MEET-CONTINUITY ON L-DIRECTED COMPLETE POSETS

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Abstract. In this paper, the definition of meet-continuity on L-directed complete posets (for short, L-dcpos) is introduced. As a generalization of meet-continuity on crisp dcpos, meet-continuity on L-dcpos, based on the generalized Scott topology, is characterized. In particular, it is shown that every continuous L-dcpo is meet-continuous and L-continuous retracts of meet-continuous L-dcpos are also meet-continuous. Then, some topological properties of meet-continuity on L-dcpos are discussed. It is shown that meet-continuity on L-dcpos is a topological invariant with respect to the generalized Scott topology, and meet-continuity on L-dcpos is hereditary with respect to generalized Scott closed subsets.

1. Introduction

In the past three decades, quantitative domain theory has formed a new branch of domain theory and has attracted much attention. Rutten's generalized (ultra)metric spaces [11], Flagg's continuity spaces [4] and Wagner's Ω-categories [12] are actually examples of quantitative domains. The main idea of these examples is to provide a metrization of the order relation in domains, which often means that one can use some other lattice instead of **2** quantify the order. This idea suggests that we can deal with quantitative domain theory via fuzzy set theory (see e.g.[19]). In[1,2,3], L-posets were introduced as the basic framework for quantitative domain theory. Recently, taking complete Hetying algebras as structures of truth values, Zhang and Fan [19,22] introduced the notions of L-dcpo, continuous L-dcpo, generalized Scott topology and L-complete lattice. Taking complete residuated lattices as structures of truth values, Yao [13,14,17] proposed the notions of L-way-below relation, L-domain and L-Scott topology. However, based on our knowledge, meet-continuity on L-dcpos is still not given in the literature.[1]

As is well-known, in domain theory, the classical definition of meet-continuity for

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[1] The term "fuzzy" currently is employed in the theories, which are built over the unit interval [0,1]. Since this paper is built over an arbitrary frame L, according to the reviewers' advice, we use "L-poset", "L-dcpo", "L-complete lattice", "L-way-below relation", " L-domain", and "L-Scott topology" to denote "fuzzy poset", "fuzzy dcpo", "fuzzy complete lattice", "fuzzy way-below relation", " fuzzy domain", and "fuzzy Scott topology" respectively.

a directed complete meet-semilattice L is that $a \wedge (\bigvee D) = \bigvee_{d \in D}(a \wedge d)$ for every $a \in L$ and every directed subset $D \subseteq L$. Therefore, in order to generalize meet-continuity from crisp posets to L-posets, one at least has to introduce a binary meet operation on L-posets. In the many-valued setting, Zhao and Zhang [23] proposed a kind of meet Ω-semilattices; Yao [15] introduced a kind of L-frames and then successfully established a fuzzy version of Parpert-Parpert-Isbell-adjunction between the category of the related L-locales and the category of stratified L-topological spaces in [16]. Based on the above works, one may be able to construct a fuzzy version of meet-continuity. On the other hand, the classical meet-continuity can be equivalently characterized by the Scott topology as $cl_\sigma(\downarrow x \cap \downarrow D) = \downarrow x$, whenever $x \leq \bigvee D$ (see e.g.[6]). It provides a suitable approach to generalizing the notion of meet-continuity on dcpos. More precisely, a dcpo X is meet-continuous if $x \leq \bigvee D$ implying $cl_\sigma(\downarrow x \cap \downarrow D) = \downarrow x$ for every $x \in X$ and every directed subset $D \subseteq X$. In our opinion, this kind of definition of meet-continuity constructs a close connection between topology and order structure. Hence, this paper is devoted to constructing meet-continuity on L-dcpos by the generalized Scott topology. We show that meet-continuity on L-dcpos generalizes meet-continuity on crisp dcpos, and that meet-continuity is a topological invariant and is a hereditary property.

The main idea of L-ideal and generalized Scott topology defined in [19] comes from the literature [4]. As shown in [4], this kind of definition of ideal and Scott topology is appropriate for quantitative domain theory including ordinary domain theory, metric domains, ultrametric domains and other examples, such as probabilistic domains and structure spaces, which may be useful in programming language semantics. As a result, our work is not only a generalization of ordinary domain theory, but also provides a reference for the studies of other quantitative domain systems.

The content of the paper is arranged as follows. In section 2, we recall some known notions and results. In section 3, based on the generalized Scott topology, we define meet-continuity on L-dcpos and then show that every continuous L-dcpo is meet-continuous, and also that L-continuous retracts of meet-continuous L-dcpos are meet-continuous. In section 4, some topological properties of meet-continuity on L-dcpos are investigated. It is shown that meet-continuity on L-dcpos is a topological invariance and also is hereditary. Finally, some conclusions are presented in section 5.

2. Preliminaries

Suppose that L is a complete lattice and $p, q \in L$. As defined in [5], p is said to be wedge-below q, denoted $p \lll q$, if for every subset $A \subseteq L$, $q \leq \bigvee A$ implies $p \leq r$ for some $r \in A$. The relation $\lll$ is called multiplicative if for every $p, q, r \in L$, $p \lll q$ and $p \lll r$ imply $p \lll q \wedge r$.

A complete lattice L is said to be completely distributive if it satisfies the complete distributivity law, i.e.,

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)}$$

holds for every family $\{x_{j,k} \mid j \in J,\ k \in K(j)\}$ in L, where M is the set of functions defined on J with the values $f(j) \in K(j)$.

Suppose that L is a completely distributive lattice and $p \in L$. If $p = a \vee b$ implies $p = a$ or $p = b$ for every $a, b \in L$, then p is said to be $\vee$-irreducible.

A frame (or complete Heyting algebra) is a complete lattice L satisfying the following infinite distributive law:

$$a \wedge (\bigvee B) = \bigvee \{a \wedge b \mid b \in B\}$$

for every $a \in L$ and every $B \subseteq L$.

Suppose that L is a frame and $a, b \in L$. We define $a \to b = \bigvee \{c \in L | a \wedge c \leq b\}$.

Throughout this paper, L denotes a frame with $\top, \bot$ as the largest and smallest elements respectively. The following definitions and theorems can be found in [7, 18, 19, 20, 22], and for convenience of the readers, proofs of some results are given.

2.1. L-dcpos.

Definition 2.1. An L-poset is a pair (X, e) such that X is a set and $e : X \times X \longrightarrow L$ is a mapping, called an L-order, that satisfies for every $x, y, z \in X$,

(1) $e(x, x) = \top$;
(2) $e(x, y) \wedge e(y, z) \leq e(x, z)$;
(3) $e(x, y) = e(y, x) = \top$ implies $x = y$.

A poset $(X, \leq)$ can be seen as an L-poset $(X, e_{\leq})$, in which $e_{\leq} : X \times X \longrightarrow L = \{\bot, \top\}$ defined as follows:

$$e_{\leq}(x, y) = \begin{cases} \top & x \leq y \\ \bot & x \not\leq y \end{cases} \tag{1}$$

for every $x, y \in X$. In the sequel, $(X, e_{\leq})$ will be always defined as above for a poset $(X, \leq)$.

Definition 2.2. Let (X, e) be an L-poset. $\varphi \in L^X$ is called an L-directed set on X if

(1) there exists $x \in X$ such that $\bot \lll \varphi(x)$;
(2) for every $x_1, x_2 \in X$ and every $a_1, a_2, a \in L$ with $a_1 \lll \varphi(x_1),\ a_2 \lll \varphi(x_2)$ and $a \lll \top$, there exists $x \in X$ such that $a \lll \varphi(x)$, $a_1 \lll e(x_1, x)$, and $a_2 \lll e(x_2, x)$.

Definition 2.3. Let (X, e) be an L-poset. $\varphi \in L^X$ is called an L-lower set if for every $x, y \in X$, $\varphi(x) \wedge e(y, x) \leq \varphi(y)$.

If φ is an L-directed set and L-lower set, then we call φ an L-ideal. We use $D_L(X)$ to denote the set of all L-directed sets on X and $I_L(X)$ the set of all L-ideals on X.

Definition 2.4. Let (X, e) be an L-poset, let $x_0 \in X$, and let $\varphi \in L^X$. Consider the following conditions:

(1) for every $x \in X, \varphi(x) \leq e(x, x_0)$;

(2) for every $y \in X, \bigwedge_{x\in X} \varphi(x) \to e(x, y) \le e(x_0, y)$.

x_0 is called a join of φ, denoted $\sqcup\varphi$, if it satisfies (1) and (2); x_0 is called an upper bound of φ if it satisfies (1).

