SYMMETRIC TRIANGULAR AND INTERVAL APPROXIMATIONS OF FUZZY SOLUTION TO LINEAR FREDHOLM FUZZY INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract. In this paper a linear Fuzzy Fredholm Integral Equation(FFIE) with arbitrary Fuzzy Function input and symmetric triangular (Fuzzy Interval) output is considered. For each variable, output is the nearest triangular fuzzy number (fuzzy interval) to the exact fuzzy solution of (FFIE).

1. Introduction

The topic of fuzzy integral equation(FIE) and fuzzy differential equation which attached growing interest for some time, in particular in relation with fuzzy control have been developed in recent years. Fuzzy differential equations (FDE) is studied in [1, 7, 19, 20, 34, 32, 43]. Prior to presenting fuzzy integral equations and their associated numerical algorithm, it is necessary to discuss an appropriate brief introduction to preliminary topics such as fuzzy number, Expected interval and fuzzy calculus. Fuzzy numbers and arithmetic operator with this numbers were first introduced and investigated bay zadeh and others [12, 26, 34]. In 1992 Heilpern represent concept of Expected interval and applied it to estimate fuzzy number. The concept of integration of fuzzy functions was introduced by Dubois and Prade[21] and investigated by Goetshel and Voxman[27], Kaleva[32], Matloka[35] and lately T. Allahviranloo[4], B.Bede and S.G. Gal [10] introduce Quadrature rules for Integral of Fuzzy-Number valued function. Wu and Ma[15] represented applications of fuzzy integration. They investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2) such that these viewed as fuzzy numbers are a collection of α-levels, $0 \leq \alpha \leq 1$. Additional related material can be fond in [27]. Recently many of authors have been proposed numerical method for solving the fuzzy Fredholm integral equation of the second kind, see [3, 12, 23, 24, 25, 26, 38, 39, 40]. Addition nonlinear Fuzzy Feredholm Integral equation has been studied by many authors, for instance, see [5, 8, 9, 11]. In this work Expected interval and symmetric fuzzy number are used to estimate fuzzy solution to linear Fuzzy Fredholm integral equation(FFIE) in other word, we propose a model for solving fuzzy Fredholm integral equation whose input is an arbitrary fuzzy function and its output is a symmetric or interval

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fuzzy function. In the frame of the analytical methods, recently Molabahrami et al [37] extended the application of an analytic method, namely homotopy analysis method, for solving(FFIT-2). we recall some fundamental results on fuzzy numbers. Section 3 introduce definition of the Fuzzy Fredholm integral equation of the second kind and show how can convert it to crisp Fredholm integral equation of the second kind. In section 4, we represent a numerical method for solving crisp Fredholm integral equation of the second kind (FIE-2). Finally, section 5 deals with algorithms that approximate fuzzy solution of (FFIE-2) by proper triangular fuzzy number whence input is approximated by nearest symmetric triangular fuzzy number or interval and then illustrate an example.

2. Notation and Basic Definitions

Definition 2.1. A fuzzy number is a fuzzy set $u : \mathbb{R} \to I = [0, 1]$,
(i) u is upper semicontinuous.
(ii) $u(x) = 0$, out side some interval $[c, d]$.
$iii)$ There are real numbers $a, b : c \leq a \leq\ b \leq d$, for which
$1.u(x)$ is monotonic increasing on $[c, a]$.
$2.u(x)$ is monotonic decreasing on $[b, d]$.
$3.u(x) = 1, a \leq x \leq b$

The set of all such fuzzy numbers is represented by E^1. Fuzzy number with linear sides and the membership function with the following form

$$u(x) = \begin{cases} 0 & if\ x < x_0 - \sigma 1, \\ \frac{x-\alpha_1}{\sigma 1} & if\ x_0 - \sigma 1 \leq x < x_0, \\ 1 & if\ x_0 \leq x \leq x_1, \\ \frac{\alpha_4 - x}{\sigma 2} & if\ x_1 \leq x < x_1 + \sigma 2, \\ 0 & if\ x_1 + \sigma 2 < x. \end{cases} \tag{1}$$

that is called *the family of all trapezoidal fuzzy numbers,* where $\sigma 1 > 0$, is left width and $\sigma 2 > 0$ is right width of u. Since the trapezoidal fuzzy number is completely characterized by four real numbers $x_0 \leq x_1$ and $\sigma 1, \sigma 2$, it is often denoted in brief as $u(x_0, x_1, \sigma 1, \sigma 2)$. The family of all trapezoidal fuzzy numbers will be denoted by $\mathbb{F}^T$. If $x_0 = x_1$, we obtain *Triangular fuzzy number* and denote it as $u(x_0, \sigma 1, \sigma 2)_T$. For a triangular fuzzy number if $\sigma 1 = \sigma 2 = \sigma$, it then is called *symmetric triangular fuzzy number* and denote it by $S(x_0, \sigma)$.

