(IC)LM-FUZZY TOPOLOGICAL SPACES

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Abstract. The aim of the present paper is to define and study (IC)LM-fuzzy topological spaces, a generalization of (weakly) induced LM-fuzzy topological spaces. We discuss the basic properties of (IC)LM-fuzzy topological spaces, and introduce the notions of interior (IC)-fication and exterior (IC)-fication of LM-fuzzy topologies and prove that **ICLM-FTop** (the category of (IC)LM-fuzzy topological spaces) is an isomorphism-closed full proper subcategory of **LM-FTop** (the category of LM-fuzzy topological spaces) and **ICLM-FTop** is a simultaneously bireflective and bicoreflective full subcategory of **LM-FTop**.

1. Introduction

Since Chang [2]introduced fuzzy theory in topology, many authors discussed various aspects of fuzzy topology. However, in a completely different direction, Höhle [6] created the notion of a topology being viewed as an L-subset of a powerset (in his case, 2^X). Then Kubiak [8] and Šostak [17] independently extended Höhle's notion to L-subsets of L^X.

In 2007, Yue studied induced and weakly induced LM-fuzzy topological spaces, and proved that **WILM-FTop** (the category of weakly induced LM-fuzzy topological spaces and continuous mappings) is a simultaneously reflective and coreflective full subcategory of **LM-FTop** (the category of LM-fuzzy topological spaces and continuous mappings) [21]. In 2011, Yao showed that, for L a frame, the category of enriched L-fuzzy topological spaces can be embedded in that of L-generalized convergence spaces as a reflective subcategory and the latter is a cartesian- closed topological category [20]. Moreover, in 2007, Li, Li and Fu defined and studied the properties of (IC)L-cotopological spaces [10], which is the generalization of Martin's weakly induced I-topological spaces [13] (particularly, Weiss's induced spaces [19] and Lowen's topologically generated spaces [11]), and showed that **ICLTop**^{c} (the category of (IC)L-cotopological spaces and continuous mappings) is simultaneously bireflective and bicoreflective in **LTop**^{c} (the category of L-cotopological spaces and continuous mappings), which implies that (IC)L-cotopological space has the similar character with weakly induced spaces. The aim of the present paper is to define and study (IC)LM-fuzzy topological spaces, a generalization of weakly induced

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and induced LM-fuzzy topological spaces. We discuss the basic properties of LM-fuzzy topological spaces, and introduce the notions of interior (IC)-fication and exterior (IC)-fication of LM-fuzzy topological spaces and prove that **ICLM-FTop** (the category of (IC)LM-fuzzy topological spaces and continuous mappings) is an isomorphism-closed full proper subcategory of **LM-FTop** and **ICLM-FTop** is a simultaneously bireflective and bicoreflective full subcategory of **LM-FTop**.

For needed categorical notions, please refer to [1, 7, 12, 15].

2. **Preliminaries**

We now give some definitions and results to be used in this paper. Throughout this paper, L always stands for a completely distributive lattice with locally multiplicative property and M is a completely distributive lattice with the least element 0 and the greatest element 1 ($0 \neq 1$). Apparently, the powerset L^X (i.e. the set of all L-subset) with the point-wise order is also a completely distributive lattice with locally multiplicative property, in which the least element and the greatest element of L^X will be written as 0_X and 1_X. An element $r \in L - \{0\}$ is called a co-prime element if, for any finite subset $K \subset L$ satisfying $r \leq \bigvee K$ (the supremum of K), there exists a $k \in K$ such that $r \leq k$. An element $s \in L - \{1\}$ is called a prime element if it is a co-prime element of L^{op} (the opposite lattice [4] of L). The set of all co-prime elements (resp., prime elements) of L will be written as $Copr(L)$ (resp., $Pr(L)$). We say a is way below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$), if for every directed (arbitrary) subset $D \subset L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. From [5], we know that $Copr(L)$ is a join-generating set of L if L is a completely distributive lattice. Hence every element in L is also the supremum of all the coprimes wedge below it. By the definition of completely distributive lattice it is easy to see that a complete lattice L is completely distributive iff the operator $\bigvee : Low(L) \to L$ taking every lower set to its supremum has a left adjoint β, and in the case, $\beta(a) = \{b \mid b \lhd a\}$ for all $a \in L$. Hence the wedge below relation has the interpolation property in a completely distributive lattice, that is, $a \lhd b$ implies there is some $c \in L$ such that $a \lhd c \lhd b$.

The way-below relation on a completely distributive lattice L is called locally multiplicative [22] if for every coprime $a \in Copr(L)$, $a \ll b$ and $a \ll c$ implies $a \ll b \wedge c$ for all $b,\ c \in L$. If a is a coprime, then $a \ll b$ if and only if $a \lhd b$ (see [5, 22]). Hence L is locally multiplicative if for every coprime a, $a \lhd b$ and $a \lhd c$ implies $a \lhd b \wedge c$ for all $b,\ c \in L$. Clearly, [0,1] is locally multiplicative.