Corollary 2.5. *Let (X, e) be an L-poset and let $\varphi \in L^X$. If the join of φ exists, then it is unique.*

Proposition 2.6. *Let (X, e) be an L-poset, let $\varphi \in L^X$, and let $x_0 \in X$. Then $x_0 = \sqcup\varphi$ iff $\bigwedge_{x\in X} \varphi(x) \to e(x, y) = e(x_0, y)$ for every $y \in X$.*

Definition 2.7. An L-poset (X, e) is called an L-directed complete poset (for short, L-dcpo) if every L-directed set on X has a join.

Let (X, e) be an L-poset, let $x \in X$, and let $\varphi \in L^X$. $\downarrow x \in L^X$ and $\downarrow\varphi \in L^X$ are defined as follows:

$$\downarrow x(y) = e(y, x),\ \downarrow\varphi(y) = \bigvee_{x'\in X} \varphi(x') \wedge e(y, x')$$

for every $y \in X$. It is simple to check that $\downarrow\varphi$ is an L-lower set and $\varphi \le \downarrow\varphi$ pointwisely.

Proposition 2.8. *The following conditions are equivalent for an L-poset (X, e) and $\varphi \in L^X$:*

(1) $\sqcup\varphi$ *exists;*

(2) $\sqcup\downarrow\varphi$ *exists.*

And if these conditions are satisfied, then $\sqcup\varphi$=$\sqcup\downarrow\varphi$.

Lemma 2.9. *If (X, e) is an L-dcpo, then $\downarrow\varphi = \varphi$ holds for every L-lower set φ.*

2.2. **Generalized Scott Topology.** Let (X, e) be an L-poset. We introduce the following notations for $x \in X$, $a \in L$, $F \subseteq X$ and $\varphi \in L^X$:

$$\uparrow_a^o x = \{y \in X \mid a \lll e(x, y)\},\ \uparrow_a x = \{y \in X \mid a \le e(x, y)\},$$
$$\downarrow_a^o x = \{y \in X \mid a \lll e(y, x)\},\ \downarrow_a x = \{y \in X \mid a \le e(y, x)\},$$
$$\uparrow_a^o F = \bigcup\{\uparrow_a^o x \mid x \in F\},\ \uparrow_a F = \bigcup\{\uparrow_a x \mid x \in F\},$$
$$\delta_a(\varphi) = \{x \in X \mid a \lll \varphi(x)\}.$$

Note. $\uparrow_a^o x$ and $\uparrow_a x$ are exactly $P_a^o(x)$ and $P_a(x)$ introduced in [19], respectively. In our opinion, the notations of $\uparrow_a^o x$ and $\uparrow_a x$ are more intuitive. Moreover, according to the reviewer's advice, we use $\delta_a(\varphi)$ to denote $\sigma_a(\varphi)$ defined in [19].

Definition 2.10. Let (X, e) be an L-dcpo and let $U \subseteq X$. Then U is generalized Scott open if for every $\varphi \in D_L(X)$, $\bigsqcup\varphi \in U$ implies the existence of $a \lll \top$ and $x \in X$ such that $a \lll \varphi(x)$ and $\uparrow_a x \subseteq U$. The collection of all generalized Scott open subsets of X is a topology, called the generalized Scott topology, denoted $\sigma_e(X)$ (for short, σ_e). The collection of all generalized Scott closed subsets of X is denoted by $\gamma_e(X)$ (for short, γ_e).

Let (X, e) be an L-dcpo. Then we have $\gamma_e(X) = \{A \subseteq X \mid X \setminus A \in \sigma_e(X)\}$ by Definition 2.10.

Corollary 2.11. *Let (X, e) be an L-poset and let $U \subseteq X$. If U is a generalized Scott open set, then for every $x \in U$, there exists $a \lll \top$ such that $\uparrow_a x \subseteq U$.*

Proof. For every $x \in X$, it is obvious that $\downarrow x \in D_L(X)$ and $\sqcup\downarrow x = x$. Since U is a generalized Scott open set, there exist $a \lll \top$ and $z \in X$ such that $a \lll e(z, x)$ and $\uparrow_a z \subseteq U$. Then for every $y \in \uparrow_a x$, we have $a \leq e(z, x) \wedge e(x, y) \leq e(z, y)$, which implies $\uparrow_a x \subseteq \uparrow_a z \subseteq U$. $\square$

Proposition 2.12. *Let (X, e) be an L-dcpo and let $U \subseteq X$. The following are equivalent:*

(1) *U is a generalized Scott closed set;*
(2) *for every $\varphi \in D_L(X)$, the property that $a \not\lll \varphi(x)$ or $\uparrow_a x \cap U \neq \emptyset$ for every $a \lll \top$ and every $x \in X$ implies, $\sqcup\varphi \in U$.*

Proof. By Definition 2.10, it follows that U is a generalized Scott closed set iff $X \setminus U$ is a generalized Scott open set iff for every $\varphi \in D_L(X)$, $\bigsqcup\varphi \in X \setminus U$ implies the existence of $a \lll \top$ and $x \in X$ such that $a \lll \varphi(x)$ and $\uparrow_a x \subseteq X \setminus U$ iff for every $\varphi \in D_L(X)$, the property that $a \not\lll \varphi(x)$ or $\uparrow_a x \cap U \neq \emptyset$ for every $a \lll \top$ and every $x \in X$ implies that $\sqcup\varphi \in U$. $\square$

By Proposition 2.12, we immediately obtain the following result.

Corollary 2.13. *Let (X, e) be an L-dcpo and let $U \subseteq X$. If for every $\varphi \in D_L(X)$, the property that $a \lll \varphi(x)$ and $\uparrow_a x \cap U \neq \emptyset$ for every $a \lll \top$ and every $x \in X$ implies that $\sqcup\varphi \in U$, then U is a generalized Scott closed set.*

Proposition 2.14. *Let L be a completely distributive lattice, let (X, e) be an L-poset, and let $x \in X$. Then for every $b \lll \top$, $\downarrow_b x$ is generalized Scott closed.*

Proof. Suppose that $\varphi \in D_L(X)$ with $a \lll \varphi(x')$ and $\uparrow_a x' \cap \downarrow_b x \neq \emptyset$ for every $a \lll \top$ and every $x' \in X$, which implies that for every $a \lll \top$ and every $x' \in X$, there exists $z \in \uparrow_a x'$ such that $z \in \downarrow_b x$, i.e., $a \leq e(x', z)$ and $b \leq e(z, x)$. Then we have $a \wedge b \leq e(x', z) \wedge e(z, x) \leq e(x', x)$. In such a case, it follows that for every $y \in X$, $b \wedge \varphi(y) = b \wedge (\bigvee_{a \lll \varphi(y)} a) = \bigvee_{a \lll \varphi(y)} a \wedge b \leq e(x', x)$, i.e., $b \leq \varphi(y) \to e(x', x)$. Thus, we have $b \leq \bigwedge_{y \in X} \varphi(y) \to e(x', x) \leq e(\sqcup\varphi, x)$, i.e., $\sqcup\varphi \in \downarrow_b x$. Therefore, $\downarrow_b x$ is generalized Scott closed by Proposition 2.12. $\square$

By Proposition 2.14, we know that if $\top \lll \top$, then $\downarrow_\top x$ is generalized Scott closed.

Definition 2.15. Let (X, e_X), and (Y, e_Y) be L-posets and let $f : X \longrightarrow Y$ be a mapping. Then f is called L-monotone if for every $x, y \in X$, $e_X(x, y) \leq e_Y(f(x), f(y))$.

Let (X, e_X), and (Y, e_Y) be L-posets and let $f : X \longrightarrow Y$ be an ordinary mapping. One can define an L-powerset operator $\widetilde{f}^{\to} : L^X \longrightarrow L^Y$ as follows: $\widetilde{f}^{\to}(\varphi)(y) = \bigvee_{x \in X} \varphi(x) \wedge e_Y(y, f(x))$ for every $\varphi \in L^X$ and every $y \in Y$.

Remark 2.16. (1) A non-empty set X can be seen as an L-poset (X, e^X), in which $e^X : X \times X \longrightarrow L = \{\perp, \top\}$ is defined as follows:

$$e^X(x,y) = \begin{cases} \top & x = y \\ \bot & x \neq y \end{cases} \tag{2}$$

for every $x, y \in X$. Given an ordinary mapping f from X to Y, for every $\varphi \in L^X$ and every $y \in Y$,

$$\widetilde{f}^{\rightarrow}(\varphi)(y) = \bigvee_{x \in X} (\varphi(x) \wedge e^Y(y, f(x))) = \bigvee\{\varphi(x) \mid x \in X, y = f(x)\}.$$

Here $\widetilde{f}^{\rightarrow}$ is exactly the L-forward powerset operator of f [8, 9, 10],. Thus, the L-forward powerset operator is a special case of the above operator.