Definition 2.2. Following [14] we represent arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$, which satisfy the following requirements.

1): $\underline{u}(r)$ is a bounded left continuous non decreasing over $[0, 1]$
2): $\overline{u}(r)$ is a bounded left continuous non increasing over $[0, 1]$
3): $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$ and $k \in \mathbb{R}$ we define addition and multiplication by k as

$$\underline{(u + v)}(r) = (\underline{u}(r) + \underline{v}(r))$$
$$\overline{(u + v)}(r) = (\overline{u}(r) + \overline{v}(r))$$

$$\underline{ku}(r) = k\underline{u}(r),\ \overline{ku}(r) = k\overline{u}(r)\ \ if\ \ k \geq 0$$

$$\overline{ku}(r) = k\underline{u}(r),\ \underline{ku}(r) = k\overline{u}(r)\ \ if\ \ k < 0.$$

Remark 2.3. Suppose $u(r) = (\underline{u}(r), \overline{u}(r))$, and $0 \leq r \leq 1$ is a fuzzy number we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2} \tag{2}$$

$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}. \tag{3}$$

It is clear that $u^d(r) \geq 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, also a fuzzy number $u \in E^1$ is said to be symmetric if $u^c(r)$ is the same real constant for all $0 \leq r \leq 1$.

Remark 2.4. Let $u(r) = (\underline{u}(r), \overline{u}(r))$ and $v(r) = (\underline{v}(r), \overline{v}(r))$ be fuzzy numbers also k, s be arbitrary real numbers, if $z = ku + sv$, then

$$z^c = ku^c + sv^c \tag{4}$$

$$z^d = |k|u^d + |s|v^d \tag{5}$$

Proof. See [3] □

For more information a bout fuzzy numbers and their application see [36, 41]

Definition 2.5. The *Expected interval*(Heilpern 1992) of fuzzy number u is defined as

$$EI(u) = [E_1^u, E_2^u] = [\int_0^1 \underline{u}(r)dr, \int_0^1 \overline{u}(r)dr], \tag{6}$$

i.e. $E_1^u = \int_0^1 \underline{u}(r)dr$ and $E_2^u = \int_0^1 \overline{u}(r)dr$.

Definition 2.6. The *width* of a fuzzy number $u(r) = (\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ is defined as follow:

$$w(u) = \int_0^1 \overline{u}(r)dr - \int_0^1 \underline{u}(r)dr. \tag{7}$$

Remark 2.7. Suppose u is a trapezoidal or triangular fuzzy number then we find from (2.6):

$$EI(u) = [\underline{u}(0.5), \overline{u}(0.5)] \text{ and } w(u) = \overline{u}(0.5) - \underline{u}(0.5).$$

Definition 2.8. The distance between two arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ is given as

$$D(u, v) = \left[\int_0^1 \left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 \left(\overline{u}(r) - \overline{v}(r)\right)^2 dr.\right]^{1/2}. \tag{8}$$

Note that an interval $I = [c_1, c_2]$ can be represented as a fuzzy number and its parametric form is $\underline{I}(r) = c_1 = \underline{c}$, $\overline{I}(r) = c_2 = \overline{c}$. Here we will propose an interval approximation operator called the nearest. Suppose u is a fuzzy number with $(\underline{u}(r), \overline{u}(r))$, we will try to find a closed interval $c(u)$ which is the nearest to u with respect to $D(.,.)$. Let $c(u) = (\underline{c}, \overline{c})$, we have to minimize

$$D(u, c(u)) = \{\int_0^1 (\underline{u}(r) - \underline{c})^2 dr + \int_0^1 (\overline{u}(r) - \overline{c})^2 dr\}^{1/2}$$

with respect to $\underline{c}$ and $\overline{c}$ refer to [2] we can obtain $c(u)$ as

$$c(u) = [\int_0^1 \underline{u}(r)dr, \int_0^1 \overline{u}(r)dr],$$

i.e. $c(u) = EI(u)$. It is clear that $\underline{c} = \int_0^1 (u^c(r))dr - \int_0^1 (u^d(r))dr$ and $\overline{c} = \int_0^1 (u^c(r))dr + \int_0^1 (u^d(r))dr$.

