ON THE FUZZY DIMENSIONS OF FUZZY VECTOR SPACES

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Abstract. In this paper, firstly, it is proved that, for a fuzzy vector space, the set of its fuzzy bases defined by Shi and Huang, is equivalent to the family of its bases defined by P. Lubczonok. Secondly, for two fuzzy vector spaces, it is proved that they are isomorphic if and only if they have the same fuzzy dimension, and if their fuzzy dimensions are equal, then their dimensions are the same, however, the converse is not true. Finally, fuzzy dimension of direct sum is considered, for a finite number of fuzzy vector spaces and it is proved that fuzzy dimension of their direct sum is equal to the sum of fuzzy dimensions of fuzzy vector spaces.

1. Introduction

After Katsaras and Liu [2] introduced the concept of fuzzy vector spaces, many scholars investigated its properties and characteristics (see [1, 4, 5, 6], etc.). For a fuzzy vector space, P. Lubczonok [6] introduced a definition of basis. In this case, the bases of a fuzzy vector space are still crisp. He defined dimension for all fuzzy vector spaces as a non-negative real number or infinity, and generalized some properties of vector spaces to fuzzy setting.

Recently, Shi and Huang [8] redefined basis and dimension of a fuzzy vector space as a fuzzy set and a fuzzy natural number, called fuzzy bases and fuzzy dimension, respectively. In this sense, Shi and Huang [8] proved that fuzzy vector spaces keep more properties of (crisp) vector spaces.

In this paper, further discussions on fuzzy dimension of fuzzy vector spaces are considered and some properties are obtained. For a fuzzy vector space, the relation between the set of its fuzzy bases and the family of its bases defined by P. Lubczonok is discussed and it is proved that there is a one-to-one correspondence between them; see Theorem 3.4. In Shi and Huang sense, a fuzzy vector space keeps completely the property of vector space such that two fuzzy vector spaces are isomorphic if and only if they have the same fuzzy dimension; see Theorem 4.5. However, the result is necessary and insufficient in P. Lubczonok sense [3]. The relation between the fuzzy dimension and dimension is investigated. For two fuzzy vector spaces, if their fuzzy dimensions are equal, then their dimensions are the same, but the converse is not true, see Theorem 4.9 and Example 4.10. Fuzzy dimension of direct sum is considered. For a finite number of fuzzy vector spaces, it is proved that fuzzy dimension of their direct sum is equal to the sum of fuzzy dimensions of each fuzzy vector space, see Theorem 5.4 and Corollary 5.5.

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2. **Preliminaries**

Throughout this paper, let F be a field. Suppose that V and U are finite-dimensional vector spaces over the field F. A fuzzy set μ on V is a mapping $\mu : V \longrightarrow [0,1]$. All fuzzy sets on V is denoted by $[0,1]^V$. Let A be a subset of V. For any $\mu \in [0,1]^V$, $a \in (0,1]$, we use the following notations:

$$\begin{aligned}
&\text{Im } \mu = \{\mu(v) : v \in V\},\\
&\text{supp}\mu = \{v \in V : \mu(v) > 0\},\\
&\mu_{[a]} = \{v \in V : \mu(v) \geq a\},\\
&\mu_{(a)} = \{v \in V : \mu(v) > a\},\\
&\mu|_A(v) = \begin{cases} \mu(v), & v \in A,\\ 0, & v \notin A.\end{cases}
\end{aligned}$$

Definition 2.1. [7] Let $\mathbb{N}$ be the set of all natural numbers. A fuzzy natural number is an antitone mapping $\lambda : \mathbb{N} \longrightarrow [0,1]$ satisfying

$$\lambda(0) = 1, \quad \bigwedge_{n\in\mathbb{N}} \lambda(n) = 0.$$

The set of all fuzzy natural number is denoted by $\mathbb{N}([0,1])$.

Definition 2.2. [7] For any $\lambda, \mu \in \mathbb{N}([0,1])$, the addition $\lambda+\mu$ of λ and μ is defined as follows: for any $n \in \mathbb{N}$,

$$(\lambda+\mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

Lemma 2.3. [7, 9] *For any $\lambda, \mu \in \mathbb{N}([0,1])$ and for any $a \in [0,1)$, one has $(\lambda + \mu)_{(a)} = \lambda_{(a)} + \mu_{(a)}$.*

Definition 2.4. [10] Let A be a fuzzy set. Define a mapping $|A| : \mathbb{N} \longrightarrow [0,1]$ such that for any $n \in \mathbb{N}$,

$$|A|(n) = \bigvee \left\{a \in (0,1] : |A_{[a]}| \geqslant n\right\}.$$

Then $|A| \in \mathbb{N}([0,1])$, which is called the cardinality of A.