(2) Let $(X, \leq_X)$, and $(Y, \leq_Y)$ be posets, let $f : X \longrightarrow Y$ be an ordinary mapping, let $A \subseteq X$, and let $B \subseteq Y$. Consider $(X, \leq_X)$ and $(Y, \leq_Y)$ respectively as the L-posets $(X, e_{\leq_X})$ and $(Y, e_{\leq_Y})$, where $L = \{\bot, \top\}$. Then for every $y \in Y$,

$$\begin{aligned} \widetilde{f}^{\rightarrow}(\chi_A)(y) = \top &\Leftrightarrow \exists x \in X, \chi_A(x) = \top, e_{\leq_Y}(y, f(x)) = \top \\ &\Leftrightarrow \exists x \in A, y \leq f(x) \Leftrightarrow y \in \downarrow f^{\rightarrow}(A). \end{aligned}$$

This indicates that the above L-powerset operator $\widetilde{f}^{\rightarrow}$ is a generalization of usual image $\downarrow f^{\rightarrow}$.

(3) An Ω-category is a set A together with a binary function $A : A \times A \longrightarrow \Omega$ such that (1) $I \leq A(a,a)$ for every $a \in A$; (2) $A(a,b) * A(b,c) \leq A(a,c)$ for every $a, b, c \in A$. Then it is easy to check that (1) Ω is an Ω-category when $\Omega(a,b) = a \rightarrow b$ for every $a, b \in \Omega$; (2) the opposite A^{op} of an Ω-category A is also an Ω-category, where $A^{op}(a,b) = A(b,a)$ for every $a, b \in A$. Let $f : A \longrightarrow B$ be an Ω-functor, i.e., f is a function between Ω-categories A and B with $A(a,b) \leq B(f(a), f(b))$ for every $a, b \in A$. Then an Ω-functor $f^{\rightarrow} : A^{op} \longrightarrow \Omega$ is defined as follows: $f^{\rightarrow}(\phi)(b) = \bigvee_{a \in A} \phi(a) * B(b, f(a))$, for every $b \in B$ and every $\phi \in \Omega^{A^{op}}$, where $\Omega^{A^{op}}$ denotes the set of all mappings from A^{op} to Ω. If $\Omega = L$, then $f^{\rightarrow}$ is exactly $\widetilde{f}^{\rightarrow}$.

Proposition 2.17. *Let $(X, e_X), (Y, e_Y)$ be two L-posets and let $f : X \rightarrow Y$ be an L-monotone mapping. Then for every $\varphi \in D_L(X)$, $\widetilde{f}^{\rightarrow}(\varphi) \in I_L(Y)$.*

Definition 2.18. Let (X, e_X) and (Y, e_Y) be L-dcpos. An L-monotone mapping $f : X \longrightarrow Y$ is L-Scott continuous if it preserves joins of L-directed sets, that is $f(\sqcup\varphi) = \sqcup\widetilde{f}^{\rightarrow}(\varphi)$ for every $\varphi \in D_L(X)$.

Proposition 2.19. *Let (X, e_X) and (Y, e_Y) be L-dcpos. If a mapping $f : X \longrightarrow Y$ preserves joins of L-directed sets, then f is L-monotone.*

Proof. For every $x, x' \in X$,

$$\begin{aligned} e_X(x, x') &= \downarrow x'(x) = \downarrow x'(x) \wedge e_Y(f(x), f(x)) \\ &\leq \bigvee_{x'' \in X} \downarrow x'(x'') \wedge e_Y(f(x), f(x'')) \\ &= \widetilde{f}^{\rightarrow}(\downarrow x')(f(x)) \leq e_Y(f(x), \sqcup\widetilde{f}^{\rightarrow}(\downarrow x')) \\ &= e_Y(f(x), f(\sqcup\downarrow x')) = e_Y(f(x), f(x')), \end{aligned}$$

which implies that $f : X \longrightarrow Y$ is L-monotone. $\square$

Theorem 2.20. *Let L be a completely distributive lattice, let (X, e_X), and (Y, e_Y) be L-dcpos, and let $f : X \longrightarrow Y$ be an L-monotone mapping. Then f is L-Scott continuous iff f is topologically continuous with respect to the generalized Scott topologies.*

Proof. Necessity. Suppose that f is L-Scott continuous. Let $V \subseteq Y$ be a generalized Scott open set and let $\varphi \in D_L(X)$ with $\sqcup\varphi \in f^{-1}(V)$. Then it follows that $\sqcup\widetilde{f}^{\rightarrow}(\varphi) = f(\sqcup\varphi)$. Since V is a generalized Scott open set, there exist $a \lll \top$ and $y \in Y$ such that $a \lll \widetilde{f}^{\rightarrow}(\varphi)(y) = \bigvee_{x\in X} \varphi(x) \wedge e_Y(y, f(x))$ and $\uparrow_a y \subseteq V$, which implies that there exists $x \in X$ such that $a \lll \varphi(x)$ and $a \lll e_Y(y, f(x))$. Then for every $x' \in \uparrow_a x$, we have $a \leq e_X(x, x') \leq e_Y(f(x), f(x'))$ and so $a \leq e_Y(y, f(x)) \wedge e_Y(f(x), f(x')) \leq e_Y(y, f(x'))$, which implies that $f(x') \in \uparrow_a y \subseteq V$, i.e., $x' \in f^{-1}(V)$. Therefore, we have $\uparrow_a x \subseteq f^{-1}(V)$. To sum up the above, it follows that $f^{-1}(V)$ is a generalized Scott open set.

Sufficiency. Suppose that f is topologically continuous w.r.t. the generalized Scott topologies. Let $\varphi \in D_L(X)$. Since f is L-monotone, it follows that

$$\begin{aligned} e_Y(\sqcup\widetilde{f}^{\rightarrow}(\varphi), f(\sqcup\varphi)) &= \bigwedge_{y\in Y} \widetilde{f}^{\rightarrow}(\varphi)(y) \rightarrow e_Y(y, f(\sqcup\varphi)) \\ &= \bigwedge_{y\in Y} (\bigvee_{x\in X} \varphi(x) \wedge e_Y(y, f(x))) \rightarrow e_Y(y, f(\sqcup\varphi)) \\ &= \bigwedge_{x\in X} \varphi(x) \rightarrow e_Y(f(x), f(\sqcup\varphi)) \\ &\geq \bigwedge_{x\in X} \varphi(x) \rightarrow e_X(x, \sqcup\varphi) = e_X(\sqcup\varphi, \sqcup\varphi) = \top. \end{aligned}$$

Conversely, let $a \lll \top$ with $a \not\leq e_Y(f(\sqcup\varphi), \sqcup\widetilde{f}^{\rightarrow}(\varphi))$, i.e., $f(\sqcup\varphi) \in Y \setminus (\downarrow_a \sqcup\widetilde{f}^{\rightarrow}(\varphi))$, which implies that $\sqcup\varphi \in f^{-1}(Y \setminus (\downarrow_a \sqcup\widetilde{f}^{\rightarrow}(\varphi)))$. Furthermore, since $f^{-1}(Y \setminus (\downarrow_a \sqcup\widetilde{f}^{\rightarrow}(\varphi)))$ is a generalized Scott open set by continuity of f and Proposition 2.14, there exist $b \lll \top$ and $x \in X$ such that $b \lll \varphi(x)$ and $\uparrow_b x \subseteq f^{-1}(Y \setminus (\downarrow_a \sqcup\widetilde{f}^{\rightarrow}(\varphi)))$. On the other hand, since φ is an L-directed set, there exists $x' \in X$ such that $a \lll \varphi(x')$ and $b \leq e_X(x, x')$, i.e., $x' \in \uparrow_b x \subseteq f^{-1}(Y \setminus (\downarrow_a \sqcup\widetilde{f}^{\rightarrow}(\varphi)))$, which implies that $f(x') \in Y \setminus (\downarrow_a \sqcup\widetilde{f}^{\rightarrow}(\varphi))$, i.e., $a \not\leq e_Y(f(x'), \sqcup\widetilde{f}^{\rightarrow}(\varphi))$. But, since $a \lll \varphi(x') = \varphi(x') \wedge e_Y(f(x'), f(x')) \leq \bigvee_{z\in X} \varphi(z) \wedge e_Y(f(x'), f(z)) = \widetilde{f}^{\rightarrow}(\varphi)(f(x'))$, it follows that $a \leq e_Y(f(x'), \sqcup\widetilde{f}^{\rightarrow}(\varphi))$, which is a contradiction. Therefore, for every $a \lll \top$, it follows that $a \leq e_Y(f(\sqcup\varphi), \sqcup\widetilde{f}^{\rightarrow}(\varphi))$, which implies that $e_Y(f(\sqcup\varphi), \sqcup\widetilde{f}^{\rightarrow}(\varphi)) = \top$. To sum up the above, we have, $f(\sqcup\varphi) = \sqcup\widetilde{f}^{\rightarrow}(\varphi)$, i.e., f is L-Scott continuous. $\square$