Now let u be a general fuzzy number as $u = (\underline{u}(r), \overline{u}(r))$ in order to obtain a symmetric triangular fuzzy number which is the nearest to u we minimize

$$D(u, S(x_0, \sigma)) = \{\int_0^1 (\underline{u}(r) - \underline{S}(x_0, \sigma))^2 dr + \int_0^1 (\overline{u}(r) - \overline{S}(x_0, \sigma))^2 dr\}^{1/2}$$

with respect to x_0 and σ. Refer to [2] we observe that
$\sigma = \frac{3}{2}\int_0^1 (\overline{u}(r) - \underline{u}(r))(1-r)dr$ and $x_0 = \frac{1}{2}\int_0^1 (\overline{u}(r) + \underline{u}(r))dr$, it means that

$$\sigma = 3\int_0^1 (1-r)u^d(r)dr, \quad x_0 = \int_0^1 u^c(r)dr. \tag{9}$$

3. **Fuzzy Integral Equation**

Prior to define fuzzy integral equation, one should recall concept of fuzzy integral equation, using the Riemann integrals as follow [27].

Definition 3.1. [22] Let $f : [a, b] \to E^1$ be fuzzy function, for each partition $P = \{t_1, ..., t_N\}$ of $[a, b]$ and for arbitrary ξ_i: $t_{i-1} \leq \xi_i \leq t_i,\ 2 \leq i \leq N$ take

$$R_p = \sum_{i=1}^{N} f(\xi_i)(t_i - t_{i-1}).$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)d(t) = limR_p, \quad max|t_i - t_{i-1}| \to 0$$

provided that this limit exists in the metric D.

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exist[27], and also,

$$(\underline{\int_a^b f(t,r)dt}) = \int_a^b \underline{f}(t,r)dt,$$

$$\overline{(\int_a^b f(t,r)dt)} = \int_a^b \overline{f}(t,r)dt.$$

The Fredholm integral equation of the second kind is [31]

$$F(t) = f(t) + \lambda \int_a^b K(t,s)F(s)ds. \tag{10}$$

Where $\lambda > 0$, $K(t,s)$ is an arbitrary kernel function over the square $a \leq t, s \leq b$ and $f(t)$ is a function of $t : a \leq t \leq b$. If $f(t)$ is a crisp function then the solution of the above equation is crisp as well. However, if $f(t)$ is a fuzzy function this equation may only possess fuzzy function. Sufficient conditions for the existence solution integral equation of the second kind , where $f(t)$ is a fuzzy function, are given in [17]. In this case in parametric form we have

$$\underline{F(t,r)} = \underline{f(t,r)} + \lambda \int_a^b \underline{K(t,s)F(s,r)}ds,$$

$$\overline{F(t,r)} = \overline{f(t,r)} + \lambda \int_a^b \overline{K(t,s)F(s,r)}ds.$$

Since for each t, both $F(t,r)$ and $f(t,r)$ are fuzzy numbers and for each s,t fixed $K(t,s)$ is an arbitrary real number, by lemma 1 we have

$$F^c(t,r) = f^c(t,r) + \lambda \int_a^b K(t,s)F^c(s,r)ds, \tag{11}$$

$$F^d(t,r) = f^d(t,r) + \lambda \int_a^b |K(t,s)|F^d(s,r)ds. \tag{12}$$

From equations (11),(12) we find

$$\int_0^1 F^c(t,r)dr = \int_0^1 f^c(t,r)dr + \lambda \int_a^b K(t,s)(\int_0^1 F^c(s,r)dr)ds, \quad a \leq t \leq b. \tag{13}$$

$$\int_0^1 F^d(t,r)dr = \int_0^1 f^d(t,r)dr + \lambda \int_a^b |K(t,s)|(\int_0^1 F^d(s,r)dr)ds, \quad a \leq t \leq b. \tag{14}$$

