\rho,A}_{\eta,M(\cdot,u_n)}[A(g(u_{n-1}))-\rho N(x_1^{n-1},x_2^{n-1},\cdots)]\\ &\quad + J^{\rho,A}_{\eta,M(\cdot,u_n)}[A(g(u_{n-1}))-\rho N(x_1^{n-1},x_2^{n-1},\cdots)]\\ &\quad - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}[A(g(u_{n-1}))-\rho N(x_1^{n-1},x_2^{n-1},\cdots)]\|\\ &\le \frac{\tau}{r-\rho m}\|A(g(u_n))-A(g(u_{n-1}))-\rho[N(x_1^n,x_2^n,\cdots)\\ &\quad - N(x_1^{n-1},x_2^{n-1},\cdots)]\|+\mu\|u_n-u_{n-1}\|\\ &\le \frac{\tau\lambda_A}{r-\rho m}\|g(u_n)-g(u_{n-1})\|+\frac{\tau\rho}{r-\rho m}\|N(x_1^n,x_2^n,\cdots)\\ &\quad - N(x_1^{n-1},x_2^{n-1},\cdots)\|+\mu\|u_n-u_{n-1}\|\\ &\le \frac{\tau\lambda_A}{r-\rho m}\|u_n-u_{n-1}+g(u_n)-g(u_{n-1})\|+(\frac{\tau\lambda_A+\mu}{r-\rho m})\|u_n-u_{n-1}\|\\ &\quad + \frac{\tau\rho}{r-\rho m}\|N(x_1^n,x_2^n,\cdots)-N(x_1^{n-1},x_2^{n-1},\cdots)\|.\end{aligned} \tag{10}$$

Using relaxed Lipschitz continuity of g and Proposition 3.2, it follows that

$$\begin{aligned}&\|u_n-u_{n-1}+g(u_n)-g(u_{n-1})\|^2\\ &\le \|u_n-u_{n-1}\|^2+2\langle g(u_n)-g(u_{n-1}),\ j(u_n-u_{n-1}+g(u_n)-g(u_{n-1}))\rangle\\ &= \|u_n-u_{n-1}\|^2+2\langle g(u_n)-g(u_{n-1}),\ j(u_n-u_{n-1})\rangle\\ &\quad +2\langle g(u_n)-g(u_{n-1}),\ j(u_n-u_{n-1}+g(u_n)-g(u_{n-1}))-j(u_n-u_{n-1})\rangle\\ &\le \|u_n-u_{n-1}\|^2-2\alpha\|u_n-u_{n-1}\|^2+4D^2\gamma_E(4\|g(u_n)-g(u_{n-1})\|/D)\\ &\le \|u_n-u_{n-1}\|^2-2\alpha\|u_n-u_{n-1}\|^2+64C\|g(u_n)-g(u_{n-1})\|^2\\ &\le (1-2\alpha+64C\lambda_g^2)\|u_n-u_{n-1}\|^2.\end{aligned} \tag{11}$$

Using Lipschitz continuity of N in all arguments with constant $\beta_i, i = 1, 2, \ldots$ respectively and using $\tilde{D}$-Lipschitz continuity of $\tilde{T}_i$ with constants $\lambda_{\tilde{T}_i},\ i = 1, 2, \ldots$ respectively, we have

$$\|N(x_1^n, x_2^n, \cdots) - N(x_1^{n-1}, x_2^{n-1}, \cdots)\| \leq \sum_{i=1}^{\infty} \beta_i \lambda_{\tilde{T}_i} \|u_n - u_{n-1}\|. \tag{12}$$

Combining (11) and (12), (10) becomes

$$\begin{aligned}\|g(u_n) - g(u_{n-1})\| \leq \big[&\frac{\tau\lambda_A}{r - \rho m}\sqrt{1 - 2\alpha + 64C\lambda_g^2} + (\frac{\tau\lambda_A}{r - \rho m} + \mu) \\ &+ \frac{\tau\rho}{r - \rho m}\sum_{i=1}^{\infty} \beta_i \lambda_{\tilde{T}_i}\big]\|u_n - u_{n-1}\|. \end{aligned} \tag{13}$$

Using (13), (9) becomes

$$\|u_{n+1} - u_n\| \leq \theta\ \|u_n - u_{n-1}\|, \tag{14}$$

where $\theta = \dfrac{1}{\sqrt{1 + 2\delta}}\big[\dfrac{\tau}{r - \rho m}(\lambda_A\sqrt{1 - 2\alpha + 64C\lambda_g^2} + \lambda_A + \rho\displaystyle\sum_{i=1}^{\infty}\beta_i\lambda_{\tilde{T}_i}) + \mu\big]$.

