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SOME RESULTS ON t-BEST APPROXIMATION IN FUZZY n-NORMED SPACES

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Abstract. The aim of this paper is to give the set of all t-best approximations on fuzzy n-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [13].

1. Introduction

In 2003, Bag and Samanta [2] introduced a definition of a fuzzy norm and proved a decomposition theorem of a fuzzy norm into a family of crisp norms. Also in 2005, Bag and Samanta [3] give some properties on fuzzy norms. The concept of 2-norm and n-norm on a linear space has been introduced and developed by Gähler in [5, 6]. Following Misiak [12] and Malčeski [11] developed the theory of n-normed space. Narayanan and Vijayabalaji [13] introduced the concept of fuzzy n-normed linear space. Vijayabalaji and Thillaigovindan [16] introduced the notion of Cauchy sequence and convergent sequence in fuzzy n-normed linear space and studied the completeness of the fuzzy n-normed linear space and Also they presented the new notion of ascending family of n-norms corresponding to an intuitionistic fuzzy n-normed linear space [17]. Many authors studied on fuzzy n-normed linear space [4]. Vaezpour and Karimi , studied on the set of all t-best approximations on fuzzy normed spaces and prove several theorems pertaining to this set [15]. In addition to this studies, Alaca introduced the concepts of 2-isometry, collinearity, 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces [1]. Recently, Goudarzi and Vaezpour developed the theory of t-best simultaneous approximation in quotient spaces and they worked on the relationship in t-proximinality and t-Chebyshevity of a given space and its quotient space [7].

In the present paper, we give the set of all t-best approximations on fuzzy n-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [15].

2. Preliminaries

Definition 2.1. [8] Let $n \in \mathbb{N}$ and let X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $||\cdot, ..., \cdot||$ on $\underbrace{X \times \cdots \times X}_{n}$ satisfying the following four properties,

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(1) $||x_1, x_2, ..., x_n|| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(2) $||x_1, x_2, ..., x_n||$ is invariant under any permutation,
(3) $||x_1, x_2, ..., \alpha x_n|| = |\alpha| \; ||x_1, x_2, ..., x_n||$ for any $\alpha \in \mathbb{R}$,
(4) $||x_1, x_2, ..., x_{n-1}, y+z|| \leq ||x_1, x_2, ..., x_{n-1}, y|| + ||x_1, x_2, ..., x_{n-1}, z||$,

is called an n-norm on X and the pair $(X, ||\cdot, ..., \cdot||)$ is called an n-normed space.

Example 2.2. Let $X = \mathbb{R}^n$ and

$$||x_1, x_2, ..., x_n||_E = abs\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \right),$$

where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, ..., n$. Then $(X, ||\cdot, ..., \cdot||_E)$ is an n-normed space which is called Euclidean n-normed space.

Definition 2.3. [13] Let X be a linear space over a real field F. A fuzzy subset N of $\underbrace{X \times \cdots \times X}_{n} \times \mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a fuzzy n-norm on X if

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$,
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(N3) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, ..., x_n$,
(N4) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, t/|c|)$, if $c \neq 0$, $c \in F$,
(N5) for all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, ..., x_n + x'_n, s+t) \geq \min\{N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x'_n, t)\},$$

(N6) $N(x_1, x_2, ..., x_n, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and

$$\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1.$$

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 2.4. [13] From (N3), it follows that in a f-n-NLS,

(N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1, x_2, ..., cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_n, t/|c|),$$

if $c \neq 0$,

(N5) for all $s, t \in \mathbb{R}$,

$$\begin{aligned} & N(x_1, x_2, ..., x_i + x'_i, ..., x_n, s+t) \\ \geq \; & \min\{N(x_1, x_2, ..., x_i, ..., x_n, s), N(x_1, x_2, ..., x'_i, ..., x_n, t)\}. \end{aligned}$$

Example 2.5. Let $(X, ||\cdot, \cdot, ..., \cdot||)$ be an n-normed space as in Definition 1. Define,

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t + ||x_1, x_2, ..., x_n||} & if \quad t > 0, \, t \in \mathbb{R}, \\ 0 & if \quad t \leq 0 \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$. Then (X, N) is a f-n-NLS.

Remark 2.6. Every fuzzy normed space induces a fuzzy metric space on it and is therefore a topological space.

