THE NUMBER OF FUZZY SUBGROUPS OF SOME NON-ABELIAN GROUPS

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Abstract. In this paper, we compute the number of fuzzy subgroups of some classes of non-abeilan groups. Explicit formulas are given for dihedral groups D_{2n}, quasi-dihedral groups QD_{2^n}, generalized quaternion groups Q_{4n} and modular p-groups M_{p^n}.

Introduction

One of the most important problems of fuzzy group theory is the classification of all the fuzzy subgroups of a finite group. Several papers have treated the problem in the particular cases of finite abelian groups. Laszlo [4] studied the construction of fuzzy subgroups of groups of order at most 6. Zhang and Zou [15] have determined the number of fuzzy subgroups of cyclic groups of order p^n, where p is a prime number. Murali and Makamba [7, 8] have considered a similar problem and computed the number of fuzzy subgroups of abelian groups of order $p^m q^n$, where p and q are distinct primes. Tărnăuceanu and Bentea [13] established recurrence relations for the number of fuzzy subgroups of two classes of finite abelian groups; finite cyclic groups and finite elementary abelian p-groups. Their result improved the Murali's works in [7, 8]. Ngcibi, Murali and Makamba [9] computed the number of fuzzy subgroups of abelian p-groups of rank two. The first step in classifying the fuzzy subgroups of a finite non-abelian groups is made by Tărnăuceanu [12]. He developed a general method to count the number of distinct fuzzy subgroups of such groups and found the number of fuzzy subgroups of a particular case of dihedral groups. In this paper, we determine the number of fuzzy subgroups of some classes of non-abelian groups including dihedral groups D_{2n}, generalized quaternion groups Q_{4n}, quasi-dihedral groups QD_{2^n} ($n \geq 4$) and modular p-groups M_{p^n} ($n \geq 3$). Note that, the groups D_{2^n}, Q_{2^n}, QD_{2^n} and M_{p^n} together with abelian p-groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$ constitute all finite p-groups of order p^n having a maximal cyclic subgroup of order p^{n-1}.

1. Preliminaries

We begin with recalling some basic notions and results of fuzzy subgroups (see [5, 6, 10] for more details). Let G be a group and $\mu : G \to [0, 1]$ be a fuzzy subset of G. Then μ is a fuzzy subgroup of G if it satisfies the following two conditions:

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(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$,
(ii) $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

Then, we have $\mu(x^{-1}) = \mu(x)$ for any $x \in G$, and $\mu(1) = \max \mu(G)$. For each $\alpha \in [0,1]$, the level subset corresponding to α is defined as $\mu_\alpha = \{x \in G : \mu(x) \geq \alpha\}$. These subsets are useful in characterization of fuzzy subgroups, in such a way that a fuzzy subset μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G.

Let $\sim$ be the natural equivalence relation on the set of all fuzzy subsets of G.

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y) \text{ for all } x, y \in G).$$

Utilizing the above equivalence relation, the fuzzy subgroups of G can be classified up to equivalence classes in such a way that two fuzzy subgroups μ and η of G are distinct if $\mu \not\sim \eta$.

The above equivalence relation has a close connection to the concept of level subgroups. In this way, suppose that G is a finite group and $\mu : G \to [0,1]$ is a fuzzy subgroup of G. Let $\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Then μ determines the following chain of subgroups of G ending in G.

$$\mu_{\alpha_1} \subseteq \mu_{\alpha_2} \subseteq \cdots \subseteq \mu_{\alpha_n} = G. \tag{1}$$

Moreover, for any $x \in G$ and $i = 1, 2, \cdots, n$, we have

$$\mu(x) = \alpha_i \Longleftrightarrow x \in \mu_{\alpha_i} \setminus \mu_{\alpha_{i-1}},$$

where by convention, we set $\mu_{\alpha_0} = \emptyset$. Volf [14] gives a necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ in such away that $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is, they determine the same chain of subgroups of type (1). Hence, there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G, which end in G. Clearly, in any group with at least two elements there are more distinct fuzzy subgroups than subgroups. Also, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of G, that is computing the number of all chains of subgroups of G that terminate in G.

The most important result that we will use frequently, is the following result for the number of fuzzy subgroups of finite cyclic groups.

Proposition 1.1. [13] *Let G be a finite cyclic group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. Then the number of all distinct fuzzy subgroups of G is given by*

$$F_G = 2^{\sum_{r=1}^{k}\alpha_r} \sum_{i_2=0}^{\alpha_2}\sum_{i_3=0}^{\alpha_3}\cdots\sum_{i_k=0}^{\alpha_k}\left(-\frac{1}{2}\right)^{\sum_{r=2}^{k} i_r}\prod_{r=2}^{k}\binom{\alpha_r}{i_r}\binom{\alpha_1+\sum_{m=2}^{r}(\alpha_m - i_m)}{\alpha_r},$$

where the above iterated sums are equal to 1 for $k = 1$.