For $A \in L^X$ and $p \in L$, let $\hat{\iota}_p(A) = \{x \mid A(x) \lhd p\}$, called the strong p-cut of A. Let 1_U denote the characteristic function of $U \in 2^X$, and $[a]$ denote the L-subset taking constant value a, where 2^X is the power set of X and $a \in L$. Each mapping $f : X \longrightarrow Y$ induces a mapping $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called L-forward powerset operator, cf. [14]), defined by

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) \mid f(x) = y\} \quad (\forall A \in L^X, \forall y \in Y).$$

The right adjoint to $f_L^{\rightarrow}$ (called L-backward powerset operator, cf. [14]) is denoted as $f_L^{\leftarrow}$ and given by

$$f_L^{\leftarrow}(B) = \bigvee\{A \in L^X \mid f_L^{\rightarrow}(A) \le B\} = B \circ f \quad (\forall B \in L^Y).$$

It is known that $f_L^{\rightarrow}$ preserves arbitrary unions and that $f_L^{\leftarrow}$ preserves arbitrary unions, arbitrary intersections, and complements when they exists (canonical examples of such morphisms are given in [14]). Let L be a locally multiplicative completely distributive lattice. It is easy to verify the following

Lemma 2.1. [9, 21] *(1) For each $p \in L$, $\hat{\iota}_p : L^X \to 2^X = \{E \mid E \subset X\}$ preserves arbitrary suprema and finite meets.*

(2) For each $p \in L$, $A \in L^X$, $B \in L^Y$ and any mapping $f : X \to Y$, we have $\hat{\iota}_p(f_L^{\rightarrow}(A)) = f^{\rightarrow}(\hat{\iota}_p(A))$, $\hat{\iota}_p(f_L^{\leftarrow}(B)) = f^{\leftarrow}(\hat{\iota}_p(B))$.

(3) $A = \bigvee_{p\in L}([p] \wedge 1_{\hat{\iota}_p(A)}) = \bigvee_{p\in Copr(L)}([p] \wedge 1_{\hat{\iota}_p(A)})$.

An LM-fuzzy topology (or a Kubiak-Šostak fuzzy topology [8, 17]) on a set X is defined to be a mapping $\tau : L^X \longrightarrow M$ satisfying:

(**FT1**) $\tau(1_X) = \tau(0_X) = 1$;

(**FT2**) $\forall A,\ B \in L^X,\ \tau(A \wedge B) \ge \tau(A) \wedge \tau(B)$;

(**FT3**) $\tau(\bigvee_{t\in T} A_t) \ge \bigwedge_{t\in T} \tau(A_t)$, for every family $\{A_t \mid t \in T\} \subset L^X$.

The value $\tau(A)$ can be interpreted as the degree of openness of A. A continuous mapping between LM-fuzzy topological spaces is a mapping $f : (X,\tau) \longrightarrow (Y,\delta)$ such that $\delta(B) \le \tau(f_L^{\leftarrow}(B))$ for all $B \in L^Y$, where $f_L^{\leftarrow}(B)$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$([8,16-17]). When $L = \{0,1\}$, this definition reduced to that of M-fuzziying topology. Let **LM-FTop** denote the category of LM-fuzzy topological spaces and continuous mappings.

In the following of this section, we give some definitions and results about LM-fuzzy topology.

Definition 2.2. [3] Let τ be an LM-fuzzy topology on X.

(1) $\mathcal{B} : L^X \longrightarrow M$ is called a base of τ if $\mathcal{B}$ satisfies the following condition:

$$\forall A \in L^X,\ \tau(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A}\ \bigwedge_{\lambda\in\Lambda} \mathcal{B}(B_\lambda),$$

where the expression $\bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A}\ \bigwedge_{\lambda\in\Lambda} \mathcal{B}(B_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(A)$.

(2) $\phi : L^X \longrightarrow M$ is called a subbase of τ if $\phi^{(\sqcap)} : L^X \longrightarrow M$ is a base of τ, where $\phi^{(\sqcap)}(A) = \bigvee_{(\sqcap)_{\lambda\in J} B_\lambda = A}\ \bigwedge_{\lambda\in J} \phi(B_\lambda)$ for all $A \in L^X$ with $(\sqcap)$ standing for "finite intersection".

Lemma 2.3. [3] *Let τ be an LM-fuzzy topology on X. Then $\phi : L^X \longrightarrow M$ is the subbase of τ if and only if $\phi^{(\sqcup)}(1_X) = 1$.*

Definition 2.4. [21] Let (X,τ) be an LM-fuzzy topological space. If $\tau(A) = \bigwedge_{r\in Copr(L)} \tau(1_{\hat{\iota}_r(A)})$ holds for all $A \in L^X$, then (X,τ) is called an induced LM-fuzzy topological space. Let **ILM-FTop** denote the category of induced LM-fuzzy topological spaces and continuous mappings.

