CONVERGENCE APPROACH SPACES AND APPROACH SPACES AS LATTICE-VALUED CONVERGENCE SPACES

G. JÄGER

Abstract. We show that the category of convergence approach spaces is a simultaneously reflective and coreflective subcategory of the category of lattice-valued limit spaces. Further we study the preservation of diagonal conditions, which characterize approach spaces. It is shown that the category of pre-approach spaces is a simultaneously reflective and coreflective subcategory of the category of lattice-valued pretopological spaces and that the category of approach spaces is a coreflective subcategory of a category of lattice-valued topological convergence spaces.

1. Introduction

Stratified lattice-valued convergence spaces were first defined in [9] for the case that the lattice is a complete Heyting algebra (or a frame). They form a Cartesian closed supercategory of the category of stratified lattice-valued topological spaces. The theory of these spaces was developped in a series of papers [10, 11, 13, 12, 14] under this restricted lattice context. Later, Wei Yao [23] generalized lattice-valued convergence spaces to the lattice context of complete residuated lattices. A similar lattice background is also used in [17], however the definitions of a lattice-valued filter in these two papers do not coincide. Looking at the definition of a stratified lattice-valued filter in [8], it becomes clear that both papers actually use two different *enriched cl-premonoids*. In [21] the lattice context was therefore generalized to this situation.

Approach spaces and convergence approach spaces were introduced by Lowen and his co-workers [18, 19, 20]. The category of approach spaces is a common supercategory for the category TOP of topological spaces and the category $\infty pqMET$ of extended pseudo-quasi metric spaces (with non-expansive maps as morphisms). The basic idea behind approach spaces is to describe spaces by a distance function between points and sets. In this way it is possible to unify several basic concepts like e.g. compactness (in TOP) and total boundedness (in $\infty pqMET$). In [20] several equivalent axiom systems for approach spaces are given, one by means of a limit function for filters, which "measures the distance of a point to be in the limit of the filter". Convergence approach spaces weaken the axioms of approach spaces

Received: October 2010; Revised: January 2011; Accepted: May 2011

Key words and phrases: L-fuzzy convergence, L-topology, L-filter, L-limit space, Approach space, Convergence approach space, Pre-approach space.

Subject Classification: 18A99, 54A20, 54A40.

using the limit function and form a Cartesian closed and hereditary supercategory of the category of approach spaces [18].

It is known, see [2], that there are close links between approach spaces and so-called probabilistic convergence spaces. In [3], on the other hand, it is shown for the case of a complete Heyting algebra that probabilistic convergence spaces can be viewed as stratified $[0,1]$-convergence spaces. It seems therefore natural to investigate if approach spaces and convergence approach spaces can be viewed as lattice-valued convergence spaces. We show in this paper that this is indeed the case. Using the extended lattice context from [21] we show that the categories of convergence approach spaces, of pre-approach spaces and of approach spaces can be nicely embedded in the category of stratified $[0,1]$-limit spaces. In fact, the categories of convergence approach spaces and pre-approach spaces turn out to be isomorphic to simultaneously reflective and coreflective subcategories of the category of stratified $[0,1]$-limit spaces and the category of approach spaces is isomorphic to a coreflective subcategory of the category of stratified $[0,1]$-limit spaces. Convergence approach spaces, pre-approach spaces and approach spaces are thus important examples of lattice-valued convergence spaces.

2. **Preliminaries**

We review the definitions and results necessary for this paper, although we will only consider quite a restricted lattice context later. We will always assume that L is a complete lattice with top element $\top$ and bottom element $\bot \neq \top$.

A *GL-monoid* $(L, \leq, *)$ is a complete lattice $(L, \leq)$ with a binary operation $* : L \times L \longrightarrow L$ satisfying the following axioms [6, 8].

(GL1) $\alpha * \top = \alpha$ for all $\alpha \in L$;
(GL2) $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in L$;
(GL3) $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ for all $\alpha, \beta, \gamma \in L$;
(GL4) if $\alpha \leq \beta$ then there is $\delta \in L$ such that $\alpha = \beta * \delta$ (Divisibility);
(GL5) $\alpha * \bigvee_{\beta_i \in B} \beta_i = \bigvee_{\beta_i \in B} (\alpha * \beta_i)$ for all $\alpha \in L, B \subseteq L$.

Prominent examples are continuous t-norms (see e.g. [22, 15]) on $[0,1]$. Here, a *t-norm* is a mapping $* : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying (GL1), (GL2), (GL3) and $\alpha \leq \beta \Rightarrow \alpha * \gamma \leq \beta * \gamma$ for all $\alpha, \beta, \gamma \in [0,1]$. A t-norm is called *continuous* if it continuous as a mapping from $[0,1] \times [0,1]$ to $[0,1]$. The most important examples are the *minimum t-norm* $\alpha * \beta = \alpha \wedge \beta$, the *product t-norm* $\alpha * \beta = \alpha\beta$ and the *Lukasiewicz t-norm* $\alpha * \beta = (\alpha + \beta - 1) \vee 0$. Further examples of GL-monoids are given by complete Heyting algebras (or frames) $(L, \leq, \wedge)$ and by MV-algebras (see [8]).

A *cl-premonoid* $(L, \leq, \otimes)$ is a complete lattice $(L, \leq)$ together with a binary operation $\otimes : L \times L \longrightarrow L$ that satisfies the following axioms [8].

(CL1) $\alpha \leq \alpha \otimes \top$ and $\alpha \leq \top \otimes \alpha$;
(CL2) $\alpha \leq \beta$ and $\gamma \leq \delta$ implies $\alpha \otimes \gamma \leq \beta \otimes \delta$;
(CL3) For all $\emptyset \neq B \subseteq L$ we have $\alpha \otimes \bigvee_{\beta \in B} \beta = \bigvee_{\beta \in B} (\alpha \otimes \beta)$ and $(\bigvee_{\beta \in B} \beta) \otimes \alpha = \bigvee_{\beta \in B} (\beta \otimes \alpha)$.

We note that the operation $\otimes$ is not necessarily commutative and we also do not require associativity. Also the distributivity in (CL3) is only required for non-empty joins. It is clear that every GL-monoid is a cl-premonoid but not necessarily vice versa.

An *enriched cl-premonoid* $(L,\leq,\otimes,*)$ is a cl-premonoid $(L,\leq\otimes)$ that is "enriched" by a GL-monoid $(L,\leq,*)$ such that the following *domination condition* is satisfied [8].

(ECL) $(\alpha_1\otimes\beta_1)*(\alpha_2\otimes\beta_2)\leq(\alpha_1*\alpha_2)\otimes(\beta_1*\beta_2)$ for all $\alpha_1,\alpha_2,\beta_1,\beta_2\in L$.

If $(L,\leq,\wedge)$ is a frame and $(L,\leq,*)$ is a GL-monoid, then both $(L,\leq,*,*)$ and $(L,\leq,\wedge,*)$ are enriched cl-premonoids. Further examples are given by certain GL-monoids which have square roots and thus allow a *monoidal mean operator*, see [8]. We do not go into details of this example as we do not need it here.

Let $(L,\leq,\otimes,*)$ be an enriched cl-premonoid and X be a set. We denote the L-sets on X by $a,b,c,...\in L^X$ and extend the operations $\wedge,\otimes,*$ pointwise from L to L^X. In particular we define for $a,b,a_j\in L^X$ $(j\in J)$
$(\bigwedge_{j\in J}a_j)(x)=\bigwedge_{j\in J}a_j(x)$, $(\bigvee_{j\in J}a_j)(x)=\bigvee_{j\in J}a_j(x)$, $(a\otimes b)(x)=a(x)\otimes b(x)$, $(a*b)(x)=a(x)*b(x)$. The order of L^X is defined as the product order, i.e. $a\leq b$ if $a(x)\leq b(x)$ for all $x\in X$. For $\alpha\in L$ and $a\in L^X$ we denote

$$\alpha_A(x)=\begin{cases}\alpha & \text{if } x\in A\\ \bot & \text{else.}\end{cases}$$

Special cases of this notation are the *characteristic function* of $A\subseteq X$, $\top_A$, and the zero function $\bot_X$. If $f:X\longrightarrow Y$ is a mapping, then we define $f^{\rightarrow}:L^X\longrightarrow L^Y$ by $f^{\rightarrow}(a)(y)=\bigvee_{x\in X:f(x)=y}a(x)$ for $a\in L^X$ and $f^{\leftarrow}:L^Y\longrightarrow L^X$ by $f^{\leftarrow}(b)=b\circ f$ for $b\in L^Y$.

A *stratified L-filter* on X (see [8]) is a mapping $\mathcal{F}:L^X\longrightarrow L$ satisfying the axioms

(LF1) $\mathcal{F}(\top_X)=\top$, $\mathcal{F}(\bot_X)=\bot$;
(LF2) $a\leq b$ implies $\mathcal{F}(a)\leq\mathcal{F}(b)$ for all $a,b\in L^X$;
(LF3) $\mathcal{F}(a)\otimes\mathcal{F}(b)\leq\mathcal{F}(a\otimes b)$ for all $a,b\in L^X$;
(LFs) $\alpha*\mathcal{F}(a)\leq\mathcal{F}(\alpha_X*a)$ for all $\alpha\in L,a\in L^X$.
The set of all stratified L-filters on X is denoted by $\mathcal{F}^s_L(X)$.