Definition 2.7. [7] A sequence $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to converge to x if given $r > 0$, $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that $N(x_1, x_2, ..., x_{n-1}, x_k - x, t) > 1 - r$ for all $k \geq n_0$.

Theorem 2.8. [16] *In a fuzzy n-normed space (X, N) a sequence $\{x_k\}$ converges to x if and only if $N(x_1, x_2, ..., x_{n-1}, x_k - x, t) \longrightarrow 1$ as $k \longrightarrow \infty$.*

3. Main Results

Definition 3.1. Let (X, N) be a fuzzy n-normed space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$\begin{aligned} B(x,r,t) &= \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) > 1 - r\} \\ B[x,r,t] &= \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) \geq 1 - r\} \end{aligned}$$

Definition 3.2. Let (X, N) be a fuzzy n-normed space. A subset A of X is said to be open if there exists $r \in (0, 1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 3.3. Let (X, N) be a fuzzy n-normed space. A subset A of X is said to be closed if for any sequence $\{x_k\}$ in A converges to $x \in A$, that is

$$\lim_{k \to \infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 1$$

for all $t > 0$ implies that $x \in A$.

Definition 3.4. Let (X, N) be a fuzzy n-normed space. A subset B of X is said to be a closure of $A \subset B$ if for any $x \in B$, there exists a sequence $\{x_k\}$ in A such that

$$\lim_{k \to \infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 1$$

for all $t > 0$. We denote the set B by $\overline{A}$.

Definition 3.5. Let (X, N) be a fuzzy n-normed space. A subset A of X is said to be compact if any sequence $\{x_k\}$ in A has a sequence converging to an element of A.

Lemma 3.6. *If (X, N) is a fuzzy normed space. Then:*
(i) the function $(x, y) \longrightarrow x + y$ is continuous,
(ii) the function $(\alpha, x) \longrightarrow \alpha x$ is continuous.

Proof. If $x_k \longrightarrow x$ and $y_k \longrightarrow y$, then as $n \longrightarrow \infty$,

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, (x_k + y_k) - (x + y), t) &\geq \\ \min \left\{ \begin{array}{c} N\left(x_1, x_2, ..., x_{n-1}, x_k - x, \frac{t}{2}\right), \\ N\left(x_1, x_2, ..., x_{n-1}, y_k - y, \frac{t}{2}\right) \end{array} \right\} &\longrightarrow 1. \end{aligned}$$

Now if $x_k \longrightarrow x$, $\alpha_k \longrightarrow \alpha$ and $\alpha_k \neq 0$ then

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, \alpha_k x_k - \alpha x, t) &= N(x_1, x_2, ..., x_{n-1}, \alpha_k(x_k - x) + x(\alpha_k - \alpha), t) \\ &\geq \min\left\{ \begin{array}{c} N\left(x_1, x_2, ..., x_{n-1}, \alpha_k(x_k - x), \frac{t}{2}\right), \\ N\left(x_1, x_2, ..., x_{n-1}, x(\alpha_k - \alpha), \frac{t}{2}\right) \end{array} \right\} \\ &= \min\left\{ \begin{array}{c} N\left(x_1, x_2, ..., x_{n-1}, x_k - x, \frac{t}{2\alpha_k}\right), \\ N\left(x_1, x_2, ..., x_{n-1}, x, \frac{t}{2(\alpha_k - \alpha)}\right) \end{array} \right\} \\ &\longrightarrow 1 \end{aligned}$$

as $k \to \infty$ and this proves (ii). □

Definition 3.7. Let (X, N) be a fuzzy n-normed space, $A \subset X$, and $A \neq \varnothing$. Let

$$d(A, x, t) = \sup\{N(x_1, x_2, ..., x_{n-1}, x - y, t) : y \in A\}.$$

where $x \in X$, $t > 0$. An element $y_0 \in A$ is said to be a $t-$ best approximation of x from A if

$$N(x_1, x_2, ..., x_{n-1}, y_0 - x, t) = d(A, x, t).$$

Definition 3.8. Let (X, N) be a fuzzy n-normed space, $A \subset X$ and, $A \neq \varnothing$. For $x \in X$, $t > 0$, we shall denote the set of all elements of t -best approximation of x from A by $P_A^t(x)$; i.e.,

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, x_2, ..., x_{n-1}, y - x, t)\}.$$

If each $x \in X$ has at least (respectively exactly) one t -best approximation in A, then A is called a $t-$ proximinal (respectively $t-$ chebyshev) set.

