

MEASURES OF FUZZY SEMICOMPACTNESS IN L -FUZZY TOPOLOGICAL SPACES

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ABSTRACT. In this paper, the notion of fuzzy semicompactness degrees is introduced in L -fuzzy topological spaces by means of the implication operation of L . Characterizations of fuzzy semicompactness degrees in L -fuzzy topological spaces are obtained, and some properties of fuzzy semicompactness degrees are researched.

1. Introduction

In 1968, Chang [3] introduced fuzzy theory into topology. In Chang's fuzzy topology, open sets are fuzzy, but the topology comprising those open sets is a crisp subset of I -power set I^X . The notion of Chang's fuzzy topology was extended by Goguen to L -topology. Later many other authors gave more results and notions concerning fuzzy topology (see [1], [2], [5]- [9], [11]- [34]).

Especially, the notion of fuzzy compactness degrees and L -fuzzy semicompactness are introduced in L -fuzzy topological spaces in [14] and [29], respectively. Based on the idea of [14], a natural problem is: Can the degrees of fuzzy semicompactness be defined in an L -fuzzy topological space?

The aim of this paper is to present a new notion of fuzzy semicompactness degrees in L -fuzzy topological spaces by means of the implication operation of L . We will also characterize the fuzzy semicompactness degrees in L -fuzzy topological spaces, and research some properties of fuzzy semicompactness degrees.

2. Preliminaries

Throughout this paper, (L, \vee, \wedge, ι) is a complete DeMorgan frame (i.e., a complete lattice with order-reversing involution satisfying joint-infinite distributive law) [10, 21]. By \perp and \top we denote the smallest element and the largest element in L , respectively. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero coprime elements in L is denoted by $M(L)$. We say that a is wedge below b in L , denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$ [4]. $\beta(b)$ is the greatest minimal family of b , $\beta^*(b) = \beta(b) \cap M(L)$. $\alpha(b)$ is the greatest maximal family of b , $\alpha^*(b) = \alpha(b) \cap P(L)$.

Received: December 2011; Revised: November 2012; Accepted: March 2013

Key words and phrases: L -fuzzy topology, Implication operation, Fuzzy semicompactness, Fuzzy semicompactness degree.

2010 AMS Classification: 54A40, 54D30, 03E72.

In a complete DeMorgan frame L , there exists a binary operation \rightarrow . Explicitly the implication is given by $a \rightarrow b = \bigvee\{c \in L : a \wedge c \leq b\}$.

It is easy to check the following properties of \rightarrow .

- (1) $(a \rightarrow b) \geq c \Leftrightarrow a \wedge c \leq b$.
- (2) $a \rightarrow b = \top \Leftrightarrow a \leq b$.
- (3) $a \rightarrow \bigwedge_i b_i = \bigwedge_i (a \rightarrow b_i)$.
- (4) $(\bigvee_i a_i) \rightarrow b = \bigwedge_i (a_i \rightarrow b)$.
- (5) $a \leq b \Rightarrow a \rightarrow c \geq b \rightarrow c, c \rightarrow a \leq c \rightarrow b$.
- (6) $(a \wedge b) \rightarrow c = a \rightarrow (b \rightarrow c)$.

We interpret $[a \leq b]$ as the degree to which $a \leq b$, then $[a \leq b] = a \rightarrow b$.

Let X be a nonempty set, L^X the set of all L -subsets on X , \perp the smallest element of L^X and \top the largest element of L^X . Then, L -fuzzy topology on a set X is a mapping $\mathcal{T} : L^X \rightarrow L$ which satisfies the following conditions:

- (1) $\mathcal{T}(\top) = \mathcal{T}(\perp) = \top$.
- (2) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V) \quad (\forall U, V \in L^X)$.
- (3) $\mathcal{T}\left(\bigvee_{j \in J} U_j\right) \geq \bigwedge_{j \in J} \mathcal{T}(U_j) \quad (\forall \{U_j\}_{j \in J} \subseteq L^X)$.