Example 2.1. For $x\in X$, $[x]:L^X\longrightarrow L$, $a\longmapsto[x](a)=a(x)$ is a stratified L-filter, the *point L-filter* of x.

The set $\mathcal{F}^s_L(X)$ is again endowed with the pointwise order, i.e. we define for $\mathcal{F},\mathcal{G}\in\mathcal{F}^s_L(X)$, $\mathcal{F}\leq\mathcal{G}$ if $\mathcal{F}(a)\leq\mathcal{G}(a)$ for all $a\in L^X$. We then have that for a family of stratified L-filters $\{\mathcal{F}_j : j\in J\}\subseteq\mathcal{F}^s_L(X)$, the greatest lower bound $\bigwedge\mathcal{F}_j$ can be calculated as $(\bigwedge\mathcal{F}_j)(a)=\bigwedge\mathcal{F}_j(a)$. For a mapping $f:X\longrightarrow Y$ and $\mathcal{F}\in\mathcal{F}^s_L(X)$ we define the *image of $\mathcal{F}$ under f* by $f^{\rightarrow}(\mathcal{F})(b)=\mathcal{F}(f^{\leftarrow}(b))$ for $b\in L^Y$. See e.g. [8] for more details on stratified L-filters.

For the rest of the paper we consider the following restricted lattice situation. We consider a strictly decreasing surjective mapping $S:[0,1]\longrightarrow[0,\infty]$. We note that $S(1)=0$. Furthermore, S is order reversing and therefore $S(\bigwedge_{i\in J}\alpha_i)=\bigvee_{i\in J}S(\alpha_i)$ and $S(\bigvee_{i\in J}\alpha_i)=\bigwedge_{i\in J}S(\alpha_i)$. We also note that the inverse mapping

$S^{-1} : [0,\infty] \longrightarrow [0,1]$ exists and is strictly decreasing and surjective and therefore shares the properties of S. Then the following operation is a continuous t-norm on $[0,1]$:

$$\alpha * \beta = S^{-1}(S(\alpha) + S(\beta)).$$

In this case, the mapping S is called an *additive generator* of the t-norm $*$ (see [15]). We will in the sequel consider the mapping S fixed and always use the t-norm generated by S. The example $S(x) = -\log(x)$, with $S(0) := \infty$, leads to the product t-norm.

We then consider $([0,1], \wedge, *)$ as enriched cl-premonoid.

For notions from category theory we refer to [1].

3. Filters and Stratified $[0,1]$-filters

We denote the set of all filters on X by $\mathbb{F}(X)$. We especially denote the *point filter* of $x \in X$ by $\dot{x} = \{A \subseteq X \;:\; x \in A\} \in \mathbb{F}(X)$.

For a set J, $\mathbb{G} \in \mathbb{F}(J)$, $\mathbb{F}_i \in \mathbb{F}(X)$, $(i \in J)$ we define the *diagonal filter* [16]

$$\kappa(\mathbb{G}, (\mathbb{F}_i)) = \bigvee_{G \in \mathbb{G}} \bigwedge_{i \in G} \mathbb{F}_i \in \mathbb{F}(X).$$

Similarly, for $\mathcal{G} \in \mathcal{F}^s_{[0,1]}(J)$, $\mathcal{F}_i \in \mathcal{F}^s_{[0,1]}(X)$, $(i \in J)$, we define the *stratified* $[0,1]$-*diagonal filter* $\mathcal{G}(\mathcal{F}_{(\cdot)}) \in \mathcal{F}^s_{[0,1]}(X)$ by (see [4, 11, 12])

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \mathcal{G}(\mathcal{F}_{(\cdot)}(a))$$

with $\mathcal{F}_{(\cdot)}(a) : \left\{ \begin{array}{ccc} J & \longrightarrow & [0,1] \\ i & \longmapsto & \mathcal{F}_i(a) \end{array} \right.$ for $a \in L^X$.

We will need two constructions that allow us to pass from filters to stratified $[0,1]$-filters and vice versa. They originate from [5].

Given $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ we define $\Phi_{\mathcal{F}} = \{A \subseteq X \;:\; \mathcal{F}(1_A) = 1\}$.

The proof of the following proposition is straightforward and is presented in order to make the paper self-contained.

Proposition 3.1. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$, $x \in X$ and $f : X \longrightarrow Y$ be a mapping. Then*

(1) $\Phi_{\mathcal{F}} \in \mathbb{F}(X)$;
(2) $\Phi_{[x]} = \dot{x}$;
(3) $\mathcal{F} \le \mathcal{G}$ *implies* $\Phi_{\mathcal{F}} \le \Phi_{\mathcal{G}}$;
(4) $\Phi_{\mathcal{F} \wedge \mathcal{G}} = \Phi_{\mathcal{F}} \wedge \Phi_{\mathcal{G}}$;
(5) $f(\Phi_{\mathcal{F}}) = \Phi_{f(\mathcal{F})}$.

Proof. (1) We know that $\mathcal{F}(1_\emptyset) = \mathcal{F}(0_X) = 0$ and hence $\emptyset \notin \Phi_{\mathcal{F}}$. As $\mathcal{F}(1_X) = 1$ we also have $X \in \Phi_{\mathcal{F}}$ and hence $\Phi_{\mathcal{F}} \neq \emptyset$. If $A \in \Phi(\mathcal{F})$ and $A \subseteq B$, then $1 = \mathcal{F}(1_A) \le \mathcal{F}(1_B)$ and therefore also $B \in \Phi_{\mathcal{F}}$. Lastly, if $A, B \in \Phi_{\mathcal{F}}$, then $\mathcal{F}(1_{A \cap B}) = \mathcal{F}(1_A \wedge 1_B) \ge \mathcal{F}(1_A) \wedge \mathcal{F}(1_B) = 1 \wedge 1 = 1$. Hence $A \cap B \in \Phi_{\mathcal{F}}$.

(2) We have $A \in \Phi_{[x]}$ if and only if $[x](1_A) = 1_A(x) = 1$. This is equivalent to

$x \in A$, i.e. to $A \in \dot{x}$.

(3) If $A \in \Phi_{\mathcal{F}}$, then $1 = \mathcal{F}(1_A) \leq \mathcal{G}(1_A)$ and hence $A \in \Phi_{\mathcal{G}}$.

(4) We have $A \in \Phi_{\mathcal{F}\wedge\mathcal{G}}$ if and only if $\mathcal{F}(1_A) \wedge \mathcal{G}(1_A) = \mathcal{F} \wedge \mathcal{G}(1_A) = 1$. This is equivalent to $\mathcal{F}(1_A) = 1$ and $\mathcal{G}(1_A) = 1$ i.e. to $A \in \Phi_{\mathcal{F}}$ and $A \in \Phi_{\mathcal{G}}$. But this is equivlaent to $A \in \Phi_{\mathcal{F}} \wedge \Phi_{\mathcal{G}}$. (Note that this proof immediately extends to arbirtrary infs.)

(5) Let first $B \in f(\Phi_{\mathcal{F}})$. Then there is $A \in \Phi_{\mathcal{F}}$ such that $f(A) \subseteq B$. Hence we have $\mathcal{F}(1_A) = 1$ and $A \subseteq f^{-1}(B)$. Therefore $f(\mathcal{F})(1_B) = \mathcal{F}(f^{-1}(1_B)) = \mathcal{F}(1_{f^{-1}(B)}) \geq \mathcal{F}(1_A) = 1$ and consequently $B \in \Phi_{f(\mathcal{F})}$. Conversely, if $B \in \Phi_{f(\mathcal{F})}$, then $1 = f(\mathcal{F})(1_B) = \mathcal{F}(f^{-1}(1_B)) = \mathcal{F}(1_{f^{-1}(B)})$ and hence $f^{-1}(B) \in \Phi_{\mathcal{F}}$. This implies $B \in f(\Phi_{\mathcal{F}})$ and the proof is complete. □

Proposition 3.2. *For $\mathcal{G} \in \mathcal{F}^s_{[0,1]}(J)$, $\mathcal{F}_i \in \mathcal{F}^s_{[0,1]}(X)$, $(i \in J)$, we have*

$$\kappa(\Phi_{\mathcal{G}}, (\Phi_{\mathcal{F}_i})) \leq \Phi_{\mathcal{G}(\mathcal{F}_{(\cdot)})}.$$

Proof. Let $A \in \kappa(\Phi_{\mathcal{G}}, (\Phi_{\mathcal{F}_i}))$. Then there are $G_1, G_2, ..., G_n \in \Phi_{\mathcal{G}}$, $F_k \in \bigwedge_{i \in G_k} \Phi_{\mathcal{F}_i}$ $(k = 1, 2, ..., n)$, such that $F_1 \cap F_2 \cap \cdots \cap F_n \subseteq A$. Then for each k we have $\mathcal{G}(1_{G_k}) = 1$ and $i \in G_k$ implies $\mathcal{F}(1_{F_k}) = 1$, i.e. $F_k \in \Phi_{\mathcal{F}_i}$. Hence $\{i \in J \,:\, F_k \in \Phi_{\mathcal{F}_i}\} \supseteq G_k$ and therefore $\mathcal{G}(1_{\{i \in J \,:\, F_k \in \Phi_{\mathcal{F}_i}\}}) = 1$. Now we have

$$\mathcal{F}_{(\cdot)}(1_{F_k})(i) = \mathcal{F}_i(1_{F_k}) \geq 1_{\{j \,:\, \mathcal{F}_j(1_{F_k}) = 1\}}(i).$$

Therefore $\mathcal{G}(\mathcal{F}_{(\cdot)})(1_{F_k}) = \mathcal{G}(\mathcal{F}_{(\cdot)}(1_{F_k})) = 1$ and it follows that

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(1_A) \geq \mathcal{G}(\mathcal{F}_{(\cdot)})(1_{F_1 \cap \ldots \cap F_n}) = \mathcal{G}(\mathcal{F}_{(\cdot)})(1_{F_1}) \wedge \cdots \wedge \mathcal{G}(\mathcal{F}_{(\cdot)})(1_{F_n}) = 1.$$

Hence $A \in \Phi_{\mathcal{G}(\mathcal{F}_{(\cdot)})}$. □

We now turn to another construction. For $\mathbb{F} \in \mathbb{F}(X)$ we define

$$\mathcal{F}^{\mathbb{F}}(a) = \bigvee \{\alpha \in [0,1] \,:\, [a \geq \alpha] \in \mathbb{F}\}$$

with $[a \geq \alpha] = \{x \in X \,:\, a(x) \geq \alpha\}$.