It follows from (8) that $\theta < 1$. Hence (14) implies that $\{u_n\}$ is a Cauchy sequence in E, and thus there exists $u \in E$ such that $u_n \to u$, as $n \to \infty$. Using Lipschitz continuity of $\tilde{T}_i$ and Algorithm 3.1, we have

$$\|x_1^0 - x_1^1\| \leq \tilde{D}(\tilde{T}_1(u_0), \tilde{T}_1(u_1)) \leq \lambda_{\tilde{T}_1}\|u_o - u_1\|;$$

$$\|x_2^0 - x_2^1\| \leq \tilde{D}(\tilde{T}_2(u_0), \tilde{T}_2(u_1)) \leq \lambda_{\tilde{T}_2}\|u_o - u_1\|;$$

....

....

It follows that $\{x_1^n\}, \{x_2^n\}, \ldots$ are all Cauchy sequences in E. Thus there exist $x_1, x_2, \ldots$ in E such that $x_1^n \to x_1, x_2^n \to x_2, \ldots$ as $n \to \infty$. Further, since $x_i^n \in \tilde{T}_i(u_n)$, we have

$$\begin{aligned} d(x_i, \tilde{T}_i(u)) &\leq \|x_i - x_i^n\| + d(x_i^n, \tilde{T}_i(u)) \\ &\leq \|x_i - x_i^n\| + \tilde{D}(\tilde{T}_i(u_n), \tilde{T}_i(u)) \\ &\leq \|x_i - x_i^n\| + \lambda_{\tilde{T}_i}\|u_n - u\| \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Hence $d(x_i, \tilde{T}_i(u)) = 0$. Since $\tilde{T}_i(u) \in CB(E)$, it follows that $x_i \in \tilde{T}_i(u)$, for $i = 1, 2, \ldots$. Thus, we have $(T_{1u})(x_1) \geq a_1(u), (T_{2u})(x_2) \geq a_2(u), \ldots$. It follows from Algorithm 3.5, that

$$g(u) = J_{M(\cdot,u)}^{\rho,A}[A(g(u)) - \rho N(x_1, x_2, \cdots)].$$

By Lemma 3.2, we conclude that $(u, x_1, x_2, \cdots)$ is a solution of problem (1). $\square$

5. **Conclusion**

The objective of this paper is to establish an existence and convergence result for a mixed variational inclusion problem involving infinite family of fuzzy mappings in real uniformly smooth Banach spaces. It is shown that problem (1) is equivalent to a fixed point problem by using the definition of resolvent operator and an iterative algorithm is constructed for solving problem (1).

We remark that there is a need to develop relationship between fuzzy sets and fuzzy variational inclusions involving infinite family of fuzzy mappings which may be useful for finding the solution of problems occuring in mathematical sciences etc..

Acknowledgements. Authors would like to express their thanks to the referee for her/his valuable comments and suggestions to improve this paper.