Definition 2.5. [2] A fuzzy vector space is a pair $\widetilde{V} = (V, \mu)$, where V is a vector space over a field F, and $\mu : V \longrightarrow [0,1]$ is a mapping satisfying $\mu(kx + ly) \geq \mu(x) \wedge \mu(y)$ for any $x, y \in V, k, l \in F$.

Lemma 2.6. [8] *If $\widetilde{V} = (V, \mu)$ is a fuzzy vector space, then there exists a finite sequence $1 = a_0 \geq a_1 > a_2 > \cdots > a_r \geq 0$ such that*

(i) *If $a, b \in (a_{i+1}, a_i]$, then $\mu_{[a]} = \mu_{[b]}$;*
(ii) *If $a \in (a_{i+1}, a_i]$ and $b \in (a_i, a_{i-1}]$, then $\mu_{[a]} \supsetneq \mu_{[b]}$;*
(iii) *If $a, b \in [a_{i+1}, a_i)$, then $\mu_{(a)} = \mu_{(b)}$;*
(iv) *If $a \in [a_{i+1}, a_i)$ and $b \in [a_i, a_{i-1})$, then $\mu_{(a)} \supsetneq \mu_{(b)}$.*

Let $\widetilde{V} = (V, \mu)$ be a fuzzy vector space. In [8], Shi and Huang redefined fuzzy basis and fuzzy dimension of a fuzzy vector space as follows. For a fuzzy vector space $\widetilde{V} = (V, \mu)$, we suppose that $\text{Im}\mu = \{a_1, a_2, \cdots, a_r\}$ and $1 \geq a_1 > a_2 > \cdots > a_r \geq 0$. Then we can obtain a nest-set of vector subspaces of V, denote by

$$\{0\} \subseteq \mu_{[a_1]} \subset \mu_{[a_2]} \subset \cdots \subset \mu_{[a_r]} = V.$$

Suppose that $\mu_{[a_1]} \neq \{0\}$, otherwise we can choose $\mu_{[a_2]}$. Let B_{a_1} be a basis of $\mu_{[a_1]}$. We can obtain a basis B_{a_2} of $\mu_{[a_2]}$ by extending B_{a_1}. Further we can obtain a basis B_{a_3} of $\mu_{[a_3]}$ by extending B_{a_2}. Analogously we can obtain a basis B_{a_r} of $\mu_{[a_r]}$ by extending $B_{a_{r-1}}$. Thus we obtain a sequence

$$B_{a_1} \subset B_{a_2} \subset B_{a_3} \subset \cdots \subset B_{a_r}, \tag{1}$$

where B_{a_i} is a basis of $\mu_{[a_i]}(1 \leq i \leq r)$. Therefore we can define a fuzzy subset β of V as follows.

$$\beta(x) = \bigvee \{a_i : x \in B_{a_i}\}.$$

Then β is called a fuzzy basis of $\widetilde{V}$ corresponding to (1). We denote all fuzzy bases of $\widetilde{V}$ obtained by above process by $\mathcal{B}_F(V)$.

Definition 2.7. [8] Let $\widetilde{V} = (V, \mu)$ be a fuzzy vector space with a fuzzy basis β. Define $\dim \widetilde{V} = |\beta|$. Then $\dim \widetilde{V}$ is called the fuzzy dimension of $\widetilde{V}$.

Corollary 2.8. [8] *Let β be a fuzzy basis of $\widetilde{V} = (V, \mu)$ obtained by* (1). *Then the following statements hold:*

(i) *For each $a \in (0, 1]$, $\beta_{[a]}$ is a basis of $\mu_{[a]}$;*
(ii) *For each $a \in [0, 1)$, $\beta_{(a)}$ is a basis of $\mu_{(a)}$.*

3. A Note on the Bases of a Fuzzy Vector Space

In [6], P. Lubczonok presented the basis of a fuzzy vector space as follows.