Again, we reproduce a straightforward proof for the sake of completeness.

Proposition 3.3. *Let $\mathbb{F}, \mathbb{G}, \mathbb{F}_i \in \mathbb{F}(X)\,(i \in J)$, $x \in X$ and $f : X \longrightarrow Y$ be a mapping. Then*

(1) $\mathcal{F}^{\mathbb{F}} \in \mathcal{F}^s_{[0,1]}(X)$;
(2) $[x] = \mathcal{F}^{\dot{x}}$;
(3) $\mathbb{F} \leq \mathbb{G}$ *implies* $\mathcal{F}^{\mathbb{F}} \leq \mathcal{F}^{\mathbb{G}}$;
(4) $\mathcal{F}^{\mathbb{F}} \wedge \mathcal{F}^{\mathbb{G}} \leq \mathcal{F}^{\mathbb{F} \wedge \mathbb{G}}$;
(5) $\bigwedge_{i \in J} \mathcal{F}^{\mathbb{F}_i} \leq \mathcal{F}^{(\bigwedge_{i \in J} \mathbb{F}_i)}$;
(6) $f(\mathcal{F}^{\mathbb{F}}) = \mathcal{F}^{f(\mathbb{F})}$.

Proof. (1) We have $\mathcal{F}^{\mathbb{F}}(0_X) = 0$ as $[0_X \geq \alpha] = \emptyset \notin \mathbb{F}$ for $\alpha > 0$. Also $\mathcal{F}^{\mathbb{F}}(1_X) = 1$ as $[1_X \geq \alpha] = X \in \mathbb{F}$ for all $\alpha \in [0,1]$. If $a \leq b$, then $[a \geq \alpha] \subseteq [b \geq \alpha]$ and hence if $[a \geq \alpha] \in \mathbb{F}$ then also $[b \geq \alpha] \in \mathbb{F}$. This implies $\{\alpha : [a \geq \alpha] \in \mathbb{F}\} \subseteq \{\alpha : [b \geq \alpha] \in \mathbb{F}\}$ and we get $\mathcal{F}^{\mathbb{F}}(a) \leq \mathcal{F}^{\mathbb{F}}(b)$. Finally we observe that $[a \geq \alpha] \cap [b \geq \beta] \subseteq [a \wedge b \geq \alpha \wedge \beta]$. Hence $\mathcal{F}^{\mathbb{F}}(a) \wedge \mathcal{F}^{\mathbb{F}}(b) = \bigvee\{\alpha : [a \geq \alpha \in \mathbb{F}\} \wedge \bigvee\{\beta : [b \geq \beta] \in \mathbb{F}\} = \bigvee\{\alpha \wedge \beta : [a \geq \alpha] \in \mathbb{F}, [b \geq \beta] \in \mathbb{F}\} \leq \bigvee\{\alpha \wedge \beta : [a \wedge b \geq \alpha \wedge \beta] \in \mathbb{F}\} \leq \bigvee\{\gamma : [a \wedge b \geq \gamma \in \mathbb{F}\} = \mathcal{F}^{\mathbb{F}}(a \wedge b)$.

(2) We have $\mathcal{F}^{\dot{x}}(a) = \bigvee\{\alpha : [a \geq \alpha] \in \dot{x}\} = \bigvee\{\alpha : [x](a) = a(x) \geq \alpha\} = [x](a)$.

(3) If $\mathbb{F} \leq \mathbb{G}$ and $[a \geq \alpha] \in \mathbb{F}$, then also $[a \geq \alpha] \in \mathbb{G}$.

(4) $\mathcal{F}^{\mathbb{F}} \wedge \mathcal{F}^{\mathbb{G}}(a) = \mathcal{F}^{\mathbb{F}}(a) \wedge \mathcal{F}^{\mathbb{G}}(a) = \bigvee\{\alpha \wedge \beta : [a \geq \alpha] \in \mathbb{F}, [a \geq \beta] \in \mathbb{G}\} \leq \bigvee\{\alpha \wedge \beta : [a \geq \alpha \wedge \beta] \in \mathbb{F}, [a \geq \alpha \wedge \beta] \in \mathbb{G}]\} = \bigvee\{\alpha \wedge \beta : [a \geq \alpha \wedge \beta] \in \mathbb{F} \wedge \mathbb{G}\} \leq \mathcal{F}^{\mathbb{F} \wedge \mathbb{G}}(a)$.

(5) We need to complete distributivity of $[0, 1]$. This means that for subsets $A_j \subseteq [0, 1]$ $(j \in J)$ we have $\bigwedge_{j \in J} \bigvee A_j = \bigvee_{(\alpha_i) \in \prod_{i \in J} A_i} \bigwedge_{j \in J} \alpha_j$. We have $\bigwedge_{i \in J} \mathcal{F}^{\mathbb{F}_i}(a) = \bigwedge_{i \in J}(\bigvee\{\alpha_i : [a \geq \alpha_i] \in \mathbb{F}_i\}) \overset{CD}{=} \bigvee_{\alpha_i : [a \geq \alpha_i] \in \mathbb{F}_i \forall i} \bigwedge_{j \in J} \alpha_j \leq \bigvee\{\bigwedge_{j \in J} \alpha_j : [a \geq \bigwedge_{j \in J} \alpha_j] \in \mathbb{F}_i \forall i\} \leq \bigvee\{\gamma : [a \geq \gamma] \in \mathbb{F}_i \forall i\} = \bigvee\{\gamma : [a \geq \gamma] \in \bigwedge_{i \in J} \mathbb{F}_i\} = \mathcal{F}^{(\bigwedge_{i \in J} \mathbb{F}_i)}$.

(6) We note that $[f^{-1}(a) \geq \alpha] = f^{-1}([a \geq \alpha])$. Hence $f(\mathcal{F}^{\mathbb{F}})(a) = \mathcal{F}^{\mathbb{F}}(f^{-1}(a)) = \bigvee\{\alpha : [f^{-1}(a) \geq \alpha] \in \mathbb{F}\} = \bigvee\{\alpha : f^{-1}([a \geq \alpha]) \in \mathbb{F}\} = \bigvee\{\alpha : [a \geq \alpha] \in f(\mathbb{F})\} = \mathcal{F}^{f(\mathbb{F})}(a)$. □

We note that (4) is a special case of (5) but that the proof of (4) does not require to complete distributivity of $[0, 1]$.

Proposition 3.4. *Let $\mathbb{F} \in \mathbb{F}(X)$ and $\mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$. Then*
(1) $\mathcal{F}^{(\Phi_\mathcal{G})} \leq \mathcal{G}$;
(2) $\Phi_{\mathcal{F}^{\mathbb{F}}} = \mathbb{F}$.

Proof. (1) We have $\mathcal{F}^{(\Phi_\mathcal{G})}(a) = \bigvee\{\alpha : [a \geq \alpha] \in \Phi_\mathcal{G}\} = \bigvee\{\alpha : \mathcal{G}(1_{[a \geq \alpha]}) = 1\} \leq \bigvee\{\alpha : \alpha * \mathcal{G}(1_{[a \geq \alpha]}) = \alpha\}$. As $\mathcal{G}$ is stratified we find $\mathcal{F}^{(\Phi_\mathcal{G})} \leq \bigvee\{\alpha : \mathcal{G}(\alpha * 1_{[a \geq \alpha]}) \geq \alpha\}$. Noting that $\alpha * 1_{[a \geq \alpha]} \leq a$ this yields $\mathcal{F}^{(\Phi_\mathcal{G})}(a) \leq \bigvee\{\alpha : \mathcal{G}(a) \geq \alpha\} = \mathcal{G}(a)$.