Definition 2.5. [21] Let (X,τ) be an LM-fuzzy topological space. If $\tau(A) \leq \bigwedge_{r\in Copr(L)} \tau(1_{\hat{\iota}_r(A)})$ holds for all $A \in L^X$, then (X,τ) is called a weakly induced LM-fuzzy topological space. Let **WILM-FTop** denote the category of weakly induced LM-fuzzy topological spaces and continuous mappings.

Definition 2.6. [16, 18] Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces and $p_t : \prod_{t\in T} X_t \longrightarrow X_t$ be the projection. Then the LM-fuzzy topology on $\prod_{t\in T} X_t$ whose subbase is defined by

$$\forall A \in L^{\prod_{t\in T} X_t},\ \phi(A) = \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(B)$$

is called the product LM-fuzzy topology of $\{\tau_t\}_{t\in T}$, denoted by $\prod_{t\in T}\tau_t$, and $(\prod_{t\in T} X_t, \prod_{t\in T}\tau_t)$ is called the product space of $\{(X_t,\tau_t)\}_{t\in T}$.

Definition 2.7. [16] Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different $X_t's$ be disjoint and $X = \bigcup_{t\in T} X_t$, and let $\tau : L^X \longrightarrow M$ be defined as follows:

$$\forall A \in L^X,\ \tau(A) = \bigwedge_{t\in T} \tau_t(A|X_t)$$

Then it is easy to verify that τ is an LM-fuzzy topology on X, and τ is called the sum LM-fuzzy topology of $\{\tau_t\}_{t\in T}$, denoted by $\bigoplus_{t\in T}\tau_t$. $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T}\tau_t)$ is called the sum space of $\{(X_t,\tau_t)\}_{t\in T}$.

Definition 2.8. [21] Let (X,τ) be an LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. It is easy to verify that $\tau/f_L^{\rightarrow} : L^Y \longrightarrow M$ is an LM-fuzzy topology on Y, where $\tau/f_L^{\rightarrow}$ is defined by

$$\forall A \in L^Y,\ \tau/f_L^{\rightarrow}(A) = \tau(f_L^{\leftarrow}(A)).$$

$\tau/f_L^{\rightarrow}$ is called the LM-fuzzy quotient topology of τ with respect to f, and $(Y, \tau/f_L^{\rightarrow})$ is called the LM-fuzzy quotient space of (X,τ) with respect to f.

Definition 2.9. [16, 18] Let (X,τ) be an LM-fuzzy topological space and $Y \subset X$. We call $(Y,\tau|Y)$ a subspace of (X,τ), where $\tau|Y : L^Y \longrightarrow M$ is defined by

$$\forall B \in L^Y,\ \tau|Y(B) = \bigvee\{\tau(A) \mid A \in L^X,\ A|Y = B\}.$$

Lemma 2.10. [21] *Let $f : (X,\tau) \longrightarrow (Y,\delta)$ be a mapping and ϕ be a subbase of δ. If $\tau(f_L^{\leftarrow}(B)) \geq \phi(B)$ for all $B \in L^Y$, then f is continuous.*

3. (IC)LM-fuzzy Topological Spaces

Definition 3.1. An LM-fuzzy topological space (X,τ) is said to be an (IC)LM-fuzzy topological space iff for all $A \in L^X,\ \tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$.

It is easy to verify that, by Definition 2.4, Definition 2.5 and Definition 3.1, the induced and weakly induced LM-fuzzy topological spaces are the (IC)LM-fuzzy topological space. However, the following example shows that an (IC)LM-fuzzy topological space needn't be a weakly induced LM-fuzzy topological space.

Example 3.2. Let $L = M$ be the locally multiplicative completely distributive lattices. We consider the map $\tau : L^X \longrightarrow M$ defined by

$$\tau(A) = (\bigvee_{x\in X} A(x)) \to ((\bigwedge_{x\in X} A(x)).$$

Then τ is an (IC)LM-fuzzy topology on X.