Definition 3.1. [6] Let $\widetilde{V} = (V, \mu)$ be a fuzzy vector space. A set $B = \{\beta_1, \beta_2, \cdots, \beta_n\}$ of vectors is called a basis of $\widetilde{V}$, if the following statements are satisfied:

(i) B is a basis of V;
(ii) For any $\{a_i\}_{i=1}^n \subseteq F$, we have

$$\mu\left(\sum_{i=1}^n a_i\beta_i\right) = \bigwedge_{i=1}^n \mu(a_i\beta_i).$$

We denote all bases of $\widetilde{V}$ by $\mathfrak{B}(V)$.

The dimension of a fuzzy vector space in P. Lubczonok sense was defined as follows.

Definition 3.2. [6] Let $\widetilde{V}=(V,\mu)$ be a fuzzy vector space. We define the dimension of $\widetilde{V}$ by

$$\dim(\widetilde{V})=\sup_{B\text{ is a basis of }V}\left(\sum_{x\in B}\mu(x)\right).$$

In order to distinguish from the definition of the fuzzy dimension in Shi and Huang sense, we denote it as $\dim_L(\widetilde{V})$ in this paper.

Lemma 3.3. [5] *Let $\widetilde{V}=(V,\mu)$ be a fuzzy vector space. A set $B=\{\beta_1,\beta_2,\cdots,\beta_n\}$ of vectors is a basis of $\widetilde{V}$ if and only if $B\cap\mu_{[a]}$ is a basis of $\mu_{[a]}$ for any $a\in[0,1]$.*

The following theorem shows that the bases of a fuzzy vector space is equivalent to its fuzzy bases.

Theorem 3.4. *Let $\widetilde{V}=(V,\mu)$ be a fuzzy vector space. Then there is a one-to-one correspondence between $\mathcal{B}_F(V)$ and $\mathfrak{B}(V)$.*

Proof. Let $\widetilde{V}=(V,\mu)$ be a fuzzy vector space. Suppose that $\text{Im}\mu=\{a_1,a_2,\cdots,a_r\}$ and $1\geq a_1>a_2>\cdots>a_r\geq 0$. Define a mapping $\psi:\mathcal{B}_F(V)\longrightarrow\mathfrak{B}(V)$ such that for any $\beta\in\mathcal{B}_F(V)$ corresponding to (1), $\psi(\beta)=B_{a_r}$. By Corollary 2.8 and Lemma 3.3, $B_{a_r}\in\mathfrak{B}(V)$. Hence, the mapping ψ is well-defined.

On the one hand, take $\beta,\beta'\in\mathcal{B}_F(V)$ and $\beta\neq\beta'$, where β,β' are obtained by the following nest sets, respectively

$$B_{a_1}\subset B_{a_2}\subset\cdots\subset B_{a_r},$$

$$B'_{a_1}\subset B'_{a_2}\subset\cdots\subset B'_{a_r}.$$

In the following, we shall show that $\psi(\beta)\neq\psi(\beta')$. Suppose that $\psi(\beta)=\psi(\beta')$. This implies $B_{a_r}=B'_{a_r}$. We note that $\beta,\beta'\in\mathcal{B}_F(V)$ and $B_{a_r},B'_{a_r}\in\mathfrak{B}(V)$. By Corollary 2.8 and Lemma 3.3, we have $\beta_{[a_i]}=B_{a_i}=B_{a_r}\cap\mu_{[a_i]}=B'_{a_r}\cap\mu_{[a_i]}=B'_{a_i}=\beta'_{[a_i]}$ for any $i\in\{1,2,\cdots,r\}$. Hence, we obtain $\beta=\beta'$. This is a contradiction. Hence, the hypothesis $\psi(\beta)=\psi(\beta')$ not holds, i.e. we have $\psi(\beta)\neq\psi(\beta')$. Therefore, ψ is injective.

On the another hand, take $B\in\mathfrak{B}(V)$. By Lemma 3.3, $B\cap\mu_{[a_i]}$ is a basis of $\mu_{[a_i]}$ for each $i\in\{1,2,\cdots,r\}$. We have

$$B\cap\mu_{[a_1]}\subset B\cap\mu_{[a_2]}\subset\cdots\subset B\cap\mu_{[a_r]}.$$

For any $v\in V$, define a fuzzy set β_0 by

$$\beta_0(v)=\bigvee\{a\in(0,1]:v\in B\cap\mu_{[a]}\}.$$

Obviously, β_0 is a fuzzy basis of $\widetilde{V}$. Thus ψ is a surjection. Therefore ψ is a one-to-one correspondence between $\mathcal{B}_F(V)$ and $\mathfrak{B}(V)$. □