2. Dihedral Groups

Tărnăuceanu [12] uses enumerations on maximal chains of subgroups of dihedral groups to obtain the following results.

Proposition 2.1. [12] *The number of all fuzzy subgroups of the dihedral group D_{2p^m} is*

$$F_{D_{2p^m}} = \frac{2^m}{p-1}\left(p^{m+1} + p - 2\right).$$

In particular, $F_{D_{2^m}} = 2^{2m-1}$.

Proposition 2.2. [12] *The number of all fuzzy subgroups of the dihedral group D_{2p^mq} is*

$$F_{D_{2p^mq}} = \frac{2^m}{(p-1)^3}[(m+2)p^{m+3}q + 2p^{m+3} - (2m+5)p^{m+2}q - \\ - 3p^{m+2} + (m+3)p^{m+1}q + p^{m+1} + (m+2)p^3 - p^2q - \\ (4m+9)p^2 + 3pq + (5m+11)p - 2p - (2m+4)]$$

In particular,

$$F(D_{2pq}) = 2(3pq + 2p + 2q + 6).$$

Recall that there is a one-to-one correspondence between the fuzzy subgroups of a group G and its chains of subgroups, which end in G. Hence, in what follows, we shall compute the number of all chains of subgroups of dihedral groups G ending in G, which results in the number of all fuzzy subgroups of G. For this we shall make use of Proposition 1.1 all over the proofs.

Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1}\rangle$ be the dihedral group of order $2n$. Then the subgroups of D_{2n} are

cyclic group $\langle a^{\frac{n}{k}}\rangle$ of order k, where k divides n,
cyclic groups $\langle a^i b\rangle$ of order 2, where $i = 0, \ldots, n-1$,
dihedral groups $\langle a^{\frac{n}{k}}, a^i b\rangle$ of order $2k$, where k divides n and $i = 0, \ldots, n/k - 1$.

Utilizing the above statements we are able to obtain the number of fuzzy subgroups of dihedral groups. To end this, we need to define a special kind of chains of subgroups. A chain of subgroups of a group is said to be cyclic if all its terms except the whole group are cyclic.

Theorem 2.3. *The number of all fuzzy subgroups of the dihedral group D_{2n} is*

$$F_{D_{2n}} = \sum_{k|n} \frac{n}{k} F_{\mathbb{Z}_{\frac{n}{k}}}(k + F_{\mathbb{Z}_k}) - (2n-1)F_{\mathbb{Z}_n} + n.$$

Proof. Let $G = D_{2n}$ and $H_n \subset H_{n-1} \subset \cdots \subset H_1 \subset H_0 = G$ be an arbitrary chain of subgroups of G, which ends in G. Clearly, the group G itself is a chain. Hence we further suppose that the chain contains at least one proper subgroup, that is $n \geq 1$. We proceed in two steps:

Case 1: All of the subgroups in the chain except G are cyclic. Then either $H_1 = \langle a^{\frac{n}{k}}\rangle$ (k divides n) or $H_1 = \langle a^i b\rangle$ ($0 \leq i < n$). In the former, we have $F_{\mathbb{Z}_k}$ possible chains and in the latter, we have just two possible chains, namely $\langle a^i b\rangle \subset G$ and $1 \subset \langle a^i b\rangle \subset G$.

Case 2: The chain contains a non-cyclic proper subgroup. Let H be the smallest non-cyclic subgroup in the chain. Then $H = \langle a^{\frac{n}{k}}, a^i b\rangle$ for some divisor k of n and

$i = 0, \ldots, n/k - 1$. The number of such chains with H fixed as the smallest non-cyclic subgroup equals the number of cyclic chains of H, which end in H times the number of chains of subgroups of G containing H, which begin and end in 1 and G, respectively. Since H is a dihedral group of order $2k$, the number of its cyclic chains is indeed determined in case (1) and equals $1+2d+\sum_{d|k} F_{\mathbb{Z}_d}$. On the other hand, for every subgroup K, such that $H \leq K \leq G$ we have $K = \langle a^{\frac{n}{k'}}, a^i b\rangle$, where k divides k' and k' divides n. Hence, there exists a bijection between the chain of subgroups of G containing H, which begin in H and end in G, and the number of chains of subgroups of $\mathbb{Z}_{\frac{n}{k}}$, which begin in 1 and end in $\mathbb{Z}_{\frac{n}{k}}$. Therefore, the number of chain of subgroups of containing H, which begin in H and end in G equals $\frac{1}{2}F_{\frac{n}{k}}$. Thus the number of chains of G with H as the smallest non-cyclic subgroup equals

$$\frac{1}{2}F_{\frac{n}{k}}\left(1+2k+\sum_{d|k} F_{\mathbb{Z}_d}\right).$$

Using the above results, it follows that

$$F_G = 1+2n+\sum_{d|n} F_{\mathbb{Z}_d} + \sum_{\substack{k|n \\ k\neq 1,n}} \frac{1}{2}\cdot\frac{n}{k}\cdot F_{\mathbb{Z}_{\frac{n}{k}}}\left(1+2k+\sum_{d|k} F_{\mathbb{Z}_d}\right).$$