Proof. Firstly, τ is an LM-fuzzy topology on X (See [7],Example 4.2.1(a)). Secondly, we prove that τ is an (IC)LM-fuzzy topology on X. It suffices to show that for all $A \in L^X,\ \tau(A) \le \tau(1_{\hat{\iota}_0(A)})$. Let us suppose that $A \ne 0_X$ (It is easy to prove the conclusion if $A = 0_X$). If there exists an $x \in X$ such that $A(x) = 0$, then $\tau(A) = 0 = \tau(1_{\hat{\iota}_0(A)})$; if $A(x) \ne 0$ for all $x \in X$, then $\tau(1_{\hat{\iota}_0(A)}) = 1$. Both of them imply that $\tau(A) \le \tau(1_{\hat{\iota}_0(A)})$ $(\forall A \in L^X)$. Then τ is an (IC)LM-fuzzy topology on X. □

Generally, the τ defined above needn't be a weakly induced LM-fuzzy topology. For example, let $X = \{x_1, x_2\}$ and $L = M = \{0, a, b, 1\}$ be the Diamond lattices. Then τ defined above is an (IC)LM-fuzzy topology on X, but it is not a weakly induced LM-fuzzy topology. In fact, let

$$A(x) = \begin{cases} a, x = x_1 \\ 1, x = x_2 \end{cases}$$

Then $\tau(A) = 1 \to a = a$ and $\tau(1_{\hat{\iota}_b(A)}) = 1 \to 0 = 0$. Hence, $\tau(A) \nleq \bigwedge_{r\in Copr(L)} \tau(1_{\hat{\iota}_r(A)})$, which implies that τ is not a weakly induced LM-fuzzy topology.

Theorem 3.3. *Let (X, τ) be an (IC)LM-fuzzy topological space and $Y \subset X$. Then the subspace $(Y, \tau|Y)$ is also an (IC)LM-fuzzy topological space.*

Proof. For all $A \in L^Y$, we have $\tau|Y(A) = \bigvee\{\tau(B) \mid B \in L^X,\ B|Y = A\} \le \bigvee\{\tau(1_{\hat{\iota}_0(B)}) \mid B \in L^X,\ 1_{\hat{\iota}_0(B)}|Y = 1_{\hat{\iota}_0(A)}\} \le \tau|Y(1_{\hat{\iota}_0(A)})$, which implies that $(Y, \tau|Y)$ is an (IC)LM-fuzzy topological space. □

Theorem 3.4. *Let (X, τ) be an (IC)LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. Then the LM-fuzzy quotient space $(Y, \tau/f_L^{\rightarrow})$ of (X, τ) with respect to f is an (IC)LM-fuzzy topological space.*

Proof. For all $A \in L^Y$, we have $\tau/f_L^{\rightarrow}(A) = \tau(f_L^{\leftarrow}(A)) \le \tau(1_{\hat{\iota}_0(f_L^{\leftarrow}(A))})$
$= \tau(f_L^{\leftarrow}(1_{\hat{\iota}_0(A)})) = \tau/f_L^{\rightarrow}(1_{\hat{\iota}_0(A)}))$, which implies that $(Y, \tau/f_L^{\rightarrow})$ is an (IC)LM-fuzzy topological space. □

Theorem 3.5. *Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of (IC)LM-fuzzy topological spaces. Then the product space $(\prod_{t\in T} X_t, \prod_{t\in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t\in T}$ is an (IC)LM-fuzzy topological space.*

Proof. Let ϕ be the subbase of τ, which is given in Definition 2.6. Then, for all $A \in L^X$, we have

$$\phi(A) = \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(B)$$

$$\leq \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(1_{\hat{\iota}_0(B)})$$

$$\leq \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(C)=1_{\hat{\iota}_0(A)}} \tau_t(C)$$

$$= \phi(1_{\hat{\iota}_0(A)}).$$

Hence, for all $A \in L^X,\ \tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$, which implies that $(\prod_{t\in T} X_t, \prod_{t\in T} \tau_t)$ is an (IC)LM-fuzzy topological space. □

Theorem 3.6. *Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of (IC)LM-fuzzy topological spaces, different X_t's be disjiont. Then the sum space $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t\in T}$ is an (IC)LM-fuzzy topological space iff for all $t \in T$, (X_t, τ_t) is an (IC)LM-fuzzy topological space.*

Proof. **Necessity** For every $t \in T,\ A_t \in L^{X_t}$, we have

$$\tau_t(A_t) = \bigwedge_{t\in T} \tau_t(A^*|X_t) = \tau(A^*) \leq \tau(1_{\hat{\iota}_0(A^*)}) = \bigwedge_{t\in T} \tau_t(1_{\hat{\iota}_0(A^*)}|X_t) = \tau_t(1_{\hat{\iota}_0(A)}),$$

where

$$A^*(x) = \begin{cases} A(x),\ x \in X_t \\ 0,\ x \notin X_t \end{cases}$$

Thus for every $t \in T,\ (X_t, \tau_t)$ is an (IC)LM-fuzzy topological space.