2.3. **Continuous L-dcpos.**

Definition 2.21. Let (X, e) be an L-dcpo, let $x, y \in X$, and let $a \in L$. If for every $\varphi \in D_L(X)$, $a \lll e(y, \sqcup\varphi)$ implies that $a \lll \varphi(z)$ and $a \lll e(x, z)$ for some $z \in X$, then x is called L_a-way below y, denoted $x \ll_a y$. x is said to be $L-$way below y, denoted $x \ll_L y$, if for every $a \lll \top$, $x \ll_a y$.

Definition 2.22. Let (X, e) be an L-dcpo and let $B \subseteq X$. If for every $x \in X$, there exists $\varphi \in D_L(B)$ such that $x = \bigsqcup \widetilde{i_B}^{\rightarrow}(\varphi)$ and $\delta_a(\widetilde{i_B}^{\rightarrow}(\varphi)) \subseteq \Downarrow_a x$ for every $a \lll \top$, then B is called a basis for X, where i_B is the embedding of B into X and $\Downarrow_a x = \{y \in X \mid y \ll_a x\}$.

Definition 2.23. An L-dcpo (X, e) is said to be continuous if it has a basis.

Theorem 2.24. *Let L be a completely distributive lattice, in which $\top$ is $\vee$-irreducible and $\lll$ is multiplicative, and let (X, e) be a continuous L-dcpo. Then for every $\perp \neq a \lll \top$ and every $x \in X$, $\Uparrow_a x = \{y \in X \mid x \ll_a y\}$ is a generalized Scott open set and $\{\Uparrow_a x \mid x \in X, \perp \neq a \lll \top\}$ is a basis for generalized Scott open sets.*

Remark 2.25. Theorem 2.24 is exactly Lemma 4.12 and Corollary 4.14 in [19], so for the proof of Theorem 2.24, the readers can refer to that of Lemma 4.12 and Corollary 4.14 in [19].

3. Meet-continuous L-dcpos

In this section, based on the generalized Scott topology, we propose the meet-continuity on L-dcpos, present equivalent descriptions of meet-continuous L-dcpos, and discuss the relationships between meet-continuity and continuous L-dcpos. Moreover, it is also shown that L-continuous retracts of meet-continuous L-dcpos are meet-continuous.

Definition 3.1. Let (X, e) be an L-dcpo and let $a \in L$. Then (X, e) is said to be L_a-meet-continuous if for every $x \in X$ and every $\varphi \in D_L(X)$, $a \lll e(x, \sqcup\varphi)$ implies $x \in cl_{\sigma_e}(\delta_a(\downarrow\varphi) \cap \downarrow_a x)$. (X, e) is said to be meet-continuous if for every $a \lll \top$, (X, e) is L_a-meet-continuous.

The definition above can be interpreted as follows: for every $x \in X$ and every directed set φ, $x \leq \sqcup\varphi$ implies $x \in cl_\sigma(\downarrow\varphi \cap \downarrow x)$, which is exactly the definition of meet-continuity of [6] for crisp dcpos.

Theorem 3.2. *Let (X, e) be an L-dcpo and let $a \in L$. Then (X, e) is L_a-meet continuous iff for every $x \in X$ and every $\varphi \in I_L(X)$, $a \lll e(x, \sqcup\varphi)$ implies $x \in cl_{\sigma_e}(\delta_a(\varphi) \cap \downarrow_a x)$.*

Proof. Necessity. Obviously.

Sufficiency. Let $\varphi \in D_L(X)$ with $a \lll e(x, \sqcup\varphi)$. Then $\downarrow\varphi \in I_L(X)$ and $\sqcup\varphi = \sqcup\downarrow\varphi$, which yields $a \lll e(x, \sqcup\downarrow\varphi)$. By the assumption and Lemma 2.9, we have $x \in cl_{\sigma_e}(\delta_a(\downarrow\downarrow\varphi) \cap \downarrow_a x) = cl_{\sigma_e}(\delta_a(\downarrow\varphi) \cap \downarrow_a x)$. Hence, (X, e) is L_a-meet-continuous. $\square$

The following lemma is a particular version of the remark made just after Proposition 3.3 in [19].

Lemma 3.3. *Let $(X, \leq)$ be a dcpo. Then*

(1) *for every $\varphi \in D_L(X)$ and every $a \lll \top$, $\delta_a(\varphi)$ is a directed set on $(X, \leq)$ and $\sqcup\varphi = \bigvee \delta_a(\varphi)$;*

(2) *for every directed subset* $D \subseteq X$, χ_D *is an* L*-directed set on* $(X, e_{\leq})$ *and* $\sqcup\chi_D = \bigvee D$;

(3) *for every subset* $A \subseteq X$ *and every* $x \in X$, $\delta_\top(\downarrow\chi_A) = \downarrow A$ *and* $\downarrow_\top x = \downarrow x$.

By Lemma 3.3, it follows that $(X, \leq)$ is a dcpo iff $(X, e_{\leq})$ is an L-dcpo.

Lemma 3.4. *Let* $(X, \leq)$ *be a dcpo. Then for every* L*-directed set* φ *on* $(X, e_{\leq})$, $D = \{y \in X \mid \downarrow\varphi(y) = \top\}$ *is an ideal on* $(X, \leq)$.

Proof. Firstly, we show that $D = \{y \in X \mid \downarrow\varphi(y) = \top\}$ is a directed set. Since φ is L-directed, it follows that $\delta_\top(\varphi) = \{y \in X \mid \varphi(y) = \top\}$ is directed on $(X, \leq)$ by Lemma 3.3 (1). Then there exists $x_0 \in X$ such that $\varphi(x_0) = \top$. Furthermore, since $\varphi(x) \leq \downarrow\varphi(x)$ for every $x \in X$, $\downarrow\varphi(x_0) = \top$, which implies $D = \{y \in X \mid \downarrow\varphi(y) = \top\}$ is non-empty. Now, let $x_1, x_2 \in D$, i.e., $\downarrow\varphi(x_1) = \downarrow\varphi(x_2) = \top$. Since $\downarrow\varphi \in D_L(X)$, there exists $z \in X$ such that $\downarrow\varphi(z) = \top$ and $\top = e_{\leq}(x_1, z) = e_{\leq}(x_2, z)$, i.e., $z \in D$ and $x_1 \leq z$, $x_2 \leq z$. Thus, D is a directed set.

We further show that D is a lower set. Let $x_1 \in D$ and let $x_2 \leq x_1$, i.e., $\downarrow\varphi(x_1) = \top$ and $e_{\leq}(x_2, x_1) = \top$. Furthermore, since $\downarrow\varphi$ is an L-lower set on $(X, e_{\leq})$, we have $\downarrow\varphi(x_1) \wedge e_{\leq}(x_2, x_1) \leq \downarrow\varphi(x_2)$ and so $\downarrow\varphi(x_2) = \top$, i.e., $x_2 \in D$. This implies that D is a lower set. $\square$

Lemma 3.5. *Let* $(X, \leq)$ *be a dcpo and let* $U \subseteq X$. *Then* U *is a generalized Scott open set on* $(X, e_{\leq})$ *iff* U *is a Scott open set on* $(X, \leq)$.