The pair (X, \mathcal{T}) is called an L -fuzzy topological space. $\mathcal{T}(U)$ is called the degree of openness of U , $\mathcal{T}^*(U) = \mathcal{T}(U')$ is called the degree of closedness of U , where U' is the L -complement of U . For any family $\mathcal{U} \subseteq L^X$, $\mathcal{T}(\mathcal{U}) = \bigwedge_{A \in \mathcal{U}} \mathcal{T}(A)$ is called the degree of openness of \mathcal{U} .

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ . For any $a \in L$, \underline{a} denotes a constant value mapping from X to L , its value is a .

Definition 2.1. [31] An L -fuzzy inclusion on X is a mapping $\tilde{c} : L^X \times L^X \rightarrow L$ defined by the equality $\tilde{c}(A, B) = \bigwedge_{x \in X} (A'(x) \vee B(x))$.

In this paper, we will write $[A \tilde{c} B]$ instead of $\tilde{c}(A, B)$.

Definition 2.2. [25] Let $a \in L \setminus \{\top\}$ and $G \in L^X$. A subfamily \mathcal{U} in L^X is said to be

- (1) an a -shading of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \not\leq a$.
- (2) a strong a -shading of G if $\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \not\leq a$.

Definition 2.3. [25] Let $a \in L \setminus \{\perp\}$ and $G \in L^X$. A subfamily \mathcal{P} in L^X is said to be

- (1) an a -remote family of G if for any $x \in X$, it follows that $G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a$.
- (2) a strong a -remote family of G if $\bigvee_{x \in X} (G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x)) \not\geq a$.
- (3) a β_a -cover of G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee_{A \in \mathcal{P}} A(x))$.
- (4) a strong β_a -cover of G if for any $x \in X$, it follows that

$$a \in \beta\left(\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{P}} A(x))\right).$$

- (5) a Q_a -cover of G if $a \leq \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{P}} A(x))$.

Definition 2.4. [28] Let \mathcal{T} be an L -fuzzy topology on X . For any $A \in L^X$, define a mapping $\mathcal{T}_s : L^X \rightarrow L$ by

$$\mathcal{T}_s(A) = \bigvee_{B \leq A} (\mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec A} \bigwedge_{x_\lambda \not\prec D \geq B} (\mathcal{T}(D'))').$$

Then \mathcal{T}_s is called the L -fuzzy semiopen operator induced by \mathcal{T} , where $\mathcal{T}_s(A)$ can be regarded as the degree to which A is semiopen and $\mathcal{T}_s^*(B) = \mathcal{T}_s(B')$ can be regarded as the degree to which B is semiclosed. For any family $\mathcal{U} \subseteq L^X$, $\mathcal{T}_s(\mathcal{U}) = \bigwedge_{A \in \mathcal{U}} \mathcal{T}_s(A)$ is called the degree of semiopenness of \mathcal{U} .

Theorem 2.5. [28] Let \mathcal{T} be an L -fuzzy topology on X and let \mathcal{T}_s be the L -fuzzy semiopen operator induced by \mathcal{T} . Then $\mathcal{T}(A) \leq \mathcal{T}_s(A)$ for any $A \in L^X$.

Definition 2.6. [28, 29] A mapping $f : X \rightarrow Y$ between two L -fuzzy topological spaces (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) is called

- (1) semicontinuous if $\mathcal{T}_2(U) \leq (\mathcal{T}_1)_s(f_L^{\leftarrow}(U))$ holds for any $U \in L^Y$.
- (2) irresolute if $(\mathcal{T}_2)_s(U) \leq (\mathcal{T}_1)_s(f_L^{\leftarrow}(U))$ holds for any $U \in L^Y$.
- (3) strongly irresolute if $(\mathcal{T}_2)_s(U) \leq \mathcal{T}_1(f_L^{\leftarrow}(U))$ holds for any $U \in L^Y$.