Definition 3.9. Let (X, N) be a fuzzy n-normed space and $A \subset X$ closed, $A \neq \varnothing$. For $t > 0$, A is said to be $t-$ boundedly compact if for each $x \in X$ and $0 < r < 1$, $B[x, r, t] \cap A$ is a compact subset of X.

Theorem 3.10. *Let (X, N) be a fuzzy n-normed space, $A \subset X$ and, $A \neq \varnothing$. Then:*

(i) $d(A + y, x + y, t) = d(A, x, t)$, *for all* $x, y \in X$ *and* $t > 0$,

(ii) $P_A^t(x + y) = P_A^t(x) + y$, *for all* $x, y \in X$ *and* $t > 0$,

(iii) $d(\alpha A, \alpha x, t) = d(A, x, \frac{t}{|\alpha|})$ *for all* $x \in X$, $t > 0$ *and* $\alpha \in \mathbb{R}\backslash\{0\}$,

(iv) $P_{\alpha A}^{|\alpha| t}(\alpha x) = \alpha P_A^t(x)$, *for all* $x \in X$, $t > 0$ *and* $\alpha \in \mathbb{R}\backslash\{0\}$,

(v) A is $t-$ proximinal (respectively $t-$ chebyshev) if and only if $A + y$ is $t-$ proximinal (respectively $t-$ chebyshev), for any given $y \in X$,

(vi) A is $t-$ proximinal (respectively $t-$ chebyshev) if and only if αA is $|\alpha|\, t-$ proximinal (respectively $|\alpha|\, t-$ chebyshev) for any given $\alpha \in \mathbb{R}\backslash\{0\}$.

Proof. (i) For any $x, y \in X$ and $t > 0$,

$$\begin{aligned} d(A + y, x + y, t) &= \sup\{N(x_1, x_2, ..., x_{n-1}, (z + y) - (x + y), t) : z \in A\} \\ &= \sup\{N(x_1, x_2, ..., x_{n-1}, z - x, t) : z \in A\} \\ &= d(A, x, t). \end{aligned}$$

(ii) Using (i), $y_0 \in P^t_{A+y}(x+y)$ if and only if $y_0 \in A+y$ and

$$d(A+y, x+y, t) = N(x_1, x_2, ..., x_{n-1}, x+y-y_0, t)$$

if and only if $y_0 - y \in A$ and

$$d(A, x, t) = N(x_1, x_2, ..., x_{n-1}, x-(y_0-y), t)$$

if and only if $y_0 - y \in P^t_A(x)$.

(iii) We have,

$$\begin{aligned} d(\alpha A, \alpha x, t) &= \sup\{N(x_1, x_2, ..., x_{n-1}, \alpha x - \alpha z, t) : z \in A\} \\ &= \sup\{N(x_1, x_2, ..., x_{n-1}, \alpha(x-z), t) : z \in A\} \\ &= \sup\{N(x_1, x_2, ..., x_{n-1}, x-z, \frac{t}{|\alpha|}) : z \in A\} \\ &= d\left(A, x, \frac{t}{|\alpha|}\right). \end{aligned}$$

(iv) From (iii) it follows that $y_0 \in P^{|\alpha|t}_{\alpha A}(\alpha x)$ if and only if $y_0 \in \alpha A$ and

$$d(\alpha A, \alpha x, |\alpha|\, t) = N(x_1, x_2, ..., x_{n-1}, \alpha x - y_0, |\alpha|\, t)$$

if and only if $y_0/\alpha \in A$ and

$$N\left(x_1, x_2, ..., x_{n-1}, x - \frac{y_0}{\alpha}, t\right) = d(A, x, t).$$

However, this is equivalent to $y_0/\alpha \in P^t_A(x)$; i.e., $y_0 \in \alpha P^t_A(x)$.

(v) is an immediate consequence of (ii), and (vi) follows from (iv). □

Corollary 3.11. *Let M be a nonempty subspace of X. Then:*

(i) $d(M, x+y, t) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $y \in M$,

(ii) $P^t_M(x+y) = P^t_M(x) + y$, for all $t > 0$, $x \in X$ and $y \in M$,

(iii) $d(M, \alpha x, |\alpha t|) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $\alpha \in \mathbb{R}\backslash\{0\}$,

(iv) $P^{|\alpha|t}_M(\alpha x) = \alpha P^t_M(x)$, for all $t > 0$, $x \in X$ and $\alpha \in \mathbb{R}\backslash\{0\}$.