Example 3.5. Let $V=\mathbb{R}^2$. We define a fuzzy set $\mu:V\longrightarrow[0,1]$ by

$$\mu(x)=\begin{cases}1, & \text{if } x\in\{(a,0):a\in\mathbb{R}\},\\ 0.2, & \text{otherwise.}\end{cases}$$

It is easy to check that $\widetilde{V} = (V, \mu)$ is a fuzzy vector space. By the definitions of fuzzy bases and bases of a fuzzy vector space, we can obtain the united forms of the fuzzy bases and bases of $\widetilde{V}$ as follows, respectively

$$\beta(x) = \begin{cases} 1, & \text{if } x = (a, 0), \\ 0.2, & \text{if } x = (0, b), \\ 0, & \textit{otherwise}, \end{cases} \qquad B = \{(a, 0), (0, b)\},$$

where $a, b \in \mathbb{R}$ and $a \neq 0$, $b \neq 0$.

We define a mapping $\psi : \mathcal{B}_F(V) \longrightarrow \mathfrak{B}(V)$ such that $\psi(\beta) = B$ for different a, b, and easily check that ψ is a one-to-one corresponding between $\mathcal{B}_F(V)$ and $\mathfrak{B}(V)$.

4. **Isomorphism of Fuzzy Vector Spaces**

In [3], Kumar introduced the concept of the isomorphism for fuzzy vector spaces and proved that two isomorphic fuzzy vector spaces have the same dimension, but the converse is not true in Lubczonok sense.

In this section, based on the fuzzy dimension of fuzzy vector spaces [8], we prove that two fuzzy vector spaces are isomorphic if and only if they have the same fuzzy dimension.

Definition 4.1. [3] Let $\widetilde{V} = (V, \mu)$ and $\widetilde{U} = (U, \lambda)$ be two fuzzy vector spaces. $\widetilde{V}$ and $\widetilde{U}$ are isomorphic if there exists an isomorphism $\phi : V \longrightarrow U$ such that $\mu(v) = \lambda(\phi(v))$ for any $v \in V$, denoted by $\widetilde{V} \cong \widetilde{U}$.

Theorem 4.2. *Let $\widetilde{V} = (V, \mu)$ and $\widetilde{U} = (U, \lambda)$ be two fuzzy vector spaces. Then the following statements are equivalent:*

(i) $\widetilde{V} \cong \widetilde{U}$;
(ii) *For each* $a \in (0, 1]$, $\mu_{[a]} \cong \lambda_{[a]}$;
(iii) *For each* $a \in [0, 1)$, $\mu_{(a)} \cong \lambda_{(a)}$.

Proof. (i) $\Longrightarrow$ (ii) By $\widetilde{V} \cong \widetilde{U}$, there exists an isomorphism $\phi : V \longrightarrow U$ such that $\mu(v) = \lambda(\phi(v))$ for any $v \in V$. Define a mapping $\psi : \mu_{[a]} \longrightarrow \lambda_{[a]}$ by

$$\psi(v) = \phi(v), \quad \text{for each} \quad v \in \mu_{[a]}.$$

For each $a \in (0, 1]$ and $v \in \mu_{[a]}$, we have $\mu(v) \geq a$, i.e. $\lambda(\phi(v)) \geq a$, then $\phi(v) \in \lambda_{[a]}$. Hence, the mapping ψ is well-defined. For any $x \in \lambda_{[a]}$, we have $\lambda(x) \geq a$. Since ϕ is an isomorphism, then there exists $v \in V$ such that $\phi(v) = x$ and $\lambda(\phi(v)) \geq a$. Hence, $\mu(v) \geq a$, it follows that $v \in \mu_{[a]}$. Then ψ is surjective. Let $\{x_i\}_{i=1}^k$ be a set of vectors in $\mu_{[a]}$. It is easy to prove that $\{x_i\}_{i=1}^k$ are linearly independent in $\mu_{[a]}$ if and only if $\{\psi(\{x_i\})\}_{i=1}^k$ is linearly independent in $\lambda_{[a]}$. Thus ψ is a one-to-one correspondence between $\mu_{[a]}$ and $\lambda_{[a]}$. Therefore, $\mu_{[a]} \cong \lambda_{[a]}$ for each $a \in (0, 1]$.