Definition 2.7. [29] Let (X, \mathcal{T}) be an L -fuzzy topological space. $G \in L^X$ is said to be L -fuzzy semicompact if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{A \in \mathcal{U}} \mathcal{T}_s(A) \wedge \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)).$$

Definition 2.8. [14] Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. The fuzzy compactness degree $cd_{\mathcal{T}}$ of G is defined as

$$cd_{\mathcal{T}}(G) = \bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}(\mathcal{U}) \rightarrow (\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)))).$$

Theorem 2.9. [21, 25] Let $f : X \rightarrow Y$ be a set mapping and $f_L^{\rightarrow} : L^X \rightarrow L^Y$ is induced by f . Then for any $\mathcal{P} \subseteq L^Y$, we have that

$$\bigwedge_{y \in Y} (f_L^{\rightarrow}(G)'(y) \vee \bigvee_{B \in \mathcal{P}} B(y)) = \bigwedge_{x \in X} (G'(x) \vee \bigvee_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x)).$$

3. Measures of Fuzzy Semicompactness

In [14], Li and Shi generalized the notion of fuzzy compactness to L -fuzzy topological spaces, and gave the definition of fuzzy compactness degrees in L -fuzzy topological spaces. Based on [14], we will generalize the notion of fuzzy semicompactness to L -fuzzy topological spaces. In order to do this, let us recall fuzzy semicompactness in L -topology [23].

Let (X, \mathcal{T}) be an L -topological space and $G \in L^X$. G is fuzzy semicompactness if and only if for every family \mathcal{U} of semiopen L -sets, it follows that

$$\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)).$$

This implies that for every family \mathcal{U} of semiopen L -sets,

$$\left[[G\tilde{\ } \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [G\tilde{\ } \bigvee \mathcal{V}] \right] = \top.$$

We know that an L -topology \mathcal{T} can be looked as a special L -fuzzy topology. Therefore, $A \in L^X$ is a semiopen set if and only if $\mathcal{T}_s(A) = \top$ [28]. Thus G is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\mathcal{T}_s(\mathcal{U}) \leq \left[[G\tilde{\ } \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [G\tilde{\ } \bigvee \mathcal{V}] \right].$$

Therefore we can naturally generalize the notion of fuzzy semicompactness degrees to L -fuzzy topological spaces as follows:

Definition 3.1. Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. The fuzzy semicompactness degree $scd_{\mathcal{T}}$ of G is defined as

$$\begin{aligned} scd_{\mathcal{T}}(G) &= \bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \rightarrow ([G\tilde{\ } \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [G\tilde{\ } \bigvee \mathcal{V}])) \\ &= \bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \rightarrow (\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \rightarrow \bigvee_{\mathcal{V} \in 2(\mathcal{U})} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)))). \end{aligned}$$

Theorem 3.2. Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \leq cd_{\mathcal{T}}(G)$.

Proof. Straightforward. \square

Theorem 3.3. Let (X, \mathcal{T}) be an L -topological space and $G \in L^X$. G is fuzzy semicompactness in (X, \mathcal{T}) if and only if $scd_{\chi_{\mathcal{T}}}(G) = \top$.

Proof. Let (X, \mathcal{T}) be an L -topological space. The mapping $\chi_{\mathcal{T}} : L^X \rightarrow L$ defined by

$$\chi_{\mathcal{T}}(A) = \begin{cases} \top, & A \in \mathcal{T}, \\ \perp, & A \notin \mathcal{T}. \end{cases}$$

is a special L -fuzzy topology. Then $A \in L^X$ is a semiopen set in L -topology \mathcal{T} if and only if $(\chi_{\mathcal{T}})_s(A) = \top$. Thus by the definition of fuzzy semicompactness and the properties of \rightarrow , we know that G is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$(\chi_{\mathcal{T}})_s(\mathcal{U}) \leq \left[[G\tilde{\ } \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [G\tilde{\ } \bigvee \mathcal{V}] \right].$$

This implies that G is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$(\chi_{\mathcal{T}})_s(\mathcal{U}) \rightarrow ([G\tilde{\ } \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [G\tilde{\ } \bigvee \mathcal{V}]) = \top.$$

By the definition of $scd_{\chi_{\mathcal{T}}}$, the conclusion is hold. \square

Theorem 3.4. Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. G is L -fuzzy semicompactness in (X, \mathcal{T}) if and only if $scd_{\mathcal{T}}(G) = \top$.