(2) We have

$$A \in \Phi_{\mathcal{F}^{\mathbb{F}}} \iff \mathcal{F}^{\mathbb{F}}(1_A) = 1 \iff \bigvee\{\alpha : A = [1_A \geq \alpha] \in \mathbb{F}\} = 1 \iff A \in \mathbb{F}.$$

□

Proposition 3.5. *For $\mathbb{G} \in \mathbb{F}(J)$, $\mathbb{F}_i \in \mathbb{F}(X)$, $(i \in J)$ we have $\mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}}) \leq \mathcal{F}^{\kappa(\mathbb{G}, (\mathbb{F}_i))}$.*

Proof. Let $a \in L^X$. Then

$$\mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}})(a) = \mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}}(a)) = \bigvee\{\alpha : [\mathcal{F}^{\mathbb{F}_{(\cdot)}}(a) \geq \alpha] \in \mathbb{G}\}.$$

Now we have $i \in [\mathcal{F}^{\mathbb{F}_{(\cdot)}}(a) \geq \alpha]$ if and only if $\mathcal{F}^{\mathbb{F}_i}(a) \geq \alpha$ if and only if $\bigvee\{\beta : [a \geq \beta] \in \mathbb{F}_i\} \geq \alpha$. Hence for every $\epsilon > 0$ there is $\beta > \alpha - \epsilon$ such that $[a \geq \beta] \in \mathbb{F}_i$ and consequently also $[a \geq \alpha - \epsilon] \in \mathbb{F}_i$ for all $\epsilon > 0$. Hence we have shown $[\mathcal{F}^{\mathbb{F}_{(\cdot)}}(a) \geq \alpha] \subseteq \{i : [a \geq \alpha - \epsilon] \in \mathbb{F}_i\}$ and therefore

$$\mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}})(a) \leq \bigvee\{\alpha : \{i : [a \geq \alpha - \epsilon] \in \mathbb{F}_i\} \in \mathbb{G}\}.$$

Now if $G = \{i : [a \geq \alpha - \epsilon] \in \mathbb{F}_i\} \in \mathbb{G}$, then

$$[a \geq \alpha - \epsilon] \in \bigwedge_{i:[a\geq\alpha-\epsilon]\in\mathbb{F}_i} \mathbb{F}_i \leq \bigvee_{G\in\mathbb{G}} \bigwedge_{i\in G} \mathbb{F}_i = \kappa(\mathbb{G}, (\mathbb{F}_i)).$$

It follows

$$\mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}})(a) \leq \bigvee\{\alpha : [a \geq \alpha - \epsilon] \in \kappa(\mathbb{G}, (\mathbb{F}_i))\} = \bigvee\{\beta + \epsilon : [a \geq \beta] \in \kappa(\mathbb{G}, (\mathbb{F}_i))\}$$

$$= \epsilon + \bigvee\{\beta : [a \geq \beta] \in \kappa(\mathbb{G}, (\mathbb{F}_i))\} = \epsilon + \mathcal{F}^{\kappa(\mathbb{G},(\mathbb{F}_i))}(a).$$

As $\epsilon > 0$ was arbitrary we obtain finally $\mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}})(a) \leq \mathcal{F}^{\kappa(\mathbb{G},(\mathbb{F}_i))}(a)$. □

We lastly establish a connection between our constructions and a definition given by Höhle. For $\mathbb{F} \in \mathbb{F}(X)$ and $a \in L^X$ we define [7]

$$\mathcal{F}_{\mathbb{F}}(a) = \bigvee_{F\in\mathbb{F}} \bigwedge_{x\in F} a(x).$$

Proposition 3.6. *Let* $\mathbb{F} \in \mathbb{F}(X)$. *Then* $\mathcal{F}^{\mathbb{F}} = \mathcal{F}_{\mathbb{F}}$.

Proof. Let $a \in L^X$. Let first $F \in \mathbb{F}$ and define $\alpha = \bigwedge_{x\in F} a(x)$. Then for $x \in F$ we have $a(x) \geq \alpha$ and hence $x \in [a \geq \alpha]$. Therefore $F \subseteq [a \geq \alpha]$ and hence $\alpha \in \{\beta \in L \mid [a \geq \beta] \in \mathbb{F}\}$. It follows $\mathcal{F}^{\mathbb{F}}(a) \geq \alpha = \bigwedge_{x\in F} a(x)$. As $F \in \mathbb{F}$ was arbitrary we conclude $\mathcal{F}^{\mathbb{F}}(a) \geq \bigvee_{F\in\mathbb{F}} \bigwedge_{x\in F} a(x) = \mathcal{F}_{\mathbb{F}}(a)$. To show the converse inequality we choose $\alpha \in L$ such that $F = [a \geq \alpha] \in \mathbb{F}$. Then $\bigwedge_{x\in[a\geq\alpha]} a(x) \geq \alpha$ and hence $\mathcal{F}_{\mathbb{F}}(a) = \bigvee_{F\in\mathbb{F}} \bigwedge_{x\in F} a(x) \geq \alpha$. This is true for all $\alpha \in L$ such that $[a \geq \alpha] \in \mathbb{F}$ and hence $\mathcal{F}_{\mathbb{F}}(a) \geq \bigvee\{\alpha \in L \mid [a \geq \alpha] \in \mathbb{F}\} = \mathcal{F}^{\mathbb{F}}(a)$. □

Proposition 3.7. *For* $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ *we have* $\Phi_{\mathcal{F}} = \bigvee_{\mathbb{G}:\mathcal{F}^{\mathbb{G}}\leq\mathcal{F}} \mathbb{G}$.

Proof. We first note that the supremum exists (in $\mathbb{F}(X)$): For $\mathbb{G} \in \mathbb{F}(X)$ with $\mathcal{F}^{\mathbb{G}} \leq \mathcal{F}$ we have by Lemma 3.4 $\mathbb{G} = \Phi_{(\mathcal{F}^{\mathbb{G}})} \leq \Phi_{\mathcal{F}}$. Hence we have an upper bound for the set of all $\mathbb{G} \in \mathbb{F}(X)$ with $\mathcal{F}^{\mathbb{G}} \leq \mathcal{F}$ and hence also a least upper bound. Next we note, as $\mathcal{F}^{(\Phi_{\mathcal{F}})} \leq \mathcal{F}$, that $\Phi_{\mathcal{F}} \leq \bigvee_{\mathbb{G}:\mathcal{F}^{\mathbb{G}}\leq\mathcal{F}} \mathbb{G}$. Conversely, let $A \in \bigvee_{\mathbb{G}:\mathcal{F}^{\mathbb{G}}\leq\mathcal{F}} \mathbb{G}$. Then there are $\mathbb{G}_1, ..., \mathbb{G}_n \in \mathbb{F}(X)$ with $\mathcal{F}^{\mathbb{G}_i} \leq \mathcal{F}$ and $G_i \in \mathbb{G}_i$ such that $G_1 \cap G_2 \cap \cdots \cap G_n \subseteq A$. Hence $1 = \mathcal{F}^{\mathbb{G}_i}(1_{G_i}) \leq \mathcal{F}(1_{G_i})$ for all $i = 1, 2, ..., n$. Moreover $1_{G_1} \cap \cdots 1_{G_n} \leq 1_A$. Hence we obtain

$$1 = \mathcal{F}(1_{G_1}) \wedge \cdots \wedge \mathcal{F}(1_{G_n}) \leq \mathcal{F}(1_{G_1} \wedge \cdots \wedge 1_{G_n}) \leq \mathcal{F}(1_A)$$

and therefore $A \in \Phi_{\mathcal{F}}$. □

4. Convergence Approach Spaces as Stratified $[0, 1]$-limit Spaces

Definition 4.1. [10, 21] A mapping $\lim : \mathcal{F}^s_{[0,1]}(X) \longrightarrow [0, 1]^X$ with

(LL1) $\lim[x](x) = 1$ for all $x \in X$;
(LL2) $\mathcal{F} \leq \mathcal{G}$ implies $\lim \mathcal{F} \leq \lim \mathcal{G}$;
(LL3) $\lim(\mathcal{F} \wedge \mathcal{G}) = \lim \mathcal{F}(x) \wedge \lim \mathcal{G}$

is called a *stratified* $[0, 1]$*-limit map*, the pair $(X, \lim)$ a *stratified* $[0, 1]$*-limit space.*

Note that in [21] such spaces are called *stratified* $[0,1]$*-strong limit spaces.*

A mapping $f : (X, \lim) \longrightarrow (X', \lim')$ is called *continuous* if $\lim \mathcal{F}(x) \leq \lim' f(\mathcal{F})(f(x))$ for all $x \in X$ and for all $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$.

The category $s[0,1]$-LIM has as objects all stratified $[0,1]$-limit spaces and as morphisms the continuous mappings.

Definition 4.2. [18] A mapping $\lambda : \mathbb{F}(X) \longrightarrow [0,\infty]^X$ with

(L1) $\lambda(\dot{x})(x) = 0$ for all $x \in X$;

(L2) $\mathbb{F} \leq \mathbb{G}$ implies $\lambda(\mathbb{G}) \leq \lambda(\mathbb{F})$;

(L3) $\lambda(\mathbb{F} \wedge \mathbb{G}) = \lambda(\mathbb{F}) \vee \lambda(\mathbb{G})$

is called a *convergence approach limit map*, the pair (X, λ) is called a *convergence approach space.*

A mapping $f : (X, \lambda) \longrightarrow (X', \lambda')$ is called a *contraction* if $\lambda'(f(\mathbb{F}))(f(x)) \leq \lambda(\mathbb{F})(x)$ for all $x \in X$ and for all $\mathbb{F} \in \mathbb{F}(X)$.

The category CAP has as objects the convergence approach spaces and as morphisms the contractions.