Sufficiency For every $A \in L^X$, we have

$$\tau(A) = \bigwedge_{t\in T} \tau_t(A|X_t) \leq \bigwedge_{t\in T} \tau_t(1_{\hat{\iota}_0(A|X_t)}) = \bigwedge_{t\in T} \tau_t(1_{\hat{\iota}_0(A)}|X_t) = \tau(1_{\hat{\iota}_0(A)}).$$

Hence $(\bigoplus_{t\in T} X_t,\ \bigoplus_{t\in T} \tau_t)$ is an (IC)LM-fuzzy topological space. □

4. (IC)-fication of LM-fuzzy Topology

Theorem 4.1. *Let (X, τ) be an LM-fuzzy topological space and $I(\tau) : L^X \longrightarrow M$ be defined by $I(\tau)(A) = \tau(A) \wedge \tau(1_{\hat{\iota}_0(A)})$ $(\forall A \in L^X)$. Then $I(\tau)$ is the largest (IC)LM-fuzzy topology on X which is contained in τ. We call $I(\tau)$ the interior (IC)-fication of τ.*

Proof. For every $A \in L^X$, we have $I(\tau)(1_{\hat{\iota}_0(A)}) = \tau(1_{\hat{\iota}_0(A)}) \geq \tau(A) \wedge \tau(1_{\hat{\iota}_0(A)}) = I(\tau)(A)$, which implies that $I(\tau)$ is the (IC)LM-fuzzy topology on X which is contained in τ. Furthermore, let $\delta \leq \tau$ and δ be an (IC)LM-fuzzy topology. Then for all $A \in L^X,\ \delta(A) \leq \delta(1_{\hat{\iota}_0(A)}) \leq \tau(1_{\hat{\iota}_0(A)})$, and thus $\delta(A) \leq \tau(A) \wedge \tau(1_{\hat{\iota}_0(A)}) = I(\tau)(A)$, i.e., $\delta \leq I(\tau)$. Hence $I(\tau)$ is the largest (IC)LM-fuzzy topology on X which is contained in τ.

Let (X, τ) be an LM-fuzzy topological space and $\phi^\tau : L^X \longrightarrow M$ be defined by

$$\phi^\tau(A) = \begin{cases} \bigvee\{\tau(B) \mid \hat{\iota}_0(B) = A\}, & A \text{ is a characteristic function, i.e., the range of } A \text{ is } \{0,1\} \\ \tau(A), & \text{otherwise} \end{cases}$$

It is easy to verify that ϕ^τ is a subbase of one LM-fuzzy topology. We denote this LM-fuzzy topology by $E(\tau)$ and call it the exterior (IC)-fication of τ (see Theorem 4.2). □

Theorem 4.2. *For an LM-fuzzy topology τ on X, $E(\tau)$ is the smallest (IC)LM-fuzzy topology on X which contains τ.*

Proof. Firstly, we show that $E(\tau)$ is the (IC)LM-fuzzy topology on X. In fact, for every $A \in L^X$, we have

$$E(\tau)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta}=B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(C_{\lambda\beta}),$$

and

$$E(\tau)(1_{\hat{\iota}_0(A)}) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = 1_{\hat{\iota}_0(A)}} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta}=B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(C_{\lambda\beta})$$

$$\geq \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta}=B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(1_{\hat{\iota}_0(C_{\lambda\beta})}).$$

We only need to consider the case that $C_{\lambda\beta}$ is not a characteristic function. Since

$$\phi^\tau(1_{\hat{\iota}_0(C_{\lambda\beta})}) = \bigvee\{\tau(B) \mid \hat{\iota}_0(B) = \hat{\iota}_0(C_{\lambda\beta})\} \geq \tau(C_{\lambda\beta}) = \phi^\tau(C_{\lambda\beta}),$$

we have

$$E(\tau)(A) \leq E(\tau)(1_{\hat{\iota}_0(A)}),$$

which implies $E(\tau)$is the (IC)LM-fuzzy topology on X which contains τ.

Secondly, we prove that $E(\tau)$ is the smallest (IC)LM-fuzzy topology on X which contains τ. Let $\tau \leq \eta$ and η be an (IC)LM-fuzzy topology on X. We need to prove that $E(\tau) \leq \eta$. It suffices to show that $\phi^\tau(A) \leq \eta(A)$ for all $A \in L^X$. If A is a characteristic function, then

$$\phi^\tau(A) = \bigvee\{\tau(B) \mid \hat{\iota}_0(B) = A\} \leq \bigvee\{\eta(B) \mid \hat{\iota}_0(B) = A\}$$

$$\leq \bigvee\{\eta(1_{\hat{\iota}_0(B)}) \mid \hat{\iota}_0(B) = A\} = \eta(A).$$

It is clearly that $\phi^\tau(A) \leq \eta(A)$ when A is not a characteristic function. Thus $E(\tau) \leq \eta$. Therefore $E(\tau)$ is the smallest (IC)LM-fuzzy topology on X which contains τ. □

Corollary 4.3. *(X, δ) is an (IC)LM-fuzzy topological space iff any two of $E(\delta)$, $I(\delta)$, δ are equal (equivalently, all the three are equal).*

Theorem 4.4. *(1) Let (X, δ) be an LM-fuzzy topological space and (Y, τ) be an (IC)LM-fuzzy topological space. Then $f : (X, \delta) \longrightarrow (Y, \tau)$ is a continuous mapping iff $f : (X, I(\delta)) \longrightarrow (Y, I(\tau)) = (Y, \tau)$ is.*

(2) Let (X, δ) be an LM-fuzzy topological space and (Y, τ) be an (IC)LM-fuzzy topological space. Then $f : (Y, \tau) \longrightarrow (X, \delta)$ is a continuous mapping iff $f : (Y, E(\tau)) = (Y, \tau) \longrightarrow (X, E(\delta))$ is.