Proof. By the definition of L -fuzzy semicompactness, we know that G is L -fuzzy semicompactness in (X, \mathcal{T}) if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$(\mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\ } \bigvee \mathcal{U}]) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}].$$

By the properties of \rightarrow , we obtain that G is L -fuzzy semicompactness in (X, \mathcal{T}) if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\mathcal{T}_s(\mathcal{U}) \rightarrow ([G\tilde{\ } \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}]) = \top.$$

By the definition of $scd_{\mathcal{T}}$, the conclusion is hold. \square

Lemma 3.5. *Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \geq a$ if and only if for any $\mathcal{U} \subseteq L^X$,*

$$\mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\ } \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}].$$

Proof. For any $a \in L$, $scd_{\mathcal{T}}(G) \geq a$, i.e.,

$$\bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \rightarrow ([G\tilde{\ } \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}])) \geq a$$

if and only if for any $\mathcal{U} \subseteq L^X$,

$$\mathcal{T}_s(\mathcal{U}) \rightarrow ([G\tilde{\ } \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}]) \geq a$$

if and only if (by the property (6) of \rightarrow) for any $\mathcal{U} \subseteq L^X$,

$$(\mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\ } \bigvee \mathcal{U}]) \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}] \geq a$$

if and only if (by the property (1) of \rightarrow) for any $\mathcal{U} \subseteq L^X$,

$$\mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\ } \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}].$$

Theorem 3.6. *Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \geq a$ if and only if for any $\mathcal{P} \subseteq L^X$,*

$$\bigvee_{F \in \mathcal{P}} \mathcal{T}_s^*(F) \vee (\bigvee_{x \in X} (G(x) \wedge \bigwedge_{F \in \mathcal{P}} F(x))) \vee a' \geq \bigwedge_{\mathcal{H} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} (G(x) \wedge \bigwedge_{F \in \mathcal{H}} F(x)).$$

Proof. It can be easily obtained by Lemma 3.5 and the definition of \mathcal{T}_s^* . \square

Theorem 3.7. *Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. Then*

$$scd_{\mathcal{T}}(G) = \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\ } \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\ } \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\}.$$

Proof. By Lemma 3.5, we know that $scd_{\mathcal{T}}(G)$ is an upper bound of

$$\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\mathcal{C}} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\mathcal{C}} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\}.$$

Since

$$scd_{\mathcal{T}}(G) = \bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \rightarrow ([G\tilde{\mathcal{C}} \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\mathcal{C}} \bigvee \mathcal{V}])),$$

then for every family $\mathcal{U} \subseteq L^X$, we have

$$\begin{aligned} scd_{\mathcal{T}}(G) &\leq \mathcal{T}_s(\mathcal{U}) \rightarrow ([G\tilde{\mathcal{C}} \bigvee \mathcal{U}] \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\mathcal{C}} \bigvee \mathcal{V}]) \\ &= (\mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\mathcal{C}} \bigvee \mathcal{U}]) \rightarrow \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\mathcal{C}} \bigvee \mathcal{V}]. \end{aligned}$$

By the property (1) of \rightarrow , we obtain that for every family $\mathcal{U} \subseteq L^X$,

$$\mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\mathcal{C}} \bigvee \mathcal{U}] \wedge scd_{\mathcal{T}}(G) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\mathcal{C}} \bigvee \mathcal{V}],$$

thus

$$scd_{\mathcal{T}}(G) \in \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\mathcal{C}} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\mathcal{C}} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\}.$$

Therefore, the conclusion is hold. \square

In order to write simply, for any mapping $\mathcal{T} : L^X \rightarrow L$, denote $\mathcal{T}_b = \{A \in L^X : \mathcal{T}(A) \geq b\}$.