Definition 3.12. For $x \in X$, $0 < r < 1$, $t > 0$,

$$\begin{aligned} S[x, r, t] &= \{y \in X : N(x_1, x_2, ..., x_{n-1}, x-y, t) = 1-r\}, \\ e^t_A(x) &= 1 - d(A, x, t). \end{aligned}$$

Theorem 3.13. *Let (X, N) be a fuzzy normed space, A be a subset of X, $x \in X\backslash\bar{A}$ and $t > 0$. Then we have,*

$$\begin{aligned} P^t_A(x) &= A \cap B[x, e^t_A(x), t] \\ &= A \cap S[x, e^t_A(x), t]. \end{aligned} \tag{1}$$

Proof. The inclusions;

$$P^t_A(x) \subseteq A \cap S[x, e^t_A(x), t] \subseteq A \cap B[x, e^t_A(x), t] \tag{2}$$

are obvious by the definitions of $P_A^t(x)$ and $e_A^t(x)$.
Conversely, let $y \in A \cap B[x, e_A^t(x), t]$, then we have, $y \in A$ and

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, y - x, t) &\geq 1 - e_A^t(x) \\ &= d(A, x, t) \\ &\geq N(x_1, x_2, ..., x_{n-1}, y - x, t). \end{aligned}$$

Therefore $y \in A$ and

$$N(x_1, x_2, ..., x_{n-1}, y - x, t) = d(A, x, t),$$

which implies that $y \in P_A^t(x)$. So, $A \cap B[x, e_A^t(x), t] \subset P_A^t(x)$, whence, by 2, we have 1, which completes the proof. □

Remark 3.14. Let (X, N) be a fuzzy n-normed space, $A \subset X$ and, $A \neq \varnothing$, $x \in X \backslash \bar{A}$ and $t > 0$. Then we have,

$$A \cap B(x, e_A^t(x), t) = \varnothing, \tag{3}$$

because, if $y_0 \in A \cap B(x, e_A^t(x), t)$ then,

$$\begin{aligned} d(A, x, t) &\geq N(x_1, x_2, ..., x_{n-1}, x - y_0, t) \\ &> d(A, x, t) \end{aligned}$$

which is impossible.

Corollary 3.15. *Let (X, N) be a fuzzy n-normed space, $A \subset X$ and, $A \neq \varnothing$, $x \in X \backslash \bar{A}$ with $P_A^t(x) \neq \varnothing$ and $0 < r < 1$ such that,*

$$\varnothing \neq A \cap B[x, r, t] \subseteq S[x, r, t]. \tag{4}$$

Then we have,

$$r = e_A^t(x),$$

and we can write $A \cap B[x, r, t] = P_A^t(x)$.

Proof. If $r < e_A^t(x)$ then by the definition of $e_A^t(x)$ we have $A \cap B[x, r, t] = \varnothing$, which contradicts 4. If $r > e_A^t(x)$, since $P_A^t(x) \neq \varnothing$, then by 1 we have

$$\varnothing \neq P_A^t(x) = A \cap B[x, e_A^t(x), t] \subseteq A \cap B(x, r, t),$$

which contradicts 4, and this completes the proof. □

Definition 3.16. Let (X, N) be a fuzzy n-normed space, $0 < r < 1$ and $t > 0$. We shall say that a set $A \subset X$ supports the cell $B[x, r, t]$, or that A is a support set of the cell $B[x, r, t]$, if we have $d(A, B[x, r, t], t) = 1$ and $A \cap B(x, r, t) = \varnothing$.

Theorem 3.17. *Let (X, N) be a fuzzy n-normed space, A a nonempty set in X, $x \in X \backslash \bar{A}$, $a_0 \in A$ and $t > 0$. We have $a_0 \in P_A^t(x)$ if and only if the set A supports the cell $B = B[x, 1 - N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t]$.*

Proof. Assume that $a_0 \in P_A^t(x)$. Hence

$$N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) = d(A, x, t).$$

Then by 3, we have $A \cap B(x, 1 - N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t) = \varnothing$, on the other hand, since $a_0 \in A \cap B[x, 1 - N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t]$, we have $d(A, B, t) = 1$. Consequently, the set A supports the cell B.