References

[1] J. P. Aubin, *Optima and equilibria*, Second ed., Springer, Berlin, 1998.
[2] S. S. Chang and N. J. Huang, *Generalized complementarity problem for fuzzy mappings*, Fuzzy Sets and Systems, **55** (1993), 227-234.
[3] S. S. Chang and Y. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems, **32** (1989), 359-367.
[4] X. P. Ding, *Algorithm of solutions for mixed implicit quasi-variational inequalities with fuzzy mappings*, Comput. Math. Appl., **38(5-6)** (1999), 231-249.
[5] X. P. Ding and J. Y. Park, *A new class of generalized nonlinear implicit quasi variational inclusions with fuzzy mapping*, J. Comput. Appl. Math., **138** (2002), 243-257.
[6] S. Heilpern, *Fuzzy mappings and fixed point theorems*, J. Math. Anal. Appl., **83** (1981), 566-569.
[7] W. Y. Heng, *Infinite family of generalized set-valued quasi-variational inclusions in Banach spaces*, Acta. Anal. Functionalis Applicata, **10** (2008), 1009-1327.
[8] N. J. Huang, *A new class of generalized set-valued implicit variational inclusions in Banach spaces with an application*, Comput. Math. Appl., **41(718)** (2001), 937-943.
[9] N. J. Huang, *A new method for a class of nonlinear variational inequalities with fuzzy mappings*, Appl. Math. Lett., **10(6)** (1997), 129-133.
[10] M. Janfada, H. Baghani and O. Baghani, *On Felbinos-type fuzzy normed linear spaces and fuzzy bounded operators*, Iranian Journal of Fuzzy Systems, **8(5)** (2011), 117-130.
[11] P. Kumam and N. Petrol, *Mixed variational-like inequality for fuzzy mappings in reflexive Banach spaces*, J. Inequal. Appl., **2009** (2009), 1-15.
[12] H. Y. Lan, *An approach for solving fuzzy implicit variational inequalities with linear membership functions*, Comput. Math. Appl., **55(3)** (2008), 563-572.
[13] H. Y. Lan, Y. J. Cho and R. U. Verma, *On nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings in Banach spaces*, Comput. Math. Appl., **51** (2006), 1529-1538.
[14] M. Lassonde, *On the use of KKM multi function in fixed point theory and related topics*, J. Math. Anal. Appl., **97** (1983), 151-201.
[15] B. S. Lee, M. F. Khan and Salahuddin, *Fuzzy nonlinear set-valued variational inclusions*, Comput. Math. Appl., **60(6)** (2010), 1768-1775.
[16] J. S. B. Nadler, *Multivalued contraction mappings*, Pacific. J. math., **30** (1969), 475-488.
[17] M. A. Noor, *Variational inequalities with fuzzy mappings (I)*, Fuzzy Sets and Systems, **55** (1989), 309-314.
[18] J. Y. Park and J. U. Jeong, *Iterative algorithm for finding approximate solutions to completely generalized strongly quasi-variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems, **115** (2000), 413-418.

[19] J. Y. Park and J. U. Jeong, *A perturbed algorithm of variational inclusions for fuzzy mappings*, Fuzzy Sets and Systems, **115** (2000), 419-424.
[20] W. V. Petryshyn, *A characterization of strictly convexity of Banach spaces and other uses of duality mappings*, J. Funct. Anal., **6** (1997), 282-291.
[21] M. H. Shih and K. K. Tan, *Generalized quasi-variational inequalities in locally convex spaces*, J. Math. Anal. Appl., **108** (1985), 333-343.
[22] E. Shivanan and E. Khorram, *Optimization of linear objective function subject to fuzzy relation inequalities constraints with max-product compozition*, Iranian Journal of Fuzzy Systems, **7(5)** (2010), 51-71.
[23] A. H. Siddiqi, R. Ahmad and S. S Irfan, *Set- valued variational inclusions with fuzzy mappings in Banach spaces*, J. Concrete. Applicable. Math., **4** (2006), 171-181.
[24] W. Takahashi, *Nonlinear variational inequalities and fixed point theorems*, J. Math. Soc. Japan, **28** (1976), 168-181.
[25] Z. Wu and J. Xu, *Generalized convex fuzzy mappings and fuzzy variational-like inequalities*, Fuzzy Sets and Systems, **160(11)** (2009), 1590-1619.
[26] C. L. Yen, *A minimax inequality and its application to variational inequalities*, Pacific J. Math., **97** (1981), 142-150.
[27] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.
[28] H. J. Zimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Publishers, Dordrecht, 1988.

R. Ahmad*, Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

E-mail address: raisain_123@rediffmail.com

M. Dilshad, Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

E-mail address: mdilshaad@gmail.com

J. C. YAO, Center for General Education, Kaohsiung Medical University, Kaohsiung 807, Taiwan

E-mail address: yaojc@kmu.edu.tw

*Corresponding author