Theorem 3.8. *Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X, a \in L \setminus \{\perp\}$. The following conditions are equivalent:*

- (1) $scd_{\mathcal{T}}(G) \geq a$.
- (2) For any $b \in P(L)$, $b \not\geq a$, each strong b -shading \mathcal{U} of G with $\mathcal{T}_s(\mathcal{U}) \not\leq b$ has a finite subfamily \mathcal{V} which is a strong b -shading of G .
- (3) For any $b \in P(L)$, $b \not\geq a$, each strong b -shading \mathcal{U} of G with $\mathcal{T}_s(\mathcal{U}) \not\leq b$, there exists a finite subfamily \mathcal{V} of \mathcal{U} and $r \in \alpha^*(b)$ such that \mathcal{V} is an r -shading of G .
- (4) For any $b \in P(L)$, $b \not\geq a$, each strong b -shading \mathcal{U} of G with $\mathcal{T}_s(\mathcal{U}) \not\leq b$, there exists a finite subfamily \mathcal{V} of \mathcal{U} and $r \in \alpha^*(b)$ such that \mathcal{V} is a strong r -shading of G .
- (5) For any $b \in M(L)$, $b \not\leq a'$, each strong b -remote family \mathcal{P} of G with $\mathcal{T}_s^*(\mathcal{P}) \not\leq b'$ has a finite subfamily \mathcal{H} which is a strong b -remote family of G .
- (6) For any $b \in M(L)$, $b \not\leq a'$, each strong b -remote family \mathcal{P} of G with $\mathcal{T}_s^*(\mathcal{P}) \not\leq b'$, there exists a finite subfamily \mathcal{H} of \mathcal{P} and $r \in \beta^*(b)$ such that \mathcal{H} is an r -remote family of G .
- (7) For any $b \in M(L)$, $b \not\leq a'$, each strong b -remote family \mathcal{P} of G with $\mathcal{T}_s^*(\mathcal{P}) \not\leq b'$, there exists a finite subfamily \mathcal{H} of \mathcal{P} and $r \in \beta^*(b)$ such that \mathcal{H} is a strong r -remote family of G .
- (8) For any $b \leq a, r \in \beta(b)$, $b, r \neq \perp$, each Q_b -cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily \mathcal{V} which is a Q_r -cover of G .
- (9) For any $b \leq a, r \in \beta(b)$, $b, r \neq \perp$, each Q_b -cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily \mathcal{V} which is a strong β_r -cover of G .

(10) For any $b \leq a, r \in \beta(b), b, r \neq \perp$, each Q_b -cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily \mathcal{V} which is a β_r -cover of G .

(11) For any $b \leq a, r \in \beta(b), b, r \neq \perp$, each strong β_b -cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily \mathcal{V} which is a Q_r -cover of G .

(12) For any $b \leq a, r \in \beta(b), b, r \neq \perp$, each strong β_b -cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily \mathcal{V} which is a strong β_r -cover of G .

(13) For any $b \leq a, r \in \beta(b), b, r \neq \perp$, each strong β_b -cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily \mathcal{V} which is a β_r -cover of G .

In Theorem 3.8 (8)-(13), if we replace $b, r \neq \perp$ and $r \in \beta(b)$ with $b \in M(L)$ and $r \in \beta^*(b)$, then the conclusions are still right.

Theorem 3.9. Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X, a \in L \setminus \{\perp\}$. If for any $c, d \in L, \beta(c \wedge d) = \beta(c) \wedge \beta(d)$. Then the following conditions are equivalent:

(1) $scd_{\mathcal{T}}(G) \geq a$.