We take $H^c(t) = \int_0^1 F^c(t,r)dr$, $h^c(t) = \int_0^1 f^c(t,r)dr$ and $H^d(t) = \int_0^1 F^d(t,r)dr$, $h^d(t) = \int_0^1 f^d(t,r)dr$. Hence (3) and (3) can be rewritten respectively as

$$H^c(t) = h^c(t) + \lambda \int_a^b K(t,s)H^c(s)ds, \tag{15}$$

$$H^d(t) = h^d(t) + \lambda \int_a^b |K(t,s)|H^d(s)ds. \tag{16}$$

First by multiplying both side of Eq(3) by $(1-r)$ we obtain

$$\int_0^1 (1-r)F^d(t,r)dr = \int_0^1 (1-r)f^d(t,r)dr + \lambda \int_a^b |K(t,s)|(\int_0^1 (1-r)F^d(s,r)dr)ds.$$

Take $G(t) = \int_0^1 (1-r)F^d(t,r)dr, \quad g(t) = \int_0^1 (1-r)f^d(t,r)dr$ then we have

$$G(t) = g(t) + \lambda \int_a^b |K(t,s)|G(s)ds. \tag{17}$$

In the last section of this paper we must solve three crisp Fredholm integral equation of the second kind, i.e. (15),(16),(17) provided that each of them have solutions that is explained in the next section.

4. **Nystrom Method for Linear (FIE-2)**

Many numerical techniques have been used successfully for crisp Fredholm integral equation of the second kind

$$x(t) = y(t) + \lambda \int_a^b K(t,s)x(s)ds \tag{18}$$

where $y(t)$ is a crisp function. In this section, we discuss in detail a straightforward generally applicable technique: *Nystrom* or *quatrature* method. In the operator form we can rewrite $x(t) = y(t) + \lambda \int_a^b K(t,s)x(s)ds$ as

$$x = y + \lambda Kx, \tag{19}$$

provided that

$$Kx = \int_a^b K(t,s)x(s)ds.$$

It is convenient to begin by considering techniques based on the using of the iteration sequences

$$x_{n+1} = y + \lambda Kx_n \tag{20}$$

with initial values and $x_0(t) = y(t)$
provided that

$$Kx_n = \int_a^b K(t,s)x_n(s)ds.$$

In other words (20) can be rewritten as

$$x_{n+1}(t) = y(t) + \lambda \int_a^b K(t,s)x_n(s)ds \tag{21}$$

with initial condition $x_0(t) = y(t)$.

Theorem 4.1. [33] *Let $K(t,s)$ be continuous for $a \leq t,s \leq b$, $\lambda > 0$ and $y(t)$ a continuous function of $a \leq t \leq b$ if $\lambda M(b-a) < 1$ where $|K(t,s)| \leq M$ then sequence (21) with initial condition is uniformly convergence on $[a,b]$.*

I. n general we shall not be able to carry out analytically the integrations that are involved. In this case we naturally turn to numerical quadrature. We introduce a quadrature rule R_N such as *Newton-cotes* of degree p with M-panel(NC(p,M)) and N-point for the interval $[a,b]$ with weights w_i and point t_i if we first ignore the error of quadrature rule R_N, then the integral equation (19) is replaced by the approximate equations

$$x_{R_N}(t) = y(t) + \lambda \sum_{j=1}^{N} w_j K(t,s_j)x_{R_N}(s_j) \qquad i = 1,...,N \tag{22}$$

and (21) by the approximate iterative scheme

$$x_{n+1,R_N}(t_i) = y(t_i) + \lambda \sum_{j=1}^{N} w_j K(t_i,s_j)x_{n,R_N}(s_j) \qquad i = 1,...,N \tag{23}$$

We write the iterations in the form

$$X_{n+1,R_N} = Y + \lambda K X_{n,R_N}$$

where

$$X_{n,R_N} = [x_{n,R_N}(t_1), x_{n,R_N}(t_2), ..., x_{n,R_N}(t_N)]$$

$$Y_n = [y_n(t_1), y_n(t_2), ..., y_n(t_N)]$$

and

$$K_{i,j} = w_j K(t_i, t_j) \quad i, j = 1, 2, ..., N.$$

Now we have some interesting consequence, according to the classic integral equation text such as [18] it will converge if in any matrix norm $\| \lambda K \| < 1$.