With the mapping S that we hold fixed throughout this paper, we define for $(X, \lambda) \in |CAP|$

$$\lim_\lambda \mathcal{F}(x) = S^{-1}(\lambda(\Phi_{\mathcal{F}})(x)).$$

The idea of using such a surjective additive generator for the t-norm goes back to [2].

Proposition 4.3. *For* $(X, \lambda) \in |CAP|$ *we have* $(X, \lim_\lambda) \in |s[0,1]$-$LIM|$.

Proof. **(LL1)** We have $\lim_\lambda [x](x) = S^{-1}(\lambda(\Phi_{[x]})(x)) = S^{-1}(\lambda(\dot{x})(x)) = S^{-1}(0) = 1$.

(LL2) If $\mathcal{F} \leq \mathcal{G}$, then $\Phi_{\mathcal{F}} \leq \Phi_{\mathcal{G}}$ and as both λ and S^{-1} are order reversing we obtain

$$\lim_\lambda \mathcal{F}(x) = S^{-1}(\lambda(\Phi_{\mathcal{F}})(x)) \leq S^{-1}(\lambda(\Phi_{\mathcal{G}})(x)) = \lim_\lambda \mathcal{G}(x).$$

(LL3) We have

$$\begin{aligned}
\lim_\lambda \mathcal{F}(x) \wedge \lim_\lambda \mathcal{G}(x) &= S^{-1}(\lambda(\Phi_{\mathcal{F}})(x)) \wedge S^{-1}(\lambda(\Phi_{\mathcal{G}})(x)) \\
&= S^{-1}(\lambda(\Phi_{\mathcal{F}})(x) \vee \lambda(\Phi_{\mathcal{G}})(x)) \;=\; S^{-1}(\lambda(\Phi_{\mathcal{F}} \wedge \Phi_{\mathcal{G}})(x)) \\
&= S^{-1}(\lambda(\Phi_{\mathcal{F} \wedge \mathcal{G}})(x)) \;=\; \lim_\lambda \mathcal{F} \wedge \mathcal{G}(x).
\end{aligned}$$

□

Proposition 4.4. *If* $f : (X, \lambda) \longrightarrow (X', \lambda')$ *is a contraction, then* $f : (X, \lim_\lambda) \longrightarrow (X', \lim_{\lambda'})$ *is continuous.*

Proof. Let $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ and $x \in X$. Then $\lim_{\lambda'} f(\mathcal{F})(f(x)) = S^{-1}(\lambda'(\Phi_{f(\mathcal{F})})(f(x)) = S^{-1}(\lambda'(f(\Phi_{\mathcal{F}}))(f(x)) \geq S^{-1}(\lambda(\Phi_{\mathcal{F}})(x)) = \lim_\lambda \mathcal{F}(x)$. □

Hence we can define a functor

$$H : \begin{cases} CAP & \longrightarrow & s[0,1]\text{-}LIM \\ (X,\lambda) & \longmapsto & (X, \lim_\lambda) \\ f & \longmapsto & f \end{cases} .$$

We will next show that this functor is injective on objects.

Proposition 4.5. *If $(X,\lambda) \neq (X,\lambda')$, then $(X, \lim_\lambda) \neq (X, \lim_{\lambda'})$.*

Proof. If $\lambda \neq \lambda'$, then there exists $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$ such that $\lambda(\mathbb{F})(x) \neq \lambda'(\mathbb{F})(x)$. But then for $\mathcal{G} = \mathcal{F}^{\mathbb{F}}$ we have $\Phi_{\mathcal{G}} = \mathbb{F}$ and hence

$$\lim_\lambda \mathcal{G}(x) = S^{-1}(\lambda(\Phi_{\mathcal{G}})(x)) = S^{-1}(\lambda(\mathbb{F})(x)) \neq S^{-1}(\lambda'(\mathbb{F})(x)) = \lim_{\lambda'} \mathcal{G}(x).$$

□

Further the functor H is full.

Proposition 4.6. *Let $(X,\lambda), (X',\lambda') \in |CAP|$ and let $f : X \longrightarrow X'$ be a mapping. If $f : (X, \lim_\lambda) \longrightarrow (X', \lim_{\lambda'})$ is continuous, then $f : (X,\lambda) \longrightarrow (X',\lambda')$ is a contraction.*

Proof. Let $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$. We again define $\mathcal{G} = \mathcal{F}^{\mathbb{F}}$. By the continuity of f we then have $\lim_\lambda \mathcal{G}(x) \leq \lim_{\lambda'} f(\mathcal{G})(f(x))$. Hence, again as $\Phi_{\mathcal{G}} = \mathbb{F}$, we get $S^{-1}(\lambda(\mathbb{F})(x)) \leq S^{-1}(\lambda'(f(\mathbb{F})(f(x)))$, from which, by S^{-1} being order reversing, we obtain $\lambda'(f(\mathbb{F}))(f(x)) \leq \lambda(\mathbb{F})(x)$. As $\mathbb{F}$ and x were arbitrary, the claim follows. □

As a result we obtain the following theorem.

Theorem 4.7. *The category CAP of convergence approach spaces is isomorphic to a full subcategory of $s[0,1]$-LIM.*

We now consider further axioms for stratified $[0,1]$-limit spaces and convergence approach spaces.

Proposition 4.8. *Let $(X,\lambda) \in |CAP|$. If (X,λ) satisfies the axiom*
(L3') $\lambda(\bigwedge_{i\in J} \mathbb{F}_i)(x) = \bigvee_{i\in J} \lambda(\mathbb{F}_i)(x)$ *for all* $\{\mathbb{F}_i : i \in J\} \subseteq \mathbb{F}(X), x \in X$;
then $(X, \lim_\lambda)$ satisfies the axiom
(LL3') $\lim_\lambda(\bigwedge_{i\in J} \mathcal{F}_i)(x) = \bigwedge_{i\in J} \lim_\lambda \mathcal{F}_i(x)$ *for all* $\{\mathcal{F}_i : i \in J\} \subseteq \mathcal{F}^s_{[0,1]}(X), x \in X$.

Proof. Similar to the proof of (LL3),We have

$$\begin{aligned} \lim_\lambda(\bigwedge_{i\in J} \mathcal{F}_i)(x) &= S^{-1}(\lambda(\Phi_{\bigwedge \mathcal{F}_i})(x)) = S^{-1}(\lambda(\bigwedge_{i\in J} \Phi_{\mathcal{F}_i})(x)) \\ = S^{-1}(\bigvee_{i\in J} \lambda(\mathcal{F}_i)(x)) &= \bigwedge_{i\in J} S^{-1}(\lambda(\Phi_{\mathcal{F}_i})(x)) = \bigwedge_{i\in J} \lim_\lambda \mathcal{F}_i(x). \end{aligned}$$

□

With the interpretation of $\lim \mathcal{F}(x)$ as the *grade of convergence of $\mathcal{F}$ to x* and $\lambda(\mathbb{F})(x)$ as the *distance that the point x is away from being a limit point of $\mathbb{F}$* ([19]), the axioms (LL3), resp. (L3), generalize the classical convergence axiom in topology that for a family of filters converging to a point x also the meet of the family converges to x.

For the next two results we remember that $\alpha * \beta = S^{-1}(S(\alpha) + S(\beta))$ is the continuous t-norm on $[0,1]$ that we consider.

Proposition 4.9. *Let* $(X,\lambda) \in |CAP|$. *If* (X,λ) *satisfies the axiom*

(L4) *If* $\mathbb{G} \in \mathbb{F}(X)$, $\mathbb{F}_y \in \mathbb{F}(X)$ *(*$y \in X$*),* $x \in X$ *then*

$$\lambda(\kappa(\mathbb{G},(\mathbb{F}_y)))(x) \leq \lambda(\mathbb{G})(x) + \bigvee_{y\in X} \lambda(\mathbb{F}_y)(y);$$

then $(X, \lim_\lambda)$ *satisfies the axiom*

(LL4) *If* $\mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$, $\mathcal{F}_y \in \mathcal{F}^s_{[0,1]}(X)$, *(*$y \in X$*),* $x \in X$ *then*

$$\lim{}_\lambda\mathcal{G}(x) * \bigwedge_{y\in X} \lim{}_\lambda\mathcal{F}_y(y) \leq \lim{}_\lambda\mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

Proof. Let $\alpha \leq \lim_\lambda\mathcal{G}(x) = S^{-1}(\lambda(\Phi_\mathcal{G})(x))$ and $\beta \leq \bigwedge_{y\in X} \lim_\lambda\mathcal{F}_y(y) \leq S^{-1}(\lambda(\Phi_{\mathcal{F}_y})(y))$ for all $y \in Y$. Then $\lambda(\Phi_\mathcal{G})(x) \leq S(\alpha)$ and $\lambda(\Phi_{\mathcal{F}_y})(y) \leq S(\beta)$ for all $y \in X$. Hence also $\bigvee_{y\in X} \lambda(\Phi_{\mathcal{F}_y})(y) \leq S(\beta)$. By (L4), we obtain from this

$$\lambda(\kappa(\Phi_\mathcal{G},(\Phi_{\mathcal{F}_y})))(x) \leq \lambda(\Phi_\mathcal{G})(x) + \bigvee_{y\in X} \lambda(\Phi_{\mathcal{F}_y})(y) \leq S(\alpha) + S(\beta).$$