(2) For any $b \in \beta(a), b \neq \perp$, each strong β_b -cover \mathcal{U} of G with $b \in \beta(\mathcal{T}_s(\mathcal{U}))$ has a finite subfamily \mathcal{V} which is a Q_b -cover of G .

(3) For any $b \in \beta(a), b \neq \perp$, each strong β_b -cover \mathcal{U} of G with $b \in \beta(\mathcal{T}_s(\mathcal{U}))$ has a finite subfamily \mathcal{V} which is a strong β_b -cover of G .

(4) For any $b \in \beta(a), b \neq \perp$, each strong β_b -cover \mathcal{U} of G with $b \in \beta(\mathcal{T}_s(\mathcal{U}))$ has a finite subfamily \mathcal{V} which is a β_b -cover of G .

4. Properties of Fuzzy Semicompactness Degrees

Theorem 4.1. Let (X, \mathcal{T}) be an L -fuzzy topological space and $G, H \in L^X$. Then $scd_{\mathcal{T}}(G \vee H) \geq scd_{\mathcal{T}}(G) \wedge scd_{\mathcal{T}}(H)$.

Proof. By Theorem 3.7 we have

$$\begin{aligned}
 scd_{\mathcal{T}}(G \vee H) &= \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [(G \vee H) \tilde{c} \bigvee \mathcal{U}] \wedge a \\
 &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(G \vee H) \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} \\
 &= \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{c} \bigvee \mathcal{U}] \wedge [H \tilde{c} \bigvee \mathcal{U}] \wedge a \\
 &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} ([G \tilde{c} \bigvee \mathcal{V}] \wedge [H \tilde{c} \bigvee \mathcal{V}]), \forall \mathcal{U} \subseteq L^X \} \\
 &\geq \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{c} \bigvee \mathcal{U}] \wedge a \\
 &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} \wedge \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [H \tilde{c} \bigvee \mathcal{U}] \wedge a \\
 &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [H \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} = scd_{\mathcal{T}}(G) \wedge scd_{\mathcal{T}}(H).
 \end{aligned}$$

□

Theorem 4.2. Let (X, \mathcal{T}) be an L -fuzzy topological space and $G, H \in L^X$. Then $scd_{\mathcal{T}}(G \wedge H) \geq scd_{\mathcal{T}}(G) \wedge \mathcal{T}_s^*(H)$.

Proof. By Theorem 3.7 we have

$$\begin{aligned}
scd_{\mathcal{T}}(G \wedge H) &= \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [(G \wedge H) \tilde{c} \bigvee \mathcal{U}] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [(G \wedge H) \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} \\
&= \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{c} (H' \vee \bigvee \mathcal{U})] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [G \tilde{c} (H' \vee \bigvee \mathcal{V})], \forall \mathcal{U} \subseteq L^X \} \\
&\geq \bigvee \{a \wedge \mathcal{T}_s^*(H) : \mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{c} \bigvee \mathcal{U}] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [G \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} = scd_{\mathcal{T}}(G) \wedge \mathcal{T}_s^*(H). \quad \square
\end{aligned}$$

Corollary 4.3. *Let (X, \mathcal{T}) be an L -fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \geq scd_{\mathcal{T}}(\perp) \wedge \mathcal{T}_s^*(G)$.*

Theorem 4.4. *Let $(X, \mathcal{T}_1), (X, \mathcal{T}_2)$ be two L -fuzzy topological spaces and satisfy $\mathcal{T}_1 \leq \mathcal{T}_2$, $G \in L^X$. Then $scd_{\mathcal{T}_2}(G) \leq scd_{\mathcal{T}_1}(G)$.*

Corollary 4.5. *Let (X, \mathcal{T}) be an L -fuzzy topological space and let \mathcal{B} be a base or subbase [7, 8, 34] of \mathcal{T} , $G \in L^X$. Then $scd_{\mathcal{T}}(G) \leq scd_{\mathcal{B}}(G)$.*