5. **Numerical Procedure**

Consider Fuzzy Fredholm integral equation of the second kind

$$F(t) = f(t) + \lambda \int_a^b K(t,s)F(s)ds \tag{24}$$

Where $\lambda > 0$, and $K(t,s)$ is an arbitrary kernel function over the square $a \leq t, s \leq b$ and $f(t)$ is a fuzzy function of $t : a \leq t \leq b$. In this work we first represent the algorithm1 for finding the nearest interval to the fuzzy solution of (FFIE-2), then represent algorithm2 for estimating the fuzzy solution of (FFIE-2) by the nearest symmetric triangular fuzzy numbers as follow:

Algorithm 5.1. (Nearest Interval for solution (FFIE-2))

Step1: : Take $h^c(t_i) = \int_0^1 f^c(t_i, r)dr$ for $t_i : a = t_1 < t_2 < ... < t_N = b$

Step2: : Take $h^d(t_i) = \int_0^1 f^d(t_i, r)dr$ for $t_i : a = t_1 < t_2 < ... < t_N = b$

Step3: : By using numerical method (Nystrom Method) solve Fredholm integral equations (3), (3), then $H^c(t_i)$ and $H^d(t_i)$ are approximated for $t_i : a = t_1 < t_2 < ... < t_N = b$

Step4): : Nearest Interval for solution of (FFIE-2) in t_i is $E(F(t_i) = [E_1, E_2]$ where $E_1 = H^c(t_i) - H^d(t_i)$ and $E_2 = H^c(t_i) + H^d(t_i)$,$i = 1, 2, ..., N$

Algorithm 5.2. (Nearest Symmetric triangular fuzzy number for solution of (FFIE-2))

Step1: : Take $h^c(t_i) = \int_0^1 f^c(t_i, r)dr$ for $t_i : a = t_1 < t_2 < ... < t_N = b$

Step2: : Take $g(t_i) = \int_0^1 (1-r)f^d(t_i, r)dr$, for $t_i : a = t_1 < t_2 < ... < t_N = b$

Step3: : By using numerical method (Nystrom method) solve Fredholm integral equations (3),(3), then $H^c(t_i)$ and $G(t_i)$ are approximated, for $t_i : a = t_1 < t_2 < ... < t_N = b$

Step4: : the Nearest Symmetric triangular fuzzy number for solution of (FFIE-2)in t_i is $S(x_0(t_i), \sigma(t_i))$ where $x_0(t_i) = H^c(t_i)$ and due to 2, $\sigma(t_i) = 3G(t_i)$, $i = 1, 2, ..., N$

Convergence and Rate of Convergence. According to our method that is represented according to the Algorithms 5.1, 5.2, we need to solve some crisp linear Fredholm integral equations of the second kind(FIE-2), convergence of our method depends on the convergence of *Nystrom* method for estimating of (FIE-2). For a given N-point quadrature rule R_N with M-panel of degree p the simplest and the most commonly used procedure for estimating the achieved accuracy is to choose a family of rule R_N and to compute an approximation solution for members of the family with increase N. Due to (21) equalities (15),(16) ,(17) are respectively replaced by

$$H^c_{n+1}(t) = h^c(t) + \lambda \int_a^b K(t,s) H^c_n(s) ds \tag{25}$$

with initial condition $H^c_0(t) = h^c(t)$,

$$H^d_{n+1}(t) = h^d(t) + \lambda \int_a^b |K(t,s)| H^d_n(s) ds \tag{26}$$

with initial condition $H^d_0(t) = h^d(t)$,

$$G_{n+1}(t) = g(t) + \lambda \int_a^b |K(t,s)| G_n(s) ds \tag{27}$$

with initial condition $G_0(t) = g(t)$.
Now we apply Quadrature rule R_N (Newton-cotes of degree p for any fixed p and increasing M) approximate solution of (25),(26),(27). Therefor we have

$$H^c_{n+1,R_N}(t_i) = h^c(t_i) + \lambda \sum_{j=1}^{N} w_j K(t_i, s_j) H^c_{n,R_N}(s_j) \qquad i = 1, ..., N \tag{28}$$

such that $H^c_{0,R_N}(t_i) = h^c(t_i)$

$$H^d_{n+1,R_N}(t_i) = h^d(t_i) + \lambda \sum_{j=1}^{N} w_j |K(t_i, s_j)| H^d_{n,R_N}(s_j) \qquad i = 1, ..., N \tag{29}$$