Proof. (1) It suffices to show necessity. Suppose that $f : (X,\delta) \longrightarrow (Y,\tau)$ is continuous. So $\tau(B) \leq \delta(f_L^{\leftarrow}(B))$ for every $B \in L^Y$. Since (Y,τ) is an (IC)LM-fuzzy topological space, we have $\tau(B) \leq \tau(1_{\hat{\iota}_0(B)})$, and hence $\tau(B) \leq \delta(f_L^{\leftarrow}(B)) \wedge \tau(1_{\hat{\iota}_0(B)}) \leq \delta(f_L^{\leftarrow}(B)) \wedge \delta(f_L^{\leftarrow}(1_{\hat{\iota}_0(B)})) = \delta(f_L^{\leftarrow}(B)) \wedge \delta(1_{\hat{\iota}_0(f_L^{\leftarrow}(B))}) = I(\delta)(f_L^{\leftarrow}(B))$, which implies that $f : (X, I(\delta)) \longrightarrow (Y, I(\tau)) = (Y,\tau)$ is a continuous mapping.

(2) It suffices to show necessity. It is sufficient to show $\phi^{\delta}(A) \leq \tau(f_L^{\leftarrow}(A))$ for all $A = 1_U \in L^X$ $(\forall U \subset X)$ by the definition of $E(\delta)$ and Lemma 2.10. Since $f : (Y,\tau) \longrightarrow (X,\delta)$ is a continuous mapping, we have $\delta(A) \leq \tau(f_L^{\leftarrow}(A))$. It is also because (Y,τ) is an (IC)LM-fuzzy topological space, thus $\tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$, and hence $\phi^{\delta}(A) = \bigvee\{\delta(B) \mid \hat{\iota}_0(B) = U\} \leq \bigvee\{\tau(f_L^{\leftarrow}(B)) \mid \hat{\iota}_0(B) = U\} \leq \bigvee\{\tau(1_{\hat{\iota}_0(f_L^{\leftarrow}(B))}) \mid \hat{\iota}_0(B) = U\} = \tau(f_L^{\leftarrow}(A))$, which implies that $f : (Y, E(\tau)) = (Y,\tau) \longrightarrow (X, E(\delta))$ is a continuous mapping. □

Let **i** : **(IC)LM-FTop**$\longrightarrow$ **LM-FTop** be the inclusion functor. By Theorem 4.1, Theorem 4.2 and Theorem 4.4, we have

Theorem 4.5. **I**, **E** *:* **LM-FTop**$\longrightarrow$**(IC)LM-FTop** *are functors,* **I**$\dashv$**i** *and* **i**$\dashv$ **E**.

Corollary 4.6. **ICLM-FTop** *is an isomorphism-closed full proper subcategory of* **LM-FTop** *which is simultaneously bireflective and bicoreflective in* **LM-FTop**, *and given an LM-fuzzy topological space (X,δ), its reflection and coreflection are given by $id_X : (X,\delta) \longrightarrow (X, I(\delta))$ and $id_X : (X, E(\delta)) \longrightarrow (X,\delta)$ respectively, where $id_X : X \longrightarrow X$ is the identity mapping.*

As every right adjoint preserves limits and every left adjoint preserves colimits, we have the following Corollaries:

Corollary 4.7. *(1) Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces. Then $\left(\prod_{t\in T} X_t, E(\prod_{t\in T}\delta_t)\right) = \left(\prod_{t\in T} X_t, \prod_{t\in T} E(\delta_t)\right)$.*

(2) Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different X_t's be disjiont. Then $\left(\bigoplus_{t\in T} X_t, I(\bigoplus_{t\in T}\delta_t)\right) = \left(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} I(\delta_t)\right)$.

Corollary 4.8. *(1) Let (X,δ) be an LM-fuzzy topological space and $Y \subset X$. Then $E(\delta|Y) = E(\delta)|Y$.*

(2) Let (X,δ) be an LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. $(Y, \delta/f_L^{\rightarrow})$ is the LM-fuzzy quotient space of (X,δ) with respect to f. Then $I(\delta/f_L^{\rightarrow}) = I(\delta)/f_L^{\rightarrow}$.