(ii) $\Longrightarrow$ (i) We show that $\text{Im}\mu = \text{Im}\lambda$ as follows. Suppose that $\text{Im}\mu = \{a_1, a_2, \cdots, a_r\}$ $(1 \geq a_1 > a_2 > \cdots > a_r \geq 0)$, $\text{Im}\lambda = \{b_1, b_2, \cdots, b_s\}(1 \geq b_1 > b_2 > \cdots > b_s \geq 0)$ and there is an element $a_i \in \text{Im}\mu \setminus \text{Im}\lambda (1 \leq i \leq r)$. Then there exists $1 \leq j \leq s$ such that $b_j < a_i < b_{j-1}$. Assume that $b_j < a_{i+1} < a_i < b_{j-1}$. By Lemma 2.6,

$$\mu_{[a_{i+1}]} \cong \lambda_{[a_{i+1}]} = \lambda_{[a_i]} \cong \mu_{[a_i]}.$$

Then $\mu_{[a_{i+1}]} = \mu_{[a_i]}$, this is a contradiction. Hence, $a_{i+1} < b_j < a_i < b_{j-1}$. By Lemma 2.6,

$$\lambda_{[b_j]} \cong \mu_{[b_j]} = \mu_{[a_i]} \cong \lambda_{[a_i]} = \lambda_{[b_{j-1}]} \subset \lambda_{[b_j]}.$$

This is a contradiction. Hence, $\text{Im}\mu \subseteq \text{Im}\lambda$. Similarly, we can prove $\text{Im}\mu \supseteq \text{Im}\lambda$. Thus $\text{Im}\mu = \text{Im}\lambda$.

So we can obtain respectively a nest-set of fuzzy vector spaces for $\widetilde{V}$ and $\widetilde{U}$ as follows:

$$\{0\} \subseteq \mu_{[a_1]} \subset \mu_{[a_2]} \subset \cdots \subset \mu_{[a_r]} = V,$$
$$\{0\} \subseteq \lambda_{[a_1]} \subset \lambda_{[a_2]} \subset \cdots \subset \lambda_{[a_r]} = U.$$

We prove that (i) holds as follows: Suppose that B_1 and A_1 are bases for $\mu_{[a_1]}$ and $\lambda_{[a_1]}$, respectively. We can obtain the bases B_2 and A_2 of $\mu_{[a_2]}$ and $\lambda_{[a_2]}$ by extending respectively B_1 and A_1. Similarly, we can obtain the bases of B_r and A_r of $\mu_{[a_r]}$ and $\lambda_{[a_r]}$ by extending B_{r-1} and A_{r-1}, respectively. Hence, we can obtain two sequences of bases as follows:

$$B_1 \subset B_2 \subset \cdots \subset B_r,$$
$$A_1 \subset A_2 \subset \cdots \subset A_r.$$

Suppose that $B_1 = \{\beta_1, \beta_2, \cdots, \beta_m\}$, $A_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. Define a mapping $\psi_{a_1} : \mu_{[a_1]} \longrightarrow \lambda_{[a_1]}$ by

$$\psi_{a_1}(v) = \sum_{i=1}^{m} b_i \alpha_i, \quad \text{for each} \quad v = \sum_{i=1}^{m} b_i \beta_i \in \mu_{[a_1]}.$$

Obviously, ψ_{a_1} is an isomorphism from $\mu_{[a_1]}$ to $\lambda_{[a_1]}$. By $\mu_{[a_2]} \cong \lambda_{[a_2]}$, the bases of vector spaces $\mu_{[a_2]}$ and $\lambda_{[a_2]}$ have the same cardinality. Suppose that $B_2 = \{\beta_1, \cdots, \beta_m, \ \beta_{m+1}, \cdots, \beta_n\}$ and $A_2 = \{\alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_n\}$ $(m < n)$. Define a mapping $\psi_{a_2} : \mu_{[a_2]} \longrightarrow \lambda_{[a_2]}$ by

$$\psi_{a_2}(v) = \sum_{i=1}^{n} b_i \alpha_i, \quad \text{for each} \quad v = \sum_{i=1}^{n} b_i \beta_i \in \mu_{[a_2]}.$$