Since $F_{\mathbb{Z}_m} = 1+\sum_{h|m} F_{\mathbb{Z}_h}$, we may simplify the above formula and obtain

$$F_G = \sum_{k|n} \frac{n}{k} F_{\mathbb{Z}_{\frac{n}{k}}}(k+F_{\mathbb{Z}_k}) - (2n-1)F_{\mathbb{Z}_n} + n.$$

The proof is complete. □

3. Generalized Quaternion Groups, Quasi-dihedral Groups and Modular p-groups

In this section, we shall use Theorem 2.3 to obtain the number of fuzzy subgroups of three remained classes of groups, namely generalized quaternion groups, quasi-dihedral groups and modular p-groups.

Let $Q_{4n} = \langle a, b : a^{2n} = 1, a^n = b^2, a^b = a^{-1}\rangle$ be the generalized quaternion group of order $4n$. Then $Z(QD_{2^n}) = \langle a^n\rangle$ and $Q_{4n}/Z(Q_{4n}) \cong D_{2n}$. Moreover, if H is a subgroup of G such that $H \cap Z(G) = 1$, then $H \leq \langle a^{2^k}\rangle \cong \mathbb{Z}_m$ is a cyclic group of odd order, in which $n = 2^k m$, for some odd integer m.

Theorem 3.1. *The number of all fuzzy subgroups of the generalized quaternion group Q_{4n} is*

$$F_{Q_{4n}} = F_{D_{2n}} + \sum_{d|m} F_{\mathbb{Z}_d} F_{D_{\frac{2n}{d}}},$$

where m is an odd integer such that $n = 2^k m$, for some k.

Proof. Let $G = Q_{4n}$. We first observe that the number of chains of subgroups of G ending in G equals to the number of chains of subgroups of D_{2n} ending in D_{2n}. Now consider the following chain of subgroups of G ending in G and containing a subgroup H, which does not contain $Z(G)$.

$$\cdots \subseteq H \subseteq \cdots \subseteq G.$$

Without loss of generality we may assume that H is a maximal term in the chain that does not contain $Z(G)$. Then $H \leq \langle a^{2^k} \rangle$, where $n = 2^k m$ for some odd integer m. Hence $H \cong \mathbb{Z}_d$ for some divisor d of m and the number of these chains with H fixed as the maximal subgroup in the chain that does not contain $Z(G)$ is equal to the number of chains of H ending in H times the number of chains of $G/L \cong D_{\frac{2n}{d}}$ ending in G/L, where $L = \langle H, a^n \rangle \cong \mathbb{Z}_{2d}$. The proof is complete. □

Let $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}-1} \rangle$ $(n \geq 4)$ be the quasi-dihedral group of order 2^n. Then $Z(QD_{2^n}) = \langle a^{2^{n-2}} \rangle$ and $QD_{2^n}/Z(QD_{2^n}) \cong D_{2^{n-1}}$. Moreover, if H is a subgroup of G such that $H \cap Z(G) = 1$, then either $H = 1$ or $H = \langle a^{2i} b \rangle$, for some $0 \leq i < 2^{n-2}$.

Theorem 3.2. *The number of all fuzzy subgroups of the quasi-dihedral* 2*-group* QD_{2^n} *(*$n \geq 4$*) is*

$$F_{QD_{2^n}} = 3 \cdot 2^{2n-3}.$$

Proof. Let $G = QD_{2^n}$. The same as for generalized quaternion groups, the number of chains of subgroups of G ending in G equals to the number of chains of subgroups of $D_{2^{n-1}}$ ending in $D_{2^{n-1}}$. It is easy to see that a chain of subgroups of G ending in G and including a subgroup that does not contain $Z(G)$ has the following forms