Theorem 4.9. *(1) Let (X,τ) be an LM-fuzzy topological space and $U \subset X$. Then $I(\tau)|U \leq I(\tau|U)$.*

(2) Let (X,δ) be an LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. $(Y, \delta/f_L^{\rightarrow})$ is the LM-fuzzy quotient space of (X,δ) with respect to f. Then $E(\delta/f_L^{\rightarrow}) \leq E(\delta)/f_L^{\rightarrow}$.

Proof. (1) It is easy to verify that $I(\tau)|U \leq I(\tau|U)$ by Theorem 3.3 and the definition of $I(\delta)$.

(2) It is easy to verify that $E(\delta/f_L^{\rightarrow}) \le E(\delta)/f_L^{\rightarrow}$ by Theorem 3.4 and the definition of $E(\delta)$. □

Remark 4.10. The following two counterexamples show that the above inequality in Theorem 4.9 cannot be replaced by equalities.

(1) Let $X = L = M = [0,1], U = [0,0.5]$, $\tau : L^X \longrightarrow M$ is defined by

$$\tau(0_X) = \tau(1_X) = \tau([0.5] \wedge 1_U) = 1 \text{ and for others } A \in L^X, \tau(A) = 0.$$

It is easy to verify that $I(\tau) : L^X \longrightarrow M$ is defined by

$$I(\tau)(0_X) = I(\tau)(1_X) = 1 \text{ and for others } A \in L^X,\ I(\tau)(A) = 0,$$

and hence $I(\tau)|U : L^U \longrightarrow M$ is defined by

$$I(\tau)|U(0_U) = I(\tau)|U(1_U) = 1 \text{ and for others } B \in L^U,\ I(\tau)|U(B) = 0.$$

On the other hand, we have $\tau|U : L^U \longrightarrow M$ defined by

$$\tau|U(0_U) = \tau|U(1_U) = \tau([0.5] \wedge 1_U) = 1 \text{ and for others } B \in L^U,\ \tau|U(B) = 0,$$

and hence $I(\tau|U) = \tau|U$. Therefore, $I(\tau)|U \le I(\tau|U)$ and $I(\tau)|U \ne I(\tau|U)$.

(2) Let $X = [-1,1], Y = L = M = [0,1]$, $\delta : L^X \longrightarrow M$ is defined by

$$\delta(0_X) = \delta(1_X) = \delta([0.5] \wedge 1_{[0.5,1]}) = \delta(1_{[-1,-0.5]}) = 1$$
$$\text{and for others } A \in L^X, \tau(A) = 0.$$

It is easy to verify that $E(\delta) : L^X \longrightarrow M$ is defined by

$$E(\delta)(0_X) = E(\delta)(1_X) = E(\delta)([0.5] \wedge 1_{[0.5,1]}) = E(\delta)(1_{[0.5,1]}) = E(\delta)(1_{[-1,-0.5]})$$
$$= E(\delta)(1_{[-1,-0.5] \cup [0.5,1]}) = 1 \text{ and for others } A \in L^X, E(\delta)(A) = 0$$

and $E(\delta)/f_L^{\rightarrow} : L^Y \longrightarrow M$ is defined by

$$E(\delta)/f_L^{\rightarrow}(0_Y) = E(\delta)/f_L^{\rightarrow}(1_Y) = E(\delta)/f_L^{\rightarrow}(1_{[0.25,1]}) = 1$$
$$\text{and for others } B \in L^Y, E(\delta)/f_L^{\rightarrow}(B) = 0.$$

On the other hand, we have $\delta/f_L^{\rightarrow} : L^Y \longrightarrow M$ is defined by

$$\delta/f_L^{\rightarrow}(0_Y) = \delta/f_L^{\rightarrow}(1_Y) = 1 \text{ and for others } B \in L^Y, \delta/f_L^{\rightarrow}(B) = 0$$

and hence $E(\delta/f_L^{\rightarrow}) = \delta/f_L^{\rightarrow}$. Therefore, $E(\delta/f_L^{\rightarrow}) \le E(\delta)/f_L^{\rightarrow}$ and $E(\delta/f_L^{\rightarrow}) \ne E(\delta)/f_L^{\rightarrow}$.

Moreover, we also have

Theorem 4.11. *Let $\{(X_t,\ \delta_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different X_t's be disjioint. Then $\bigoplus_{t\in T} E(\delta_t) = E(\bigoplus_{t\in T} \delta_t)$.*

Proof. $E(\delta) \le \bigoplus_{t\in T} E(\delta_t)$ by Theorem 3.6 and the definition of $E(\delta)$. Conversely, let $\lambda \in Copr(L)$ and $\lambda \lhd \bigoplus_{t\in T} E(\delta_t)(A)$ $(\forall A \in L^X)$, i.e.,

$$\lambda \lhd \bigoplus_{t\in T} E(\delta_t)(A) = \bigwedge_{t\in T} E(\delta_t)(A|X_t)$$

$$= \bigwedge_{t\in T} \bigvee_{\bigvee_{\lambda\in\Lambda^t} D^t_\lambda = A|X_t} \bigwedge_{\lambda\in\Lambda^t} \bigvee_{(\sqcap)_{\beta\in\Lambda^t_\lambda} E^t_{\beta\lambda}=D^t_\lambda} \bigwedge_{\beta\in\Lambda^t_\lambda} \phi^{\delta_t}(E^t_{\beta\lambda}).$$