Proof. Necessity. Suppose that U is a generalized Scott open set on $(X, e_{\leq})$. Then for every $x \in U$ with $x \leq y$, there exists $a \lll \top$ such that $\uparrow_a x \subseteq U$ by Corollary 2.11. In such a case, $y \in \uparrow_\top x \subseteq \uparrow_a x \subseteq U$. Thus, U is an upper set on $(X, \leq)$. Let D be a directed set on $(X, \leq)$ such that $\bigvee D \in U$. By Lemma 3.3 (2), we have χ_D is an L-directed set on $(X, e_{\leq})$ and $\sqcup\chi_D = \bigvee D$, which implies $\sqcup\chi_D \in U$. Then there exist $a \lll \top$ and $y \in X$ such that $a \lll \chi_D(y)$ and $\uparrow_a y \subseteq U$, which means that $\chi_D(y) = \top$ and $y \in U$, i.e., $y \in D \cap U$. To sum up the above, it follows that U is a Scott open set on $(X, \leq)$.

Sufficiency. Suppose that U is a Scott open set on $(X, \leq)$. Let φ is an L-directed set on $(X, e_{\leq})$ such that $\sqcup\varphi \in U$. By Lemma 3.3 (1), we have $\delta_\top(\varphi) = \{x \in X \mid \varphi(x) = \top\}$ is a directed set on $(X, \leq)$ and $\sqcup\varphi = \bigvee \delta_\top(\varphi)$, which implies that $\bigvee \delta_\top(\varphi) \in U$. Then there exists $x \in X$ such that $x \in \delta_\top(\varphi) \cap U$, i.e., there exists $x \in U$ such that $\top \lll \top = \varphi(x)$. Furthermore, since U is an upper set on $(X, \leq)$, we have $\uparrow_\top x = \{y \in X \mid x \leq y\} \subseteq U$. Hence, U is a generalized Scott open set on $(X, e_{\leq})$. $\square$

Theorem 3.6. *Let* $(X, \leq)$ *be a dcpo. Then* $(X, \leq)$ *is meet-continuous iff* $(X, e_{\leq})$ *is* $L_\top$*-meet-continuous.*

Proof. Suppose that $(X, e_{\leq})$ is $L_\top$-meet-continuous. For every $x \in X$ and every directed subset $D \subseteq X$ with $x \leq \bigvee D$, we have, $\top \lll e_{\leq}(x, \sqcup\chi_D) = \top$ and $\chi_D \in D_L(X)$ by Lemma 3.3 (2). Then we have $x \in cl_{\sigma_e}(\delta_\top(\downarrow\varphi) \cap \downarrow_\top x)$ by the $L_\top$-meet-continuity of $(X, e_{\leq})$. Furthermore, since $\sigma_e(X) = \sigma(X)$, $\delta_\top(\downarrow\chi_D) = \downarrow D$ and $\downarrow_\top x = \downarrow x$ by Lemma 3.3 (3), 3.5, we have $x \in cl_\sigma(\downarrow D \cap \downarrow x)$. Therefore, $(X, \leq)$ is meet-continuous.

Suppose that $(X,\leq)$ is meet-continuous. For every $x\in X$ and every $\varphi\in D_L(X)$ with $\top\lll e_{\leq}(x,\sqcup\varphi)$, we have $x\leq\sqcup\varphi$. Let $D=\{y\in X\mid{\downarrow}\varphi(y)=\top\}$. Then by Lemma 3.3 (1), 3.4 and Proposition 2.8, D is an ideal on $(X,\leq)$ and $\bigvee D=\bigvee{\downarrow}D=\sqcup\varphi=\sqcup{\downarrow}\varphi$. Thus, it follows that $x\in cl_{\sigma}({\downarrow}D\cap{\downarrow}x)$. Furthermore, since $\sigma_e(X)=\sigma(X)$, $\delta_\top({\downarrow}\varphi)={\downarrow}D$ and ${\downarrow}_\top x={\downarrow}x$ by Lemma 3.3 (3), 3.5, we have $x\in cl_{\sigma_e}(\delta_\top({\downarrow}\varphi)\cap{\downarrow}_\top x)$. Therefore, $(X,e_{\leq})$ is $L_\top$-meet-continuous. $\square$

The theorem above means that meet-continuity on L-dcpos generalizes meet-continuity on crisp dcpos. Next, we give an equivalent characterization of meet-continuity on L-dcpos.

Theorem 3.7. *Let L be a completely distributive lattice, in which $\lll$ is multiplicative, let (X,e) be an L-dcpo, and let $a\lll\top$. Then (X,e) is L_a-meet-continuous iff ${\uparrow}^o_a(U\cap{\downarrow}_a x)\in\sigma_e(X)$ for every $U\in\sigma_e(X)$ and every $x\in X$.*

Proof. Necessity. Let $U\in\sigma_e(X)$ and let $x\in X$. Suppose $\varphi\in I_L(X)$ with $\sqcup\varphi\in{\uparrow}^o_a(U\cap{\downarrow}_a x)$. Then there exists $y\in U\cap{\downarrow}_a x$ such that $a\lll e(y,\sqcup\varphi)$. By L_a-meet-continuity of X, we have $U\cap\delta_a(\varphi)\cap{\downarrow}_a y\neq\emptyset$. This implies $\delta_a(\varphi)\cap{\uparrow}^o_a(U\cap{\downarrow}_a x)\supseteq\delta_a(\varphi)\cap{\uparrow}^o_a(U\cap{\downarrow}_a y)\neq\emptyset$, i.e., there exists $z\in\delta_a(\varphi)$ and $z\in{\uparrow}^o_a(U\cap{\downarrow}_a x)$. Then there exists $z_0\in U\cap{\downarrow}_a x$ such that $z\in{\uparrow}^o_a z_0$. Furthermore, since $a\lll\varphi(z)\wedge e(z_0,z)\leq\varphi(z_0)$, we have $a\lll\varphi(z_0)$ and ${\uparrow}^o_a\, z_0\subseteq{\uparrow}^o_a(U\cap{\downarrow}_a x)$. Hence, ${\uparrow}^o_a(U\cap{\downarrow}_a x)\in\sigma_e(X)$.

Sufficiency. Let $x\in X$ and let $\varphi\in I_L(X)$ with $a\lll e(x,\sqcup\varphi)$. If $x\notin cl_{\sigma_e}(\delta_a(\varphi)\cap{\downarrow}_a x)$, then there exists $U\in\sigma_e(X)$ such that $x\in U$ and $U\cap\delta_a(\varphi)\cap{\downarrow}_a x=\emptyset$. This implies ${\uparrow}^o_a(U\cap{\downarrow}_a x)\cap\delta_a(\varphi)=\emptyset$. Since ${\uparrow}^o_a(U\cap{\downarrow}_a x)\in\sigma_e(X)$, there exist $y\in X$ and $b\lll 1$ such that $b\lll\varphi(y)$ and ${\uparrow}^o_b y\subseteq{\uparrow}_b y\subseteq{\uparrow}^o_a(U\cap{\downarrow}_a x)$. Note that since $\varphi\in I_L(X)$, there exists $z\in X$ such that $a\lll\varphi(z)$, $b\lll e(y,z)$. Hence, ${\uparrow}^o_a(U\cap{\downarrow}_a x)\cap\delta_a(\varphi)\neq\emptyset$, which is a contradiction. $\square$

Corollary 3.8. *Let L be a completely distributive lattice, in which $\lll$ is multiplicative, and let (X,e) be an L-dcpo. Then (X,e) is meet-continuous iff ${\uparrow}^o_a(U\bigcap{\downarrow}_a x)\in\sigma_e(X)$ for every $U\in\sigma_e(X)$, every $x\in X$ and every $a\lll\top$.*

Theorem 3.9. *Let $\lll$ be multiplicative on L. Then every continuous L-dcpo is meet-continuous.*