$$1 \subseteq \cdots \subseteq G, \tag{2}$$

$$\langle a^{2i} b \rangle \subseteq \cdots \subseteq G, \tag{3}$$

$$1 \subseteq \langle a^{2i} b \rangle \subseteq \cdots \subseteq G. \tag{4}$$

The number of chains of type (2) is the same as the number of chains of subgroups of G that contain $Z(G)$ and ending in G that is $F_{D_{2^{n-1}}}$. On the other hand, the number of chains of types (3) and (4) is the same and it equals the number of chains of the form $H \subseteq \cdots \subseteq G$ such that $\langle a^{2i} b \rangle \subset H$ plus one for the chain $\{G\}$ with one subgroup. If $\langle a^{2i} b \rangle \subset H$, then $H = \langle a^{2^{n-1-s}}, a^{2i} b \rangle$ for some $1 \leq s \leq n-2$. Also, if K is a subgroup of G such that $H \subseteq K \subseteq G$, then $K = \langle a^{2^{n-1-t}}, a^{2i} b \rangle$ for some $1 \leq t \leq s$. Hence the number of chains $H \subseteq \cdots \subseteq G$ such that $\langle a^{2i} b \rangle \subset H$ equals the number of chains of $\mathbb{Z}_{2^s}$, which begin in 1 and end in $\mathbb{Z}_{2^s}$ that is 2^{s-1}. Therefore, the number of chains of types (3) and (4), when i ranges over $\{1, \ldots, 2^{n-2}\}$ is equal to

$$2^{n-2} \cdot 2 \cdot \left(1 + \sum_{s=1}^{n-2} 2^{s-1} \right) = 2^{2n-3},$$

from which the result follows. □

Let $M_{p^n} = \langle a, b : a^{p^{n-1}} = b^p = 1, a^b = a^{p^{n-2}+1} \rangle$ $(n \geq 3)$ be the modular p-group of order p^n. Then $Z(M_{p^n}) = \langle a^{p^{n-2}} \rangle$ and $M_{p^n}/Z(M_{p^n}) \cong \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$. Moreover, if H is a subgroup of G such that $H \cap Z(G) = 1$, then either $H = 1$ or $H = \langle a^{ip^{n-2}} b \rangle$, for some $0 \leq i < p$.

Theorem 3.3. *The number of all fuzzy subgroups of the modular* p*-group* M_{p^n} *(*$n \geq 3$ *and* $p^n \neq 8$*) is*

$$F_{M_{p^n}} = 2^{n-1}(n-1)p + 2^n,$$

Proof. Let $G = M_{p^n}$. The same as before, the number of chains of subgroups of G that contain $Z(G)$ and ending in G equals to the number of chains of subgroups of $G/Z(G) \cong \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$ ending in $G/Z(G)$, which is equal to $2^{n-2}(n-2)p + 2^{n-1}$, by [9, Theorem 2.1]. A simple observation shows that a chain of subgroups of G ending in G and including a subgroup that does not contain $Z(G)$ has the following forms

$$1 \subseteq \cdots \subseteq G, \tag{5}$$

$$\langle a^{ip^{n-2}} b\rangle \subseteq \cdots \subseteq G, \tag{6}$$

$$1 \subseteq \langle a^{ip^{n-2}} b\rangle \subseteq \cdots \subseteq G. \tag{7}$$

The number of chains of type (5) is the same as the number of chains of subgroups of G that contain $Z(G)$ and ending in G that is $2^{n-2}(n-2)p+2^{n-1}$. Moreover, the number of chains of types (6) and (7) is the same and it equals the number of chains of the form $H \subseteq \cdots \subseteq G$ such that $\langle a^{ip^{n-2}} b\rangle \subset H$ plus one for the chain $\{G\}$ with one subgroup. If $\langle a^{ip^{n-2}} b\rangle \subset H$, then $H = \langle a^{p^{n-1-s}}, a^{ip^{n-2}} b\rangle$ for some $1 \leq s \leq n-2$. Also, if K is a subgroup of G such that $H \subseteq K \subseteq G$, then $K = \langle a^{p^{n-1-t}}, a^{ip^{n-2}} b\rangle$ for some $1 \leq t \leq s$. Hence the number of chains $H \subseteq \cdots \subseteq G$ such that $\langle a^{ip^{n-2}} b\rangle \subset H$ equals the number of chains of $\mathbb{Z}_{p^s}$, which begin with 1 and end in $\mathbb{Z}_{p^s}$ that is 2^{s-1}. Therefore, the number of chains of types (6) and (7), when i ranges over $\{0, \ldots, p-1\}$ is equal to

$$2p\left(1 + \sum_{s=1}^{n-2} 2^{s-1}\right) = 2^{n-1}p,$$

from which the result follows. □

We note that modular p-groups M_{p^n} have the same maximal subgroup structure as the abelian p-group $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$. Hence by the equality (2) in [12],

$$F_{M_{p^n}} = F_{\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p} = 2^{n-1}(n-1)p + 2^n.$$

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