Conversely, suppose $a_0 \notin P_A^t(x)$, hence

$$N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) < d(A, x, t),$$

and let $0 < \varepsilon < 1$ such that

$$N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) < d(A, x, t) - \varepsilon.$$

Then there exists $a \in A$ such that

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) &< d(A, x, t) - \varepsilon \\ &< N(x_1, x_2, ..., x_{n-1}, a - x, t), \end{aligned}$$

hence $a \in B(x, 1 - N(x_1, x_2, ..., x_{n-1}, a_0 - x), t)$. Consequently, A does not support the cell B. □

Remark 3.18. We recall that a set A in a topological space τ is said to be countably compact, if every countable open cover of A has a finite subcover, or, equivalently, if for every decreasing sequence $A_1 \supset A_2 \supset ....$ of non-void closed subset of A we have $\cap_{n=1}^{\infty} A_n \neq \varnothing$.

Theorem 3.19. *Let (X, N) be a fuzzy normed space, τ be an arbitrary topology on X and $t > 0$. If A is a nonempty set of X such that for $A \cap B[x, r, t]$ is $\tau-$ countably compact, then A is $t-$ proximinal.*

Proof. For all $n \in \mathbb{N}$, $0 < 1 - d(A, x, t) + \frac{d(A,x,t)}{n+1} < 1$. Put

$$A_n^t = A \cap B\left[x, 1 - d(A, x, t) + \frac{d(A, x, t)}{n + 1}, t\right], \ (n = 1, 2, ...).$$

Since for every $n \in \mathbb{N}$, $d(A, x, t)\left(1 - \frac{1}{n+1}\right) < d(A, x, t)$, obviously $A_1^t \supset A_2^t \supset \cdots$ and each $A_n^t \neq \varnothing$. Hence there exists $a_n^t \in A$ such that

$$d(A, x, t)\left(1 - \frac{1}{n + 1}\right) < N(x_1, x_2, ..., x_{n-1}, a_n^t - x, t).$$

It follows that $a_n^t \in A_n^t$. Now, since each A_n^t is $\tau-$countably compact and $\tau-$closed, we conclude that there exists $a_0 \in \bigcap_{n=1}^{\infty} A_n^t$. Then we have

$$\begin{aligned} d(A, x, t) &\geq N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) \\ &\geq d(A, x, t)\left(1 - \frac{1}{n + 1}\right), \end{aligned}$$

$(n = 1, 2, ...)$, whence $a_0 \in P_A^t(x)$ which completes the proof. □

Definition 3.20. Let (X,N) be a fuzzy n-normed space, $A \subset X$ and, $A \neq \varnothing$. An element $y_0 \in A$ is said to be an $F-$best approximation of $x \in X$ from A if it is a $t-$best approximation of x from A, for every $t > 0$, i.e.,

$$y_0 \in \bigcap_{t\in(0,\infty)} P_A^t(x).$$

The set of all elements of $F-$best approximations of X from A is denoted by $FP_A(x)$, i.e.,

$$FP_A(x) = \bigcap_{t\in(0,\infty)} P_A^t(x).$$

If each $x \in X$ has at least (respectively exactly) one $F-$best approximation in A, then A is called a $F-$proximinal (respectively $F-$Chebyshev) set.

Example 3.21. Let $X = \mathbb{R}^3$. Define $N : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0,\infty) \longrightarrow [0,1]$ by

$$N(x_1,x_2,x_3,t) = \begin{cases} \left(\exp \frac{||x_1,x_2,x_3||_\infty}{t}\right)^{-1} & if \quad t > 0, t \in \mathbb{R} \\ 0 & if \quad t \leq 0 \end{cases}.$$

where $||x_1,x_2,x_3||_\infty = \max\limits_{1\leq i\leq 3} \sum\limits_{j=1}^{3} |x_{ij}|$. Then $(X,N,*)$ is a fuzzy 3-normed space. Let

$$A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2\}$$

and $x_3 = (0,0,4), x_1 = (1,0,0), x_2 = (0,1,0)$. Then for every $t > 0$,

$$\begin{aligned} N(x_1,x_2,(0,-1,1)-(0,0,4),t) &= N(x_1,x_2,(0,1,1)-(0,0,4),t) \\ &= \left(\exp \frac{4}{t}\right)^{-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & d(A,(0,0,4),t) = \sup\{N((x_1,x_2,(x,y,z)-(0,0,4)),t) : x^2+y^2 \leq 1, 0 \leq z \leq x^2+y^2\} \\ = & \sup\left\{ \begin{array}{c} \left(\exp \frac{\max(|x_{11}|+|x_{12}|+|x_{13}|,|x_{21}|+|x_{22}|+|x_{23}|,|x_{31}|+|x_{32}|+|x_{33}-4|)}{t}\right)^{-1} : \\ x^2+y^2 \leq 1, 0 \leq z \leq x^2+y^2 \end{array} \right\} \\ = & \left(\exp \frac{4}{t}\right)^{-1}. \end{aligned}$$