Then for all $t \in T$, there exists $\{D^t_\lambda\}_{\lambda\in\Lambda^t} \subset L^{X_t}$ such that

(i) $\bigvee_{\lambda\in\Lambda^t} D^t_\lambda = A|X_t$;

(ii) For each $\lambda \in \Lambda^t$, there exists $\{E^t_{\lambda\beta}\}_{\beta\in\Lambda^t_\lambda} \subset L^{X_t}$ such that $(\sqcap)_{\beta\in\Lambda^t_\lambda} E^t_{\beta\lambda} = D^t_\lambda$;

(iii) For each $\beta \in \Lambda^t_\lambda$, we have $\lambda \leq \phi^{\delta_t}(E^t_{\beta\lambda})$.

Let $(D^t_\lambda)^* \in L^X,\ (E^t_{\beta\lambda})^* \in L^X$ be defined as follows:

$$(D^t_\lambda)^*(x) = \begin{cases} D^t_\lambda(x),\ x \in X_t \\ 0,\ x \notin X_t \end{cases}$$

$$(E^t_{\beta\lambda})^*(x) = \begin{cases} E^t_{\beta\lambda}(x),\ x \in X_t \\ 0,\ x \notin X_t \end{cases}$$

Then we have

$$\bigvee_{t\in T} \bigvee_{\lambda\in\Lambda^t} (D^t_\lambda)^* = A, (\sqcap)_{\beta\in\Lambda^t_\lambda} (E^t_{\beta\lambda})^* = (D^t_\lambda)^* \text{ and } \phi^{\delta_t}(E^t_{\beta\lambda}) = \phi^{\bigoplus_{t\in T}\delta_t}((E^t_{\beta\lambda})^*).$$

Hence $\lambda \leq \phi^{\bigoplus_{t\in T}\delta_t}((E^t_{\beta\lambda})^*)$. Note that

$$E(\bigoplus_{t\in T}\delta_t)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\beta\lambda} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^{\bigoplus_{t\in T}\delta_t}(C_{\beta\lambda}).$$

We have $\lambda \leq E(\bigoplus_{t\in T}\delta_t)(A)$, and hence $\bigoplus_{t\in T} E(\delta_t) \leq E(\delta)$. □

Remark 4.12. Let $\{(X_t, \delta_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces. Then $\big(\prod_{t\in T} X_t, I(\prod_{t\in T}\delta_t)\big) \geq \big(\prod_{t\in T} X_t, \prod_{t\in T} I(\delta_t)\big)$, and the inequality cannot be replaced by equality.

Proof. It is easy to verify that $\big(\prod_{t\in T} X_t, I(\prod_{t\in T}\delta_t)\big) \geq \big(\prod_{t\in T} X_t, \prod_{t\in T} I(\delta_t)\big)$ by Theorem 3.5 and the definition of $I(\delta)$. The following example shows that the inequality cannot be replaced by equality.

Let $X = L = M = [0, 1]$, $\tau : L^X \longrightarrow M$ is defined by

$$\tau(0_X) = \tau(1_X) = \tau([0.5] \wedge 1_{[0,0.5]}) = \tau(1_{(0.5,1]}) = 1$$

$$\text{and for others } A \in L^X, \tau(A) = 0.$$

Then $I(\tau) : L^X \longrightarrow M$ is defined by

$$I(\tau)(0_X) = I(\tau)(1_X) = I(\tau)(1_{(0.5,1]}) = 1 \text{ and for others } A \in L^X, I(\tau)(A) = 0,$$

and hence

$$I(\tau) \times I(\tau)(([0.5] \wedge 1_{X\times[0,0.5]\cup[0,0.5]\times X}) \vee 1_{(0.5,1]\times(0.5,1]}) = 0.$$

On the other hand, we have

$$\tau \times \tau(([0.5] \wedge 1_{X\times[0,0.5]\cup[0,0.5]\times X}) \vee 1_{(0.5,1]\times(0.5,1]}) = 1$$

and

$$I(\tau \times \tau)(([0.5] \wedge 1_{X\times[0,0.5]\cup[0,0.5]\times X}) \vee 1_{(0.5,1]\times(0.5,1]}) = 1.$$

Therefore $I(\tau) \times I(\tau) \leq I(\tau \times \tau)$ and $I(\tau) \times I(\tau) \neq I(\tau \times \tau)$. $\square$

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