Obviously, ψ_{a_2} is an isomorphism between $\mu_{[a_2]}$ and $\lambda_{[a_2]}$, and $\psi_{a_2}(v) = \psi_{a_1}(v)$ for each $v \in \mu_{[a_1]}$. Further, we can obtain an isomorphism ψ_{a_3} between $\mu_{[a_3]}$ and $\lambda_{[a_3]}$ such that $\psi_{a_3}(v) = \psi_{a_2}(v)$ for any $v \in \mu_{[a_2]}$. Analogously, we can obtain an isomorphism ψ_{a_r} from $\mu_{[a_r]}$ to $\lambda_{[a_r]}$ such that $\psi_{a_r}(v) = \psi_{a_{r-1}}(v)$ for each $v \in \mu_{[a_{r-1}]}$. We note that $\psi_{a_r}$is an isomorphism from V to U. By the deposition theorem of fuzzy sets, for each $v \in V$,

$$\begin{aligned} \mu(v) &= \bigvee \left\{a \in (0,1] : v \in \mu_{[a]}\right\} \\ &= \bigvee \left\{a \in (0,1] : \psi_a(v) \in \lambda_{[a]}\right\} \\ &= \bigvee \left\{a \in (0,1] : \psi_{a_r}(v) \in \lambda_{[a]}\right\} \\ &= \lambda\left(\psi_{a_r}(v)\right). \end{aligned}$$

Thus $\widetilde{V} \cong \widetilde{U}$.

Similarly, we can prove (i) $\Leftrightarrow$ (iii). □

Lemma 4.3. [7] *Let A be a fuzzy set on V. Then $|A|_{[a]} = |A_{[a]}|$ for any $a \in (0, 1]$.*

Lemma 4.4. [8] *Let $\widetilde{V} = (V, \mu)$ be a fuzzy vector space. We have $(\dim \widetilde{V})_{(a)} = (\dim \mu)_{(a)} = \dim \mu_{(a)}$ for each $a \in [0, 1)$.*

Theorem 4.5. *Let $\widetilde{V} = (V, \mu)$ and $\widetilde{U} = (U, \lambda)$ be two fuzzy vector spaces. Then $\widetilde{V} \cong \widetilde{U}$ if and only if $\dim \widetilde{V} = \dim \widetilde{U}$.*

Proof. Suppose that $\widetilde{V} \cong \widetilde{U}$. By Theorem 4.2, $\mu_{(a)} \cong \lambda_{(a)}$ for each $a \in [0, 1)$. This implies that $\dim \mu_{(a)} = \dim \lambda_{(a)}$ for each $a \in [0, 1)$. By Lemma 4.4, $(\dim \mu)_{(a)} = (\dim \lambda)_{(a)}$ for each $a \in [0, 1)$. Hence, $\dim \widetilde{V} = \dim \widetilde{U}$.

Conversely, suppose that $\dim\widetilde{V} = \dim\widetilde{U}$, β_V and β_U are fuzzy bases for $\widetilde{V}$ and $\widetilde{U}$, respectively. Since $\dim\widetilde{V} = \dim\widetilde{U}$,we have $|\beta_V| = |\beta_U|$. For each $a \in (0, 1]$, we have $|\beta_V|_{[a]} = |\beta_U|_{[a]}$. By Lemma 4.3, $|(\beta_V)_{[a]}| = |(\beta_U)_{[a]}|$. By Corollary 2.8, we can obtain $\mu_{[a]} \cong \lambda_{[a]}$ for each $a \in (0, 1]$. By Theorem 4.2, we have $\widetilde{V} \cong \widetilde{U}$. □

Remark 4.6. For a fuzzy vector space, the condition in Theorem 4.5 is necessary and insufficient under the definition of dimension in P. Lubczonok sense. The relation between the fuzzy dimension and dimension is shown in the following theorem and example.

Lemma 4.7. [6] *Let $\widetilde{V} = (V, \mu)$ be a fuzzy vector space. If $\widetilde{V}$ is finite dimensional, then $\dim_L(\widetilde{V}) = \sum_{x \in B} \mu(x)$, where B is any basis for $\widetilde{V}$.*

Lemma 4.8. *Let $\widetilde{V} = (V, \mu)$ be a fuzzy vector space and $\beta \in \mathcal{B}_F(V)$. Then $\beta(x) = \mu(x)$ for any $x \in \text{supp}\beta$.*