such that $H^d_{0,R_N}(t_i) = h^d(t_i)$

$$G_{n+1,R_N}(t_i) = g(t_i) + \lambda \sum_{j=1}^{N} w_j |K(t_i, s_j)| G_{n,R_N}(s_j) \qquad i = 1, ..., N \tag{30}$$

such that $G_{0,R_N}(t_i) = g(t_i)$.
By Theorem1, if $|K(t,s)| < M$ and $\lambda M(b-a) < 1$, then each of the sequences (28),(29),(30) converge to $H^c_{R_N}$, $H^d_{R_N}$, G_{R_N} respectively that are estimation of exact solution of (15), (16), (17) i.e., Algorithems 5.1, 5.2 are convergence. Now let $||e_1|| = \max_{a\le t\le b} |H^c(t) - H^c_{R_N}(t)|$, $||e_2|| = \max_{a\le t\le b} |H^d(t) - H^d_{R_N}(t)|$ and $||e_3|| = \max_{a\le t\le b} |G(t) - G_{R_N}(t)|$. Refereing to [18] and recall that $H_1 = (I - \lambda K)^{-1}$, $H_2 = (I - \lambda|k|)^{-1}$ we have

$$||e_1|| \le (1 + ||\lambda H_1||)||C_1(\lambda K(s,t) H^c_{R_N}(t)||N^{-p}$$
$$||e_2|| \le (1 + ||\lambda H_2||)||C_2(\lambda |K(s,t)| H^d_{R_N}(t)||N^{-p}$$
$$||e_3|| \le (1 + ||\lambda H_2||)||C_3(\lambda |K(s,t)| G_{R_N}(t)||N^{-p}$$

where $C_1(\lambda K(s,t)H^c_{R_N}(t)$ is a constant dependent on the function $\lambda K(s,t)H^c_{R_N}(t)$, $C_2(\lambda K(s,t)H^d_{R_N}(t)$ is a constant dependent on the function $\lambda|K(s,t)|H^d_{R_N}(t)$ and $C_3(\lambda|K(s,t)|G_{R_N}(t)$ is a constant dependent on the function $\lambda|K(s,t)|G_{R_N}(t)$. Then convergence rate of our method is the same as that of Nystroms method based on Newton-cotes of degree p.

Example 5.3. [22]. Consider the fuzzy Fredholm integral equation with

$$\underline{f}(t,r) = sin(\frac{t}{2})(\frac{13}{15}(r^2+r) + \frac{2}{15}(4-r^3-r)),$$

$$\overline{f}(t,r) = sin(\frac{t}{2})(\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r)).$$

and the kernel is

$$K(t,s) = 0.1sin(\frac{t}{2})sin(s), \quad 0 \le s,t \le 2\pi, \quad \lambda = 1$$

and $a = 0,\ b = 2\pi$. The exact solution in this case is given by

$$\underline{F}(t,r) = (r^2+r)sin(\frac{t}{2}),$$

$$\overline{F}(t,r) = (4-r^3-r)sin(\frac{t}{2}),$$

therefore, the Expected interval of exact solution is

$$EI(F)(t) = [\frac{5}{6}\sin(\frac{t}{2}), \frac{13}{4}\sin(\frac{t}{2})].$$

One can see

$$h^c(t_i) = \frac{49}{24}\sin(\frac{t_i}{2}), \quad i = 1,2,...,N \tag{31}$$

$$h^d(t_i) = \frac{473}{60}\sin(\frac{t_i}{2}) \quad i = 1,2,...,N. \tag{32}$$

According to the step 3 of Algorithm 5.1 we will have two crisp Fredholm integral equations as

$$H^c(t) = h^c(t) + \int_0^{2\pi} 0.1sin(\frac{t}{2})sin(s)H^c(s)ds \tag{33}$$

$$H^d(t) = h^d(t) + \int_0^{2\pi} |0.1sin(\frac{t}{2})sin(s)|H^d(s)ds \tag{34}$$

Now we apply $Nystroms$ method for each one, and get two sequence as follows

$$H^c_{n+1}(t_i) = h^c(t_i) + \int_0^{2\pi} 0.1sin(\frac{t}{2})sin(s)H^c_n(s)ds, \quad i = 1,2,...,N \tag{35}$$

$$H^d_{n+1}(t_i) = h^d(t_i) + \int_0^{2\pi} |0.1sin(\frac{t}{2})sin(s)|H^d_n(s)ds, \quad i = 1,2,...,N \tag{36}$$