Proof. Let $a\lll\top$, let $x\in X$, and let $\varphi\in D_L(X)$ with $a\lll e(x,\sqcup\varphi)$. Since (X,e) is continuous, (X,e) has a basis $B\subseteq X$ such that $x=\sqcup\widetilde{i_B}^{\rightarrow}(\phi)$ for some $\phi\in D_L(X)$ and $\delta_b(\widetilde{i_B}^{\rightarrow}(\phi))\subseteq{\Downarrow}_b x$ for every $b\lll\top$. Let U be a generalized Scott open neighborhood of x. Then $\sqcup\widetilde{i_B}^{\rightarrow}(\phi)\in U$. Since $U\in\sigma_e(X)$, there exist $z\in X$ and $c\lll\top$ such that $c\lll\widetilde{i_B}^{\rightarrow}(\phi)(z)$ and ${\uparrow}_c\, z\subseteq U$. Furthermore, since $\widetilde{i_B}^{\rightarrow}(\phi)\in D_L(X)$, there exists $z_0\in X$ such that $a\lll\widetilde{i_B}^{\rightarrow}(\phi)(z_0)$ and $c\lll e(z,z_0)$, which implies $z_0\in U$. Note that for every $b\lll\top$, we have $\delta_b(\widetilde{i_B}^{\rightarrow}(\phi))\subseteq{\Downarrow}_b x$, and hence, $z_0\ll_a x$, which means $z_0\in{\downarrow}_a x$. Since $z_0\ll_a x$, $a\lll e(x,\sqcup\varphi)$ implies that there exists $y\in X$ such that $a\lll\varphi(y)$ and $a\lll e(z_0,y)$. Considering ${\downarrow}\varphi\in I_L(X)$, we have ${\downarrow}\varphi(z_0)\geq{\downarrow}\varphi(y)\wedge e(z_0,y)\ggg a$, which implies $z_0\in\delta_a({\downarrow}\varphi)$. Therefore, we have, $z_0\in U\cap\delta_a({\downarrow}\varphi)\cap{\downarrow}_a x$, i.e., $U\cap\delta_a({\downarrow}\varphi)\cap{\downarrow}_a x\neq\emptyset$. By the arbitrariness of U, we have $x\in cl_{\sigma_e}({\downarrow}_a x\cap\delta_a({\downarrow}\varphi))$. This implies that (X,e) is L_a-meet-continuous. Therefore, (X,e) is meet-continuous by the arbitrariness of a. $\square$

Definition 3.10 (21)**.** Let (X, e_X), (Y, e_Y) be L-dcpos and let $s : (X, e_X) \longrightarrow (Y, e_Y)$, $r : (Y, e_Y) \longrightarrow (X, e_X)$ be L-Scott continuous mappings. Then a pair (s, r) of mappings is called an L-continuous section-retraction-pair if $r \circ s = id_X$. In such a case, we call (X, e_X) an L-continuous retract of (Y, e_Y).

Theorem 3.11. *Let $\lll$ be multiplicative on L and let (X, e_X) be an L-continuous retract of a meet-continuous L-dcpo (Y, e_Y). Then (X, e_X) is also meet-continuous.*

Proof. Let $a \lll \top$, let $x \in X$, and let $\varphi \in D_L(X)$ with $a \lll e_X(x, \sqcup\varphi)$. Let U be a generalized Scott open neighborhood of x. Since (X, e_X) is an L-continuous retract of (Y, e_Y), there exist L-Scott continuous mappings $s : (X, e_X) \longrightarrow (Y, e_Y)$, $r : (Y, e_Y) \longrightarrow (X, e_X)$ such that $r \circ s = id_X$. By the L-Scott continuity of s, it is known that s is an L-monotone mapping and $s(\sqcup\phi) = \sqcup\widetilde{s}^{\rightarrow}(\phi)$ for every $\phi \in D_L(X)$, and hence, $a \lll e_Y(s(x), s(\sqcup\varphi)) = e_Y(s(x), \sqcup\widetilde{s}^{\rightarrow}(\varphi))$ and $\widetilde{s}^{\rightarrow}(\varphi) \in D_L(Y)$ by Proposition 2.17. Since $x = r \circ s(x) \in U \in \sigma_e(X)$ and r is L-Scott continuous, it follows that $s(x) \in r^{-1}(U) \in \sigma_e(Y)$. Therefore, by the meet-continuity of Y, we have $s(x) \in cl_{\sigma_e(Y)}(\delta_a(\downarrow\widetilde{s}^{\rightarrow}(\varphi)) \cap \downarrow_a s(x))$ and $r^{-1}(U) \cap \delta_a(\downarrow\widetilde{s}^{\rightarrow}(\varphi)) \cap \downarrow_a s(x)) = r^{-1}(U) \cap \delta_a(\widetilde{s}^{\rightarrow}(\varphi)) \cap \downarrow_a s(x)) \neq \emptyset$. Choose $y \in r^{-1}(U) \cap \delta_a(\widetilde{s}^{\rightarrow}(\varphi)) \cap \downarrow_a s(x))$, i.e., $a \lll \widetilde{s}^{\rightarrow}(\varphi)(y)$, $a \lll e_Y(y, s(x))$ and $r(y) \in U$. Since $\widetilde{s}^{\rightarrow}(\varphi)(y) = \bigvee_{z \in X} \varphi(z) \wedge e_Y(y, s(z))$, there exists $z \in X$ such that $a \lll \varphi(z)$ and $a \lll e_Y(y, s(z))$. Furthermore, since the pair (s, r) is an L-continuous retract, we have, $a \lll e_X(r(y), r \circ s(x)) = e_X(r(y), x)$, $a \lll e_X(r(y), r \circ s(z)) = e_X(r(y), z)$, and hence, $a \lll \downarrow\varphi(r(y)) = \bigvee_{z \in X} \varphi(z) \wedge e_X(r(y), z)$. From the above discussion, it follows that $r(y) \in U \cap \delta_a(\downarrow\varphi) \cap \downarrow_a x)$. By the arbitrariness of U, we have, $x \in cl_{\sigma_e(X)}(\delta_a(\downarrow\varphi) \cap \downarrow_a x)$, i.e., (X, e_X) is L_a-meet-continuous. Thus, (X, e_X) is meet-continuous by the arbitrariness of a. $\square$

On crisp dcpos, meet-continuity forces closer relationships between the Scott and Lawson topologies. So we present the notions of generalized lower topology and generalized Lawson topology, and discuss their relationships with the generalized Scott topology on meet-continuous L-dcpos.

Definition 3.12. Let (X, e) be an L-dcpo and let $a \in L$. We call the topology generated by the complements $X \backslash \uparrow_a x$ $(x \in X)$ the L_a-generalized lower topology and denote it by $\omega_a(X)$. $\omega_e(X) = \bigcup\{\omega_a(X) \mid a \in L, a \lll \top\}$ is called then the generalized lower topology.

Remark 3.13. It is obvious that $cl_{\omega_a}(\{x\}) = \uparrow_a x$, $\uparrow_a F = \bigcup\{\uparrow_a x \mid x \in X\}$ is a generalized lower closed set for every finite subset $F \subseteq X$ and $\omega_e(X) \subseteq \sigma_e(X^{op})$.

Theorem 3.14. *If (X, e) is an L-dcpo, then the generalized lower topology is T_0.*

Proof. Let $x, y \in X$ with $x \neq y$. Since (X, e) is an L-dcpo, there exists $a \lll \top$ such that $a \not\lll e(x, y)$ or $a \not\lll e(y, x)$, i.e., $x \in X \backslash \uparrow_a y$ or $y \in X \backslash \uparrow_a x$. By the definition of generalized lower topology, we have, $X \backslash \uparrow_a y$, $X \backslash \uparrow_a x \in \omega_e(X)$. However, $y \notin X \backslash \uparrow_a y$, $x \notin X \backslash \uparrow_a x$. So the generalized lower topology is T_0. $\square$

Definition 3.15. Let (X, e) be an L-dcpo. We call the topology generated by $\sigma_e(X) \cup \omega_e(X)$ the generalized Lawson topology and denote it by $\lambda_e(X)$.

Theorem 3.16. *If (X,e) is an L-dcpo, then the generalized Lawson topology is T_1.*

Proof. For every $x \in X$ and every $a \lll \top$, we have $\{x\} = \downarrow_a x \bigcap \uparrow_a x$. Now, $\downarrow_a x$ is a generalized Scott closed set, while $\uparrow_a x$ is a generalized lower closed set. Hence, $\{x\}$ is a generalized Lawson closed set, which implies that the generalized Lawson topology is T_1. □

Lemma 3.17. *Let L be a completely distributive lattice, let $a \in L$, and let (X,e) be a meet-continuous L-dcpo. Then the following assertions hold:*

(1) *if $U \in \lambda_e(X)$, then $\uparrow_a^o U \in \sigma_e(X)$;*
(2) *if $Y = \uparrow_a^o Y$ for $Y \subseteq X$, then $int_{\sigma_e} Y = int_{\lambda_e} Y$;*
(3) *if $Y = \downarrow_a^o Y$ for $Y \subseteq X$, then $cl_{\sigma_e} Y = cl_{\lambda_e} Y$.*

Proof. (1) Suppose $\sqcup\varphi \in \uparrow_a^o U$, where $U \in \lambda_e(X)$ and $\varphi \in D_L(X)$. Then we can find $x \in U$ with $a \lll e(x, \sqcup\varphi)$. From the definition of the generalized Lawson topology, there exist a generalized Scott open set V and a finite set $F \subseteq X$ such that $x \in V \backslash \uparrow_a F \subseteq U$. By the meet-continuity of (X,e), it follows that $x \in cl_{\sigma_e}(\delta_a(\downarrow\varphi) \cap \downarrow_a x)$, i.e., $V \cap \delta_a(\downarrow\varphi) \cap \downarrow_a x \neq \emptyset$. In addition, since $x \in V \backslash \uparrow_a F$, we have $V \backslash \uparrow_a F \cap \delta_a(\downarrow\varphi) \cap \downarrow_a x = V \cap \delta_a(\downarrow\varphi) \cap \downarrow_a x \neq \emptyset$. Then $U \cap \delta_a(\varphi) \cap \downarrow_a x \neq \emptyset$, i.e., there exists $y \in U$ such that $a \lll \varphi(y)$ and $\uparrow_a^o y \subseteq \uparrow_a^o U$. Therefore, $\uparrow_a^o U \in \sigma_e(X)$.