Proposition 3.2 then yields $\lambda(\Phi_{\mathcal{G}(\mathcal{F}_{(\cdot)})})(x) \leq S(\alpha) + S(\beta)$ and hence

$$\lim{}_\lambda\mathcal{G}(\mathcal{F}_{(\cdot)})(x) = S^{-1}(\lambda(\Phi_{\mathcal{G}(\mathcal{F}_{(\cdot)})})(x)) \geq S^{-1}(S(\alpha) + S(\beta)) = \alpha * \beta.$$

□

Proposition 4.10. *Let* $(X,\lambda) \in |CAP|$. *If* (X,λ) *satisfies the axiom*

(LF) *If* $J \neq \emptyset$, $\psi : J \longrightarrow X$, $\mathbb{G} \in \mathbb{F}(J)$, $\mathbb{F}_i \in \mathbb{F}(X)$ *(*$i \in J$*),* $x \in X$ *then*

$$\lambda(\kappa(\mathbb{G},(\mathbb{F}_i))(x) \leq \lambda(\psi(\mathbb{G}))(x) + \bigvee_{i\in J} \lambda(\mathbb{F}_i)(\psi(i));$$

then $(X, \lim_\lambda)$ *satisfies the axiom*

(LLF) *If* $J \neq \emptyset$, $\psi : J \longrightarrow X$, $\mathcal{G} \in \mathcal{F}^s_{[0,1]}(J)$, $\mathcal{F}_i \in \mathcal{F}^s_{[0,1]}(X)$, *(*$i \in J$*),* $x \in X$ *then*

$$\lim{}_\lambda\psi(\mathcal{G})(x) * \bigwedge_{i\in J} \lim{}_\lambda\mathcal{F}_i(\psi(i)) \leq \lim{}_\lambda\mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

Proof. Let $\alpha \leq \lim_\lambda\psi(\mathcal{G})(x) = S^{-1}(\lambda(\Phi_{\psi(\mathcal{G})})(x))$ and $\beta \leq \bigwedge_{i\in J} \lim_\lambda\mathcal{F}_i(\psi(i)) \leq S^{-1}(\lambda(\Phi_{\mathcal{F}_i})(\psi(i)))$ for all $i \in J$. Then $\lambda(\psi(\Phi_\mathcal{G})(x) = \lambda(\Phi_{\psi(\mathcal{G})})(x)) \leq S(\alpha)$ and $\lambda(\Phi_{\mathcal{F}_i})(\psi(i)) \leq S(\beta)$ for all $i \in J$. Hence also $\bigvee_{i\in J} \lambda(\Phi_{\mathcal{F}_i})(\psi(i)) \leq S(\beta)$. By (LF), we obtain

$$\lambda(\kappa(\Phi_\mathcal{G},(\Phi_{\mathcal{F}_i})))(x) \leq \lambda(\psi(\Phi_\mathcal{G}))(x) + \bigvee_{i\in J} \lambda(\Phi_{\mathcal{F}_i})(\psi(i)) \leq S(\alpha) + S(\beta).$$

Proposition 3.2 then yields $\lambda(\Phi_{\mathcal{G}(\mathcal{F}_{(\cdot)})})(x) \leq S(\alpha) + S(\beta)$ and hence

$$\lim{}_\lambda\mathcal{G}(\mathcal{F}_{(\cdot)})(x) = S^{-1}(\lambda(\Phi_{\mathcal{G}(\mathcal{F}_{(\cdot)})})(x)) \geq S^{-1}(S(\alpha) + S(\beta)) = \alpha * \beta.$$

□

The axioms (LL4), resp. (L4) generalize a diagonal axiom due to Kowalsky [16] to lattice-valued convergence spaces [11], resp. to convergence approach spaces [18, 19]. Note that classically, i.e. for $L = \{0,1\}$, a convergence space satisfying (LL1),(LL2),(LL3') and (LL4) is a topological space [16]. Similarly, (LLF), resp. (LF) generalize a diagonal axiom due to Fischer to lattice-valued convergence spaces [12], resp. to convergence approach spaces [2]. Classically a convergence space satisfying (LL1), (LL2) and (LLF) is a topological space.

5. **Reflectiveness**

For $(X, \lim) \in |s[0,1]\text{-}LIM|$ we define

$$\lambda_{\lim}(\mathbb{F})(x) = S(\bigvee\{\lim \mathcal{F}(x) \ : \ \Phi_{\mathcal{F}} \le \mathbb{F}\}).$$

Proposition 5.1. *For $(X, \lim) \in |s[0,1]\text{-}LIM|$ we have $(X, \lambda_{\lim}) \in |CAP|$.*

Proof. **(L1)** We have $\Phi_{[x]} = \dot{x}$. Hence $\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \dot{x}\} \ge \lim[x](x) = 1$. Therefore $\lambda_{\lim}(\dot{x})(x) \le S(1) = 0$.

(L2) Let $\mathbb{F} \le \mathbb{G}$. If $\Phi_{\mathcal{F}} \le \mathbb{F}$ then $\Phi_{\mathcal{F}} \le \mathbb{G}$ and hence $\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\} \subseteq \{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{G}\}$. Therefore $\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\} \le \bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{G}\}$ and we get

$$\lambda_{\lim}(\mathbb{G})(x) = S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{G}\}) \le S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\}) = \lambda_{\lim}(\mathbb{F})(x).$$

(L3) We have $\{\lim \mathcal{F} \wedge \mathcal{G}(x) : \Phi\mathcal{F} \le \mathbb{F}, \Phi_{\mathcal{G}} \le \mathbb{G}\} \subseteq \{\lim \mathcal{F} \wedge \mathcal{G}(x) : \Phi_{\mathcal{F}} \wedge \Phi_{\mathcal{G}} \le \mathbb{F} \wedge \mathbb{G}\} \subseteq \{\lim \mathcal{H}(x) : \Phi_{\mathcal{H}} \le \mathbb{F} \wedge \mathbb{G}\}$. Hence

$$\begin{aligned}
\lambda_{\lim}(\mathbb{F} \wedge \mathbb{G})(x) &= S(\bigvee\{\lim \mathcal{H}(x) : \Phi_{\mathcal{H}} \le \mathbb{F} \wedge \mathbb{G}\}) \\
&\le S(\bigvee\{\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}, \Phi_{\mathcal{G}} \le \mathbb{G}\}) \\
&= S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\} \wedge \bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{G}\}) \\
&= S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\}) \vee S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{G}\}) \\
&= \lambda_{\lim}(\mathbb{F})(x) \vee \lambda_{\lim}(\mathbb{G})(x).
\end{aligned}$$

□

Proposition 5.2. *If $f : (X, \lim) \longrightarrow (X', \lim')$ is continuous, then $f : (X, \lambda_{\lim}) \longrightarrow (X', \lambda_{\lim'})$ is a contraction.*

Proof. If $\Phi_{\mathcal{F}} \le \mathbb{F}$ then $\Phi_{f(\mathcal{F})} = f(\Phi_{\mathcal{F}}) \le f(\mathbb{F})$. Hence $\{\lim' \mathcal{G}(f(x)) : \Phi_{\mathcal{G}} \le f(\mathbb{F})\} \supseteq \{\lim' f(\mathcal{F})(f(x)) : \Phi_{f(\mathcal{F})} \le f(\mathbb{F})\} \supseteq \{\lim' f(\mathcal{F})(f(x)) : \Phi_{\mathcal{F}} \le \mathbb{F}\}$. By continuity we moreover obtain $\lim \mathcal{F}(x) \le \lim' f(\mathcal{F})(f(x))$ and hence

$$\bigvee\{\lim' \mathcal{G}(f(x)) : \Phi_{\mathcal{G}} \le f(\mathbb{F})\} \ge \bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\}.$$

Therefore $\lambda_{\lim'}(f(\mathbb{F}))(f(x)) = S(\bigvee\{\lim' \mathcal{G}(f(x)) : \Phi_{\mathcal{G}} \le f(\mathbb{F})\}) \le S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\}) = \lambda_{\lim}(\mathbb{F})(x)$. □

Proposition 5.3. *Let $(X, \lim) \in |s[0,1]\text{-}LIM|$, $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ and $x \in X$. Then $\lim_{(\lambda_{\lim})} \mathcal{F}(x) \ge \lim \mathcal{F}(x)$.*

Proof. We have

$$\lim_{(\lambda_{\lim})} \mathcal{F}(x) = S^{-1}(\lambda_{\lim}(\Phi_{\mathcal{F}})(x)) = S^{-1}(S(\bigvee\{\lim \mathcal{G}(x) : \Phi_{\mathcal{G}} \le \Phi_{\mathcal{F}}\})) \ge \lim \mathcal{F}(x).$$

□

Proposition 5.4. *Let* $(X,\lambda) \in |CAP|$, $\mathbb{F} \in \mathbb{F}(X)$ *and* $x \in X$. *Then* $\lambda_{(\lim_\lambda)}(\mathbb{F})(x) = \lambda(\mathbb{F})(x)$.