Theorem 4.6. *Let $f : X \rightarrow Y$ be a set mapping, \mathcal{T}_1 be an L -fuzzy topology on X , \mathcal{T}_2 be an L -fuzzy topology on Y , and $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be an L -fuzzy strong irresolute mapping. Then for any $G \in L^X$, $cd_{\mathcal{T}_1}(G) \leq scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G))$.*

Proof. For any $G \in L^X$, we have

$$\begin{aligned}
scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G)) &= \bigvee \{a \in L : (\mathcal{T}_2)_s(\mathcal{U}) \wedge [f_L^{\rightarrow}(G) \tilde{c} \bigvee \mathcal{U}] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [f_L^{\rightarrow}(G) \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} \\
&\geq \bigvee \{a \in L : \mathcal{T}_1(f_L^{\rightarrow}(\mathcal{U})) \wedge [G \tilde{c} \bigvee f_L^{\rightarrow}(\mathcal{U})] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [G \tilde{c} \bigvee f_L^{\rightarrow}(\mathcal{V})], \forall \mathcal{U} \subseteq L^X \} \geq cd_{\mathcal{T}_1}(G). \quad \square
\end{aligned}$$

Theorem 4.7. *Let $f : X \rightarrow Y$ be a set mapping, \mathcal{T}_1 be an L -fuzzy topology on X , \mathcal{T}_2 be an L -fuzzy topology on Y , and $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be an L -fuzzy irresolute mapping. Then for any $G \in L^X$, $scd_{\mathcal{T}_1}(G) \leq scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G))$.*

Proof. For any $G \in L^X$, we have

$$\begin{aligned}
scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G)) &= \bigvee \{a \in L : (\mathcal{T}_2)_s(\mathcal{U}) \wedge [f_L^{\rightarrow}(G) \tilde{c} \bigvee \mathcal{U}] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [f_L^{\rightarrow}(G) \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \} \\
&\geq \bigvee \{a \in L : (\mathcal{T}_1)_s(f_L^{\rightarrow}(\mathcal{U})) \wedge [G \tilde{c} \bigvee f_L^{\rightarrow}(\mathcal{U})] \wedge a \\
&\leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} [G \tilde{c} \bigvee f_L^{\rightarrow}(\mathcal{V})], \forall \mathcal{U} \subseteq L^X \} \geq scd_{\mathcal{T}_1}(G). \quad \square
\end{aligned}$$

Theorem 4.8. *Let $f : X \rightarrow Y$ be a set mapping, \mathcal{T}_1 be an L -fuzzy topology on X , \mathcal{T}_2 be an L -fuzzy topology on Y , and $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be an L -fuzzy semicontinuous mapping. Then for any $G \in L^X$, $s cd_{\mathcal{T}_1}(G) \leq cd_{\mathcal{T}_2}(f_L^\rightarrow(G))$.*

Proof. For any $G \in L^X$, we have

$$\begin{aligned} cd_{\mathcal{T}_2}(f_L^\rightarrow(G)) &= \bigvee \{a \in L : \mathcal{T}_2(\mathcal{U}) \wedge [f_L^\rightarrow(G) \tilde{c} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [f_L^\rightarrow(G) \tilde{c} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \\ &\geq \bigvee \{a \in L : (\mathcal{T}_1)_s(f_L^\leftarrow(\mathcal{U})) \wedge [G \tilde{c} \bigvee f_L^\leftarrow(\mathcal{U})] \wedge a \\ &\leq \bigvee_{\mathcal{V} \in 2(\mathcal{U})} [G \tilde{c} \bigvee f_L^\leftarrow(\mathcal{V})], \forall \mathcal{U} \subseteq L^X\} \geq s cd_{\mathcal{T}_1}(G). \end{aligned}$$

□

Acknowledgements. This article is supported by the National Natural Science Foundation of P.R. China (Grant No. 11071151) and the Natural Science Foundation of Shaanxi Province, P.R. China (Grant No. 2010JM1005).

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