(2) Suppose $Y = \uparrow_a^o Y$ for $Y \subseteq X$. Since $\sigma_e \subseteq \lambda_e$, we have $int_{\sigma_e} Y \subseteq int_{\lambda_e} Y$. From (1), it holds that $int_{\lambda_e} Y \subseteq \uparrow_a^o(int_{\lambda_e} Y) = int_{\sigma_e}(\uparrow_a^o(int_{\lambda_e} Y) \subseteq int_{\sigma_e} \uparrow_a^o Y = int_{\sigma_e} Y$. Hence, $int_{\sigma_e} Y = int_{\lambda_e} Y$.

(3) Similar to the proof of (2). □

Lemma 3.18. *Let (X,e) be an L-dcpo. Then $\downarrow_\top x \subseteq U$ for every $x \in U \in \gamma_e(X)$.*

Proof. Suppose $\downarrow_\top x \nsubseteq U$. Then there exists $y_0 \in \downarrow_\top x$ such that $y_0 \in X \backslash U \in \sigma_e(X)$. Thus, there exists $b \lll \top$ such that $\uparrow_b y_0 \subseteq X \setminus U$, which implies $x \in X \setminus U$. It is a contradiction. □

Theorem 3.19. *Let L be a completely distributive lattice, in which $\lll$ is multiplicative, and let $\top \lll \top$. If (X,e) is an $L_\top$-meet-continuous L-dcpo, then $\sigma_e(X)^{op}$ is a complete Heyting algebra.*

Proof. Let $\gamma_e(X) = \{A \subseteq X \mid X \setminus A \in \sigma_e(X)\}$ be the generalized Scott closed set lattice (ordered by set-theoretic inclusion). Then we have $\sigma_e(X)^{op} \cong \gamma_e(X)$. Let $\{A_i | i \in I\} \subseteq \gamma(X)$ be a family of generalized Scott closed sets such that $K \subseteq \bigvee_{i \in I} A_i$. Since $\bigvee_{i\in I} A_i = cl_{\sigma_e}(\bigcup_{i \in I} A_i)$, it follows that $K \subseteq \bigcup_{i\in I} A_i$. Assume that there exists $x \in K$ such that $x \notin \bigvee_{i \in I}(K \cap A_i)$. Note that since $\bigvee_{i\in I}(K \cap A_i) = cl_{\sigma_e}(\bigcup_{i\in I}(K \cap A_i))$, we can find a generalized Scott open neighborhood U of x such that $U \cap \bigcup_{i \in I}(K \cap A_i) = U \cap K \cap \bigcup_{i\in I} A_i = \emptyset$. By Theorem 3.7, $\uparrow_\top^o(U \cap \downarrow_\top x) \in \sigma_e(X)$. Since $x \in \uparrow_\top^o(U \cap \downarrow_\top x)$ and $x \in K \subseteq cl_\sigma(\bigcup_{i\in I} A_i)$, we have $(\bigcup_{i\in I} A_i) \cap \uparrow_\top^o(U \cap \downarrow_\top x) \neq \emptyset$, which implies that there exist $y \in X$ and $i_0 \in I$ such that $y \in A_{i_0} \cap \uparrow_\top^o(U \cap \downarrow_\top x)$. It means that there exists $z \in U$ such that $z \in \downarrow_\top x \cap \downarrow_\top y$. Note that since both K and A_{i_0} are generalized Scott closed sets, it follows that $\downarrow_\top x \subseteq K$ and $\downarrow_\top y \subseteq A_{i_0}$ by Lemma 3.18. Then we have $z \in U \cap K \cap A_{i_0} \subseteq U \cap K \cap \bigcup_{i\in I} A_i$. It is a contradiction. Hence, $K = \bigvee_{i\in I}(K \cap A_i)$.

This implies that $\gamma_e(X)$ is a meet-continuous lattice. Thus, $\gamma_e(X)$ is a complete Heyting algebra. $\square$

4. **Topological Invariance and Heredity of Meet-continuous L-dcpos**

In this section, we discuss the topological invariance of meet-continuity with respect to generalized Scott topology and its heredity with respect to generalized Scott closed subsets.

Let (X, e) be an L-dcpo and let $A \subseteq X$. For every $\varphi \in L^A$, we define $\widetilde{\varphi} \in L^X$

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & x \in A \\ \perp & x \in X - A. \end{cases} \tag{3}$$

Lemma 4.1. *Let (X, e) be an L-dcpo and let $A \subseteq X$. If $\varphi \in D_L(A)$, then $\widetilde{\varphi} \in D_L(X)$ and $\sqcup\varphi = \sqcup\widetilde{\varphi}$.*

Proof. Firstly, we show $\widetilde{\varphi} \in D_L(X)$.

(i) By the definition of $\widetilde{\varphi}$ and $\varphi \in D_L(A)$, there exists $x \in X$ such that $\perp \lll \widetilde{\varphi}(x)$.

(ii) For every $x_1, x_2 \in X$ and every $a_1, a_2, a \in L$ such that $a_1 \lll \widetilde{\varphi}(x_1)$, $a_2 \lll \widetilde{\varphi}(x_2)$ and $a \lll \top$, we have $x_1, x_2 \in A$. Otherwise, if $x_1 \in A'$, then $\widetilde{\varphi}(x_1) = \perp \not\ggg a$. It is a contradiction. Note that since $\varphi \in D_L(A)$, there exists $x \in A \subseteq X$ such that $a \lll \varphi(x) = \widetilde{\varphi}(x)$, $a_1 \lll e(x_1, x)$ and $a_2 \lll e(x_2, x)$.

We further show that $\sqcup\varphi = \sqcup\widetilde{\varphi}$. For every $y \in X$, $e(\sqcup\widetilde{\varphi}, y) = \bigwedge_{x\in X} \widetilde{\varphi}(x) \to e(x, y) = \bigwedge_{x\in A} \widetilde{\varphi}(x) \to e(x, y) = \bigwedge_{x\in A} \varphi(x) \to e(x, y)$, i.e., $\sqcup\varphi = \sqcup\widetilde{\varphi}$. $\square$

Proposition 4.2. *Let (X, e) be an L-dcpo. If $A \neq \emptyset$ is a generalized Scott closed set, then (A, e) is also an L-dcpo. Moreover, a subset of A is generalized Scott closed in A iff it is generalized Scott closed in X.*

Proof. Firstly, we show that (A, e) is an L-dcpo.

For every $\varphi \in D_L(A)$, by Lemma 4.1, we know that $\widetilde{\varphi} \in D_L(X)$ and $\sqcup\varphi = \sqcup\widetilde{\varphi}$. Now we show $\sqcup\varphi \in A$. Suppose that $\sqcup\varphi \notin A$, i.e., $\sqcup\varphi \in X \setminus A$. Then there exist $a \lll \top$ and $y \in X$ such that $a \lll \widetilde{\varphi}(y)$ and $\uparrow_a y \subseteq X \setminus A$. This implies $y \in X \setminus A$, i.e., $\widetilde{\varphi}(y) = \perp \not\ggg a$. It is a contradiction.

Secondly, we show that for every $B \subseteq A$, if $B \in \gamma_e(X)$, then $B \in \gamma_e(A)$.

Let $\varphi \in D_L(A)$ which satisfies $a \not\lll \varphi(y)$ or $\uparrow_a x \cap B \neq \emptyset$ for every $a \lll \top$ and every $y \in A$. Then we have $\varphi(y) = \widetilde{\varphi}(y)$ and $\widetilde{\varphi} \in D_L(X)$ by Lemma 4.1. Furthermore, since $B \in \gamma_e(X)$, we have $\sqcup\varphi = \sqcup\widetilde{\varphi} \in B$ by Proposition 2.12 and Lemma 4.1. Therefore, we have $B \in \gamma_e(A)$ again by Proposition 2.12.