Proof. We have

$$\lambda_{(\lim_\lambda)}(\mathbb{F})(x) = S(\bigvee\{\lim_\lambda \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\})$$
$$= \bigwedge S(\{S^{-1}(\lambda(\Phi_{\mathcal{F}})(x)) : \Phi_{\mathcal{F}} \le \mathbb{F}\}) = \bigwedge\{\lambda(\Phi_{\mathcal{F}})(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\}.$$

By (L2) and as $\Phi_{(\mathcal{F}^{\mathbb{F}})} = \mathbb{F}$ we obtain that $\bigwedge\{\lambda(\Phi_{\mathcal{F}})(x) : \Phi_{\mathcal{F}} \le \mathbb{F}\} = \lambda(\mathbb{F})(x)$ and the proof is complete. □

Proposition 5.5. *CAP is isomorphic to a reflective subcategory of $s[0,1]$-LIM.*

Proof. Let $(X,\lim) \in |s[0,1]\text{-}LIM|$. By Proposition 5.1 then $(X,\lambda_{\lim}) \in |CAP|$. By Proposition 5.3, $id_X : (X,\lim) \longrightarrow (X, \lim_{(\lambda_{\lim})})$ is continuous. Further, using Proposition 5.2, for $(Y,\lambda_Y) \in |CAP|$ and a continuous mapping $f : (X,\lim) \longrightarrow (Y, \lim_{(\lambda_Y)})$ also $f : (X,\lambda_{\lim}) \longrightarrow (Y, \lambda_{(\lim_{\lambda_Y})})$ is continuous. Hence by Proposition 5.4, $f : (X,\lambda_{\lim}) \longrightarrow (Y,\lambda_Y)$ is continuous. Therefore id_X is a CAP-reflection. □

We look into further axioms.

Proposition 5.6. *Let* $(X,\lim) \in |s[0,1]\text{-}LIM|$. *If* $(X,\lim)$ *satisfies the axiom*
(LL3') $\lim \bigwedge_{i\in J} \mathcal{F}_i(x) = \bigwedge_{i\in J} \lim \mathcal{F}_i(x)$ *for all* $\{\mathcal{F}_i : i \in J\} \subseteq \mathcal{F}^s_{[0,1]}(X), x \in X$;
then $(X,\lambda_{\lim})$ *satisfies the axiom*
(L3') $\lambda_{\lim}(\bigwedge_{i\in J} \mathbb{F}_i)(x) = \bigvee_{i\in J} \lambda_{\lim}(\mathbb{F}_i)(x)$ *for all* $\{\mathbb{F}_i : i \in J\} \subseteq \mathbb{F}(X), x \in X$.

Proof. We have $\lambda_{\lim}(\bigwedge_{i\in J}\mathbb{F}_i)(x) = S(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \le \bigwedge_{i\in J}\mathbb{F}_i\})$. If $\Phi_{\mathcal{F}_i} \le \mathbb{F}_i$ for all $i \in J$, then $\Phi_{(\bigwedge \mathcal{F}_i)} = \bigwedge_{i\in J}\Phi_{\mathcal{F}_i} \le \bigwedge_{i\in J}\mathbb{F}_i$. Hence $\{\lim(\bigwedge_{i\in J}\mathcal{F}_i)(x) : \Phi_{\mathcal{F}_i} \le \mathbb{F}_i \forall i \in J\} \subseteq \{\lim(\bigwedge_{i\in J}\mathcal{F}_i)(x) : \Phi_{\bigwedge \mathcal{F}_i} \le \bigwedge_{i\in J}\mathbb{F}_i\} \subseteq \{\lim \mathcal{H}(x) : \Phi_{\mathcal{H}} \le \bigwedge_{i\in J}\mathbb{F}_i\}$. Therefore we obtain $\lambda_{\lim}(\bigwedge_{i\in J}\mathbb{F}_i)(x) = S(\bigvee\{\lim \mathcal{H}(x) : \Phi_{\mathcal{H}} \le \bigwedge_{i\in J}\mathbb{F}_i\}) \le S(\bigvee\{\bigwedge_{i\in J}\lim \mathcal{F}_i(x) : \Phi_{\mathcal{F}_i} \le \mathbb{F}_i \forall i\})$. As $[0,1]$ is completely distributive, we have

$$\bigwedge_{i\in J}\bigvee\{\lim \mathcal{F}_i(x) : \Phi_{\mathcal{F}_i} \le \mathbb{F}_i\} = \bigvee_{\Phi_{\mathcal{F}_i} \le \mathbb{F}_i \forall i}\ \bigwedge_{j\in J}\lim \mathcal{F}_j(x).$$

Hence

$$\lambda_{\lim}(\bigwedge_{i\in J}\mathbb{F}_i)(x) = S(\bigvee\{\lim \mathcal{H}(x) : \Phi_{\mathcal{H}} \le \bigwedge_{i\in J}\mathbb{F}_i\}) \le S(\bigwedge_{i\in J}\bigvee\{\lim \mathcal{F}_i(x) : \Phi_{\mathcal{F}_i} \le \mathbb{F}_i\})$$
$$= \bigvee_{i\in J} S(\bigvee\{\lim \mathcal{F}_i(x) : \Phi_{\mathcal{F}_i} \le \mathbb{F}_i\}) = \bigvee_{i\in J}\lambda_{\lim}(\mathbb{F}_i)(x).$$

The other inequality follows from (L2): As $\bigwedge_{i\in J}\mathbb{F}_i \le \mathbb{F}_j$ for all $j \in J$ we have $\lambda_{\lim}(\mathbb{F}_j)(x) \le \lambda_{\lim}(\bigwedge_{i\in J}\mathbb{F}_i)(x)$ for all $j \in J$ and hence $\bigvee_{j\in J}\lambda_{\lim}(\mathbb{F}_j)(x) \le \lambda_{\lim}(\bigwedge_{i\in J}\mathbb{F}_i)(x)$. □

If we denote the category of all convergence approach spaces that satisfy (L3') by PAP (convergence approach spaces which satisfy (L3') are called *pre-approach spaces* in [18]) and the category of stratified $[0,1]$-limit spaces that satisfy (LL3') by $s[0,1]$-$PLIM$, then we deduce from the Propositions 4.8 and 5.6 the following result.

Proposition 5.7. *PAP is isomorphic to a reflective subcategory of $s[0,1]$-$PLIM$.*

6. Coreflectiveness

Let again $(X, \lim) \in |s[0,1]\text{-}LIM|$. Define for $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$

$$\lambda^{\lim}(\mathbb{F})(x) = S(\lim \mathcal{F}^{\mathbb{F}}(x)).$$

Proposition 6.1. *For $(X, \lim) \in |s[0,1]\text{-}LIM|$ we have $(X, \lambda^{\lim}) \in |CAP|$.*

Proof. (L1) We have $\lambda^{\lim}(\dot{x})(x) = S(\lim \mathcal{F}^{\dot{x}}(x)) = S(\lim[x](x)) = S(1) = 0$.
(L2) If $\mathbb{F} \le \mathbb{G}$, then $\mathcal{F}^{\mathbb{F}} \le \mathcal{F}^{\mathbb{G}}$ and hence $\lambda^{\lim}(\mathcal{G})(x) = S(\lim \mathcal{F}^{\mathbb{G}}(x)) \le S(\lim \mathcal{F}^{\mathbb{F}}(x)) = \lambda^{\lim}(\mathbb{F})(x)$.
(L3) We have $\lambda^{\lim}(\mathbb{F} \wedge \mathbb{G}) = S(\lim \mathcal{F}^{\mathbb{F} \wedge \mathbb{G}}(x)) \le S(\lim(\mathcal{F}^{\mathbb{F}} \wedge \mathcal{F}^{\mathbb{G}})(x)) = S(\lim \mathcal{F}^{\mathbb{F}}(x) \wedge \lim \mathcal{F}^{\mathbb{G}}(x)) = S(\lim \mathcal{F}^{\mathbb{F}}(x)) \vee S(\lim \mathcal{F}^{\mathbb{G}}(x)) = \lambda^{\lim}(\mathbb{F})(x) \vee \lambda^{\lim}(\mathbb{G})(x)$. □

Proposition 6.2. *If $f : (X, \lim) \longrightarrow (X', \lim')$ is continuous, then $f : (X, \lambda^{\lim}) \longrightarrow (X', \lambda^{\lim'})$ is a contraction.*

Proof. We have for $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$, $\lambda^{\lim'}(f(\mathbb{F}))(f(x)) = S(\lim' \mathcal{F}^{f(\mathbb{F})}(f(x))) = S(\lim' f(\mathcal{F}^{\mathbb{F}})(f(x))) \le S(\lim \mathcal{F}^{\mathbb{F}}(x)) = \lambda^{\lim'}(\mathbb{F})(x)$. □

Proposition 6.3. *Let $(X, \lim) \in |s[0,1]\text{-}LIM|$, $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ and $x \in X$. Then $\lim^{(\lambda_{\lim})} \mathcal{F}(x) \le \lim \mathcal{F}(x)$.*

Proof. We have by Lemma 3.7 and (LL2)

$$\overset{(\lambda_{\lim})}{\lim} \mathcal{F}(x) = S^{-1}(\lambda^{\lim}(\Phi_{\mathcal{F}})(x)) = S^{-1}(S(\lim \mathcal{F}^{(\Phi_{\mathcal{F}})})(x))) = \lim \mathcal{F}^{(\Phi_{\mathcal{F}})}(x) \le \lim \mathcal{F}(x).$$

□

Proposition 6.4. *Let $(X, \lambda) \in |CAP|$, $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$. Then $\lambda^{(\lim_\lambda)}(\mathbb{F})(x) = \lambda(\mathbb{F})(x)$.*

Proof. We have

$$\lambda^{\lim_\lambda}(\mathbb{F})(x) = S(\lim_\lambda \mathcal{F}^{\mathbb{F}}(x)) = S(S^{-1}(\lambda(\Phi_{(\mathcal{F}^{\mathbb{F}})})(x))) = \lambda(\mathbb{F})(x).$$

□

Proposition 6.5. *CAP is isomorphic to a coreflective subcategory of $s[0,1]$-LIM.*

Proof. We note that for $(X, \lim) \in |s[0,1]\text{-}LIM|$, $(X, \lambda^{\lim}) \in |CAP|$ by Proposition 6.1. By Proposition 6.3, also $id_X : (X, \lim_{(\lambda^{\lim})}) \longrightarrow (X, \lim)$ is continuous. Further, for a space $(Y, \lambda_Y) \in |CAP|$ and a continuous mapping $f : (Y, \lim_{\lambda_Y^{\lim}}) \longrightarrow (X, \lim)$ we find that $f : (Y, \lambda_Y) \longrightarrow (X, \lambda^{\lim})$ is a contraction (using the Propositions 6.2 and 6.4). Hence id_X is a CAP-coreflection. □

We look again into further axioms.