So, for every $t > 0$, $a_0 = (0,-1,1)$ and $a_1 = (0,1,1)$ are $t-$best approximations of $(0,0,4)$ from A. Hence $a_0 = (0,-1,1)$ and $a_1 = (0,1,1)$ are $F-$best approximations of $x = (0,0,4)$ from A. Therefore A is not an $F-$Chebyshev set.

Example 3.22. Let $X = \mathbb{R}^3$. Define $N : X \times X \times X \times \mathbb{R} \longrightarrow [0,1]$ by

$$N(x_1,x_2,x_3,t) = \begin{cases} \frac{t}{t+||x_1,x_2,x_3||} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases}.$$

Then (X,N) is a fuzzy 3-normed space where $||x_1,x_2,x_3||_\infty = \max\limits_{1\leq i\leq 3} \sum\limits_{j=1}^{3} |x_{ij}|$. Let

$$A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}.$$

Then, for every $a = (x, y, z) \in \mathbb{R}^3$ where $x^2 + y^2 + z^2 > 1$, there exits a unique $a_0 = (x_0, y_0, z_0) \in A$ (especially in ∂A) which is an $F-$best approximation of a from A. So A is an $F-$proximinal set.

Remark 3.23. For an arbitrary set $A \subset X$ we shall denote by ∂A the boundary of A, and by $\mathcal{M}_A$ the set of all elements of the $F-$best approximation of the elements $x \in X$ from A, i.e.,

$$\mathcal{M}_A = \underset{x \in X}{\cup} FP_A(x).$$

Theorem 3.24. *Let (X, N) be a fuzzy n-normed space, $A \neq \varnothing$, and A be a $F-$best proximinal set in X. Then*

$$\partial A \subset \overline{\mathcal{M}_A}.$$

Proof. If $\partial A = \varnothing$, the proof is obvious. If $\partial A \neq \varnothing$, let $a_0 \in \partial A$, $0 < \varepsilon < 1$ and $t > 0$ be arbitrary. Then there exists $0 < \varepsilon' < 1$ such that $\min\{(1 - \varepsilon'), (1 - \varepsilon')\} > 1 - \varepsilon$ and the cell $B(a_0, \varepsilon', t/2)$ contains at least one element $x \in X \backslash A$. Let $\pi_A(x) \in FP_A(x)$ (it exists, since by hypothesis, A is $F-$proximinal). Then we have,

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, a_0 - \pi_A(x), t) &\geq \min\left\{ \begin{array}{c} N\left(x_1, x_2, ..., x_{n-1}, a_0 - x, \frac{t}{2}\right) \\ N\left(x_1, x_2, ..., x_{n-1}, x - \pi_A(x), \frac{t}{2}\right) \end{array} \right. \\ &= \min\left\{ \begin{array}{c} N\left(x_1, x_2, ..., x_{n-1}, a_0 - x, \frac{t}{2}\right) \\ N\left(x_1, x_2, ..., x_{n-1}, A - x, \frac{t}{2}\right) \end{array} \right. \\ &\geq \min\left\{ \begin{array}{c} N\left(x_1, x_2, ..., x_{n-1}, a_0 - x, \frac{t}{2}\right) \\ N\left(x_1, x_2, ..., x_{n-1}, a_0 - x, \frac{t}{2}\right) \end{array} \right. \\ &\geq \min\{(1 - \varepsilon'), (1 - \varepsilon')\} \\ &> 1 - \varepsilon. \end{aligned}$$

So, $B(a_0, \varepsilon, t) \cap \mathcal{M}_A \neq \varnothing$ and since $\varepsilon > 0$ is arbitrary, we obtain $a_0 \in \overline{\mathcal{M}_A}$ which completes the proof. □

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