Finally, we show that for every $B \subseteq A$, if $B \in \gamma_e(A)$, then $B \in \gamma_e(X)$.

Suppose $B \notin \gamma_e(X)$. By Proposition 2.12. there exists $\varphi \in D_L(X)$ such that for every $a \lll \top$ and every $x \in X$, $a \not\lll \varphi(x)$ or $\uparrow_a x \cap B \neq \emptyset$, but $\sqcup\varphi \notin B$. Then we have $\sqcup\varphi \in X \setminus A \in \sigma_e(X)$ or $\sqcup\varphi \in A \setminus B \in \sigma_e(A)$. If $\sqcup\varphi \in X \setminus A \in \sigma_e(X)$, then there exist $z \in X$ and $b \lll \top$ such that $b \lll \varphi(z)$ and $\uparrow_b z \subseteq X \setminus A$. This implies that there exist $b \lll \top$ and $z \in X$ such that $b \lll \varphi(z)$ and $\uparrow_b z \cap B = \emptyset$, which is a contradiction. Similarly, we can prove that the case $\sqcup\varphi \in A \setminus B \in \sigma_e(A)$. $\square$

By Proposition 4.2, we have the following theorem.

Theorem 4.3. *Let (X,e) be an L-dcpo. Then for every generalized Scott closed subset A, the relative generalized Scott topology on A agrees with the generalized Scott topology of the L-dcpo (A,e).*

Lemma 4.4. *Let (X,e) be an L-dcpo and let A be a non-empty generalized Scott closed set. If $x \in A$ and $\varphi \in D_L(A)$, then $cl_{\sigma_e}(\delta_a(\downarrow\widetilde{\varphi})\cap\downarrow_a x) = cl_{\sigma_e(A)}(\delta_a(\downarrow\varphi)\cap\downarrow_a x)$, where $a \in L$.*

Proof. Straightforward. □

Proposition 4.5. *Let $a \in L$, let (X,e) be an L_a-meet-continuous L-dcpo, and let A be a non-empty generalized Scott closed set. Then (A,e) is also an L_a-meet-continuous L-dcpo. In particular, $\downarrow_b x$ is an L_a-meet-continuous L-dcpo for every $x \in X$ and every $b \lll \top$.*

Proof. Let $\varphi \in D_L(A)$ with $a \lll e(x,\sqcup\varphi)$. From Lemma 4.1, we have $\sqcup\varphi = \sqcup\widetilde{\varphi}$ and $\sqcup\varphi \in A$. Then, we have $x \in cl_{\sigma_e}(\delta_a(\widetilde{\varphi})\cap\downarrow_a x) = cl_{\sigma_e(A)}(\delta_a(\varphi)\cap\downarrow_a x)$ by Lemma 4.4 and the L_a-meet-continuity of (X,e). Therefore, (A,e) is an L_a-meet-continuous L-dcpo by Definition 3.1. □

Corollary 4.6. *Let (X,e) be a meet-continuous L-dcpo and let A be a non-empty generalized Scott closed set. Then (A,e) is also a meet-continuous L-dcpo. In particular, $\downarrow_a x$ is a meet-continuous L-dcpo for every $x \in X$ and every $a \lll \top$.*

Lemma 4.7. *Let (X,e_X), (Y,e_Y) be L-dcpos and let $\sigma_e(X),\sigma_e(Y)$ be generalized Scott topologies of (X,e_X), (Y,e_Y), respectively. If $f: X \longrightarrow Y$ is a homeomorphism w.r.t. generalized Scott topology, then for every $x_1,x_2 \in X$, $e_X(x_1,x_2) = e_Y(f(x_1),f(x_2))$.*

Proof. Straightforward. □

Theorem 4.8. *Let (X,e_X), (Y,e_Y) be L-dcpos and let $(X,\sigma_e(X)) \cong (Y,\sigma_e(Y))$, where $\sigma_e(X),\sigma_e(Y)$ are generalized Scott topologies of (X,e_X), (Y,e_Y), respectively. If (X,e_X) is meet-continuous, then (Y,e_Y) is also meet-continuous.*

Proof. Let $a \lll \top$, $y \in Y$ and $\sqcup\varphi \in D_L(Y)$ with $a \lll e_Y(y,\sqcup\varphi)$. We show that for every $U \in \sigma_e(Y)$ with $y \in U$, $U\cap\downarrow_a y\cap\delta_a(\downarrow\varphi) \neq \emptyset$. Since $(X,\sigma_e(X)) \cong (Y,\sigma_e(Y))$, there exists a homeomorphism $f: X \longrightarrow Y$. Then there exists $x \in X$ such that $y = f(x)$, $x \in f^{-1}(U) \in \sigma_e(X)$ and $\widetilde{f^{-1}}^{\rightarrow}(\varphi) \in I_L(X)$, which satisfies $f^{-1}(\sqcup\varphi) = \sqcup\widetilde{f^{-1}}^{\rightarrow}(\varphi)$. By Lemma 4.7, we have $a \lll e_Y(y,\sqcup\varphi) = e_Y(f(x), f(\sqcup\widetilde{f^{-1}}^{\rightarrow}(\varphi))) = e_X(x,\sqcup\widetilde{f^{-1}}^{\rightarrow}(\varphi))$. Furthermore, since (X,e_X) is meet-continuous, it follows that $f^{-1}(U)\cap\downarrow_a x\cap\delta_a(\downarrow\widetilde{f^{-1}}^{\rightarrow}(\varphi)) = f^{-1}(U)\cap\downarrow_a x\cap\delta_a(\widetilde{f^{-1}}^{\rightarrow}(\varphi)) \neq \emptyset$, i.e., there exists $x_0 \in X$ such that $x_0 \in f^{-1}(U)$, $a \leq e_X(x_0,x)$ and $a \lll \widetilde{f^{-1}}^{\rightarrow}(\varphi)(x_0)$. Note that $\widetilde{f^{-1}}^{\rightarrow}(\varphi)(x_0) = \bigvee_{y'\in Y}(\varphi(y') \wedge e_X(x_0, f^{-1}(y')) = \bigvee_{y'\in Y}(\varphi(y') \wedge e_Y(f(x_0),y') = \downarrow\varphi(f(x_0))$, so we have $a \lll \downarrow\varphi(f(x_0))$. Hence, $f(x_0) \in U$ and $a \leq e_Y(f(x_0),y) = e_Y(f(x_0),f(x)) = e_X(x_0,x)$ by the continuity of f and Lemma 4.7. To sum up the above, we have $U\cap\downarrow_a y\cap\delta_a(\downarrow\varphi) \neq \emptyset$. Therefore, (Y,e_Y) is L_a-meet-continuous, and hence, (Y,e_Y) is meet-continuous by the arbitrariness of a. □

5. Conclusions and Further Work

Taking frames as the structure of truth values, we proposed the notion of meet-continuity on L-dcpos, based on the generalized Scott topology. Moreover, we showed that every continuous L-dcpo defined in [19] is meet-continuous, and also that L-continuous retracts of meet-continuous L-dcpos are meet-continuous. Additionally, we introduced the notions of generalized lower topology and generalized Lawson topology. As a result of our study, it follows that meet-continuity forces closer relationships between the generalized Scott and generalized Lawson topologies. Finally, we investigated some topological properties of meet-continuity on L-dcpos. In particular, it is shown that meet-continuity is a topological invariant and is a hereditary property.

Further Work: Similar to Flagg's logical approach to quantitative domain theory in [4], Zhang and Fan [19] defined L-dcpos and built the generalized Scott topology (a crisp topology) on L-dcpos. As shown in [4], this kind of definition of dcpos and Scott topology is appropriate for quantitative domain theory. Moreover, many nice results have been obtained in [19, 21, 22], for example, the generalized Scott topology is not only a generalization of Scott topology on ordinary domain, but also keeps a lot of good properties of Scott topology. Therefore, this paper carried L-dcpos and the generalized Scott topology as its fundamental definitions and concepts. On the other hand, Yao and Shi [13,14,17] redefined L-dcpos and proposed the notion of L-Scott topology. The L-Scott topology is a stratified L-topology and has some advantage. Naturally, one may be able to construct a meet-continuity of L-dcpos by L-Scott topology just like the meet-continuity of crisp dcops by crisp Scott topology.

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