Proposition 6.6. *Let $(X, \lim) \in |s[0,1]\text{-}LIM|$. If $(X, \lim)$ satisfies the axiom*
(LL3') $\lim \bigwedge_{i \in J} \mathcal{F}_i(x) = \bigwedge_{i \in J} \lim \mathcal{F}_i(x)$ *for all $\{\mathcal{F}_i : i \in J\} \subseteq \mathcal{F}^s_{[0,1]}(X), x \in X$;*
then $(X, \lambda^{\lim})$ satisfies the axiom
(L3') $\lambda^{\lim}(\bigwedge_{i \in J} \mathbb{F}_i)(x) = \bigvee_{i \in J} \lambda^{\lim}(\mathbb{F}_i)(x)$ *for all $\{\mathbb{F}_i : i \in J\} \subseteq \mathbb{F}(X), x \in X$.*

Proof. We use Proposition 3.6 (5): $\bigvee_{i\in J}\mathcal{F}^{\mathbb{F}_i} \le \mathcal{F}^{\bigvee_{i\in J}\mathbb{F}_i}$. We have

$$\lambda^{\lim}(\bigwedge_{i\in J}\mathbb{F}_i)(x) = S(\lim\mathcal{F}^{\bigwedge_{i\in J}\mathbb{F}_i}(x)) \le S(\lim\bigwedge_{i\in J}\mathcal{F}^{\mathbb{F}_i}(x))$$
$$\stackrel{(LL3')}{=} S(\bigwedge_{i\in J}\lim\mathcal{F}^{\mathbb{F}_i}(x)) = \bigvee_{i\in J} S(\lim\mathcal{F}^{\mathbb{F}_i}(x)) = \bigvee_{i\in J}\lambda^{\lim}(\mathbb{F}_i)(x).$$

□

Proposition 6.7. *PAP is isomorphic to a coreflective subcategory of $s[0,1]$-$PLIM$.*

Proposition 6.8. *Let $(X,\lim)\in|s[0,1]\text{-}LIM|$. If $(X,\lim)$ satisfies the axiom*
(LL4) *If $\mathcal{G}\in\mathcal{F}^s_{[0,1]}(X)$, $\mathcal{F}_y\in\mathcal{F}^s_{[0,1]}(X)$, $(y\in X)$, $x\in X$ then*

$$\lim\mathcal{G}(x) * \bigwedge_{y\in X}\lim\mathcal{F}_y(y) \le \lim\mathcal{G}(\mathcal{F}_{(\cdot)})(x);$$

then $(X,\lambda^{\lim})$ satisfies the axiom
(L4) *If $\mathbb{G}\in\mathbb{F}(X)$, $\mathbb{F}_y\in\mathbb{F}(X)$ $(y\in X)$, $x\in X$ then*

$$\lambda^{\lim}(\kappa(\mathbb{G},(\mathbb{F}_y))(x) \le \lambda^{\lim}(\mathbb{G})(x) + \bigvee_{y\in X}\lambda^{\lim}(\mathbb{F}_y)(y).$$

Proof. Let $\lambda^{\lim}(\mathbb{G})(x)\le\alpha$ and $\bigvee_{y\in X}\lambda^{\lim}(\mathbb{F}_y)(y)\le\beta$. Then $\lambda^{\lim}(\mathbb{F}_y)(y)\le\beta$ for all $y\in X$. Hence $S(\lim\mathcal{F}^{\mathbb{G}}(x))\le\alpha$ and $S(\lim\mathcal{F}^{\mathbb{F}_y}(y))\le\beta$ for all $y\in X$. But then $S^{-1}(\alpha)\le\lim\mathcal{F}^{\mathbb{G}}(x)$ and $S^{-1}(\beta)\le\lim\mathcal{F}^{\mathbb{F}_y}(y)$ for all $y\in X$. Therefore

$$S^{-1}(\alpha)*S^{-1}(\beta)\le\lim\mathcal{F}^{\mathbb{G}}(x)*\bigwedge_{y\in X}\lim\mathcal{F}^{\mathbb{F}_y}(y)\le\lim\mathcal{F}^{\mathbb{G}}(\mathcal{F}^{\mathbb{F}_{(\cdot)}})(x)\le\lim\mathcal{F}^{\kappa(\mathbb{G},(\mathbb{F}_y))}(x).$$

Hence

$$\lambda^{\lim}(\kappa(\mathbb{G},(\mathbb{F}_y)))(x) = S(\lim\mathcal{F}^{\kappa(\mathbb{G},(\mathbb{F}_y))}(x)) \le S(S^{-1}(\alpha)*S^{-1}(\beta))$$
$$= S(S^{-1}(S(S^{-1}(\alpha))+S(S^{-1}(\beta)))) = \alpha+\beta.$$

□

Proposition 6.9. *Let $(X,\lim)\in|s[0,1]\text{-}LIM|$. If $(X,\lim)$ satisfies the axiom*
(LLF) *If $J\neq\emptyset$, $\psi: J\longrightarrow X$, $\mathcal{G}\in\mathcal{F}^s_{[0,1]}(J)$, $\mathcal{F}_i\in\mathcal{F}^s_{[0,1]}(X)$, $(i\in J)$, $x\in X$ then*

$$\lim\psi(\mathcal{G})(x) * \bigwedge_{i\in J}\lim\mathcal{F}_i(\psi(i)) \le \lim\mathcal{G}(\mathcal{F}_{(\cdot)})(x);$$

then $(X,\lambda^{\lim})$ satisfies the axiom
(LF) *If $J\neq\emptyset$, $\psi: J\longrightarrow X$, $\mathbb{G}\in\mathbb{F}(J)$, $\mathbb{F}_i\in\mathbb{F}(X)$ $(i\in J)$, $x\in X$ then*

$$\lambda^{\lim}(\kappa(\mathbb{G},(\mathbb{F}_i))(x) \le \lambda^{\lim}(\psi(\mathbb{G}))(x) + \bigvee_{i\in J}\lambda^{\lim}(\mathbb{F}_i)(\psi(i)).$$

Proof. The proof is similar to the proof of Proposition 6.8. □

The category AP of approach spaces (see Lowen [18, 19]) has as objects the convergence approach spaces that satisfy the axioms (L3') and (L4). Equivalently we can describe the objects as the convergence approach spaces that satisfy the axiom (LF), see [2]. If we denote by $s[0,1]$-TCS the category of stratified $[0,1]$-limit spaces that satisfy the axioms (LL3') and (LL4) (or, equivalently, the axiom (LLF), see [12]), then the Propositions 6.8 and 6.9, together with Propositions 4.9 and 4.10 yield the following result.

Proposition 6.10. *AP is isomorphic to a coreflective subcategory of* $s[0,1]$-TCS.

We note that it is at present unknown if AP is also a reflective subcategory of $s[0,1]$-TCS.

7. Conclusions

We showed in this paper that the category CAP of convergence approach spaces is isomorphic to a simultaneously reflective and coreflective subcategory of $s[0,1]$-LIM, the category of stratified $[0,1]$-limit spaces if we consider the enriched cl-premonoid $([0,1],\wedge,*)$ with a continuous t-norm with surjective additive generator $S : [0,1] \longrightarrow [0,\infty]$. In the same way we saw that the categories of pre-approach spaces and of approach spaces are isomorphic to certain subcategories of stratified $[0,1]$-limit spaces. This gives us important examples for lattice-valued convergence spaces.

Another example for lattice-valued convergence spaces is given by stratified lattice-valued topological spaces [8].That $s[0,1]$-TOP, the category of stratified $[0,1]$-topological spaces is a reflective subcategory of the category of $s[0,1]$-LIM (see [10]), poses an open question:

Can approach spaces also be viewed as stratified $[0,1]$*-topological spaces?*

The answer to this question is to date unknown. The major difficulty seems to be the axiom (LP) for stratified $[0,1]$-limit spaces. This axiom is the lattice-valued version of the classical axiom that a filter converges to a point if and only if it is finer than the neighbourhood filter of the point. In the lattice-valued case with a lattice different to $L = \{0,1\}$, this axiom is no longer equivalent to the axiom (LL3'), see [11].

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Gunther Jäger, Department of Statistics, Rhodes University, 6140 Grahamstown, South Africa

E-mail address: g.jager@ru.ac.za