with initial value

$$H_0^c(t_i) = h^c(t_i), \quad H_0^d(t_i) = h^d(t_i), \quad i = 1, 2, ..., N$$

The approximation of both Left bound and right bound of the nearest interval to the solution after 5 iterations , using the Simpson rule with 8 integration nodes are compared in Figure.1, with the exact solution at $t = \pi$. The exact and approximate expected interval of fuzzy solution after 5 iteration at $times\ t = 0, \frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}, \frac{2\pi}{5}, ..., \pi$ are given in Table 1, where $E_1^{F(t)}$ is a Left bound of $EI(F(t))$ and $E_2^{F(t)}$ is a Right bound of $EI(F(t))$

t	$E_1^{F(t)}$	Approx of $E_1^{F(t)}$	$E_2^{F(t)}$	Approx of$E_2^{F(t)}$
0	0	0	0	0
$\frac{\pi}{10}$	0.13035	0.13035	0.50839	0.50839
$\frac{\pi}{5}$	0.25750	0.25750	1.00427	1.00427
$\frac{3\pi}{10}$	0.37831	0.37831	1.47543	1.47542
$\frac{4\pi}{10}$	0.48980	0.48980	1.91025	1.91025
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
π	0.833333	0.83330	3.24999	3.25001

TABLE 1

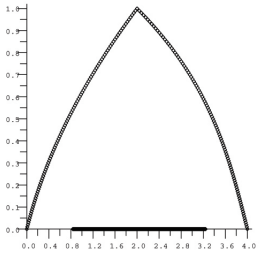


FIGURE 1. The Exact and Interval Approximation of Fuzzy Solution at $t = \pi$

The exact solution at $t = \pi$ is $F(t) = (r^2 + r, 4 - r^3 - r)$ and due to Table1 the approximation of the nearest interval to fuzzy solution at $t = \pi$ is $c(F(\pi)) = [0.83330, 3.25001]$ therefore

$$D(F(\pi), c(F(\pi))) = [\int_0^1 (r^2 + r - 0.83330)^2 dr + \int_0^1 ((4 - r^3 - r) - 3.25001)^2 dr]^{0.5} = 0.80782$$

In this example, we apply Algorithm 5.2 for finding the nearest symmetric triangular fuzzy number to exact solution at $t = \pi$. According to the step 3 of Algorithm 5.2, we will have two crisp Fredholm integral equation as

$$H^c(t) = h^c(t) + \int_0^{2\pi} 0.1sin(\frac{t}{2})sin(s)H^c(s)ds \tag{37}$$

$$G(t) = g(t) + \int_0^{2\pi} |0.1sin(\frac{t}{2})sin(s)|G(s)ds \tag{38}$$

Now we apply $Nystroms$ method for each, and get two sequence as follow

$$H^c_{n+1}(t_i) = h^c(t) + \int_0^{2\pi} 0.1sin(\frac{t}{2})sin(s)H^c_n(s)ds \tag{39}$$

with initial value

$$H^c_0(t_i) = h^c(t_i), \quad i = 1, 2, ..., N$$

$$G_{n+1}(t_i) = g(t) + \int_0^{2\pi} |0.1sin(\frac{t}{2})sin(s)|G_n(s)ds, \quad i = 0, 1, ..., N \tag{40}$$

with initial value

$$G_0(t_i) = g(t_i), \quad i = 1, 2, ..., N$$

In this example we apply Algorithm 5.2 for finding the nearest symmetric triangular fuzzy number to exact solution at arbitrary time t_i. For instance the exact fuzzy solution at $t = \pi$ is $F(t) = (r^2 + r, 4 - r^3 - r)$ and the symmetric triangular fuzzy number that approximate fuzzy solution after 5 iteration at $t = \pi$ is $S(2.04166, 2.30004)$ that are given in Figure2.

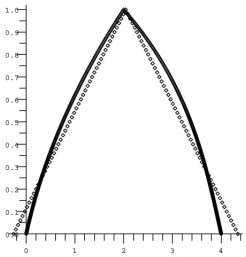


FIGURE 2. Solid and Dot Lines Represent, Respectively, the Exact and the Symmetric Triangular Approximation of the Fuzzy Solution at $t = \pi$, Due to (8) We Conclude That $D(F(\pi, r), S(2.04166, 2.30004)) = 2.2218$

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