Proof. Suppose that β is a fuzzy basis for $\widetilde{V}$ obtained by (1). For any $x \in \text{supp}\beta$, let $b = \beta(x) = \bigvee\{a \in (0, 1] : x \in B_a\}$. We have $b \in \{a_1, a_2, \cdots, a_r\}$. Without lost the generality, let $b = a_i$. Hence, $x \in B_{a_i} \subseteq \mu_{[a_i]}$. Therefore, we have $\mu(x) \geq a_i = b$. If $\mu(x) > a_i$, then $x \in \mu_{[a_{i-1}]} \subseteq \mu_{[a_i]}$. By the definition of fuzzy bases, we have $x \in B_{a_i} \cap \mu_{[a_{i-1}]} = B_{a_{i-1}}$. This implies $\beta(x) \geq a_{i-1} > a_i = b$. This contradict $\beta(x) = b$. Thus, we have $\mu(x) = b = \beta(x)$. □

Theorem 4.9. *Let $\widetilde{V} = (V, \mu)$ and $\widetilde{U} = (U, \lambda)$ be two fuzzy vector spaces. If $\dim(\widetilde{V}) = \dim(\widetilde{U})$, then $\dim_L(\widetilde{V}) = \dim_L(\widetilde{U})$.*

Proof. By $\dim(V) = \dim(\widetilde{U})$, we have $| \beta_v | = | \beta_u |$ for any $\beta_v \in \mathcal{B}_F(V)$ and $\beta_u \in \mathcal{B}_F(U)$. By Lemma 4.3, we obtain $| (\beta_v)_{[a]} | = | (\beta_u)_{[a]} |$ for each $a \in (0, 1]$. Thus, we have $| \text{supp}(\beta_v) | = | \text{supp}(\beta_u) |$. By Lemma 4.8, we know that $\mu(x) = \beta_v(x)$ for any $x \in \text{supp}(\beta_v)$. Similarly, we have $\lambda(y) = \beta_u(y)$ for any $y \in \text{supp}(\beta_u)$. By Corollary 2.8, we can obtain that $\text{supp}(\beta_v)$ and $\text{supp}(\beta_u)$ are the bases for some subspaces of V and U, respectively. We can extend $\text{supp}(\beta_v)$ and $\text{supp}(\beta_u)$ to B_v, B_u which are the bases of $\widetilde{V}$ and $\widetilde{U}$. Obviously, for any $x \in B_v - \text{supp}(\beta_v)$, $y \in B_u - \text{supp}(\beta_u)$, we have

$$\beta_v(x) = \mu(x) = \beta_u(y) = \lambda(y) = 0.$$

Hence, we have

$$\begin{aligned}\dim_L(\widetilde{V}) &= \sum_{x\in B_v}\mu(x) = \sum_{x\in \mathrm{supp}(\beta_v)}\mu(x)\\ &= \sum_{x\in \mathrm{supp}(\beta_v)}\beta_v(x) = \sum_{y\in \mathrm{supp}(\beta_u)}\beta_u(y)\\ &= \sum_{y\in \mathrm{supp}(\beta_u)}\lambda(y) = \sum_{x\in B_u}\lambda(x)\\ &= \dim_L(\widetilde{U}).\end{aligned}$$

□

The following example shows that the converse of Theorem 4.9 is not true.

Example 4.10. Suppose that $\widetilde{V}$ is the fuzzy vector space defined in Example 3.5. We define another fuzzy set $\lambda : \mathbb{R}^2 \longrightarrow [0,1]$ by

$$\lambda(x) = \begin{cases} 0.8, & \text{if } x \in \{(0,a) : a \in \mathbb{R}\},\\ 0.4, & \text{otherwise.}\end{cases}$$

In Example 3.6 of [3], Kumar checked that $\widetilde{U} = (V,\lambda)$ is a fuzzy vector space, and showed that

$$\dim_L(\widetilde{V}) = \dim_L(\widetilde{U}) = 1.2.$$

However, we can obtain

$$\dim(\widetilde{V})(n) = \begin{cases} 1, & n = 0,1,\\ 0.2, & n = 2,\\ 0, & n \geq 3,\end{cases} \qquad \dim(\widetilde{U})(n) = \begin{cases} 1, & n = 0,\\ 0.8, & n = 1,\\ 0.4, & n = 2,\\ 0, & n \geq 3.\end{cases}$$

Hence, $\dim(\widetilde{V}) \neq \dim(\widetilde{U})$. Therefore, the converse of Theorem 5.4 is not true.

5. **Fuzzy Dimension of Direct Sum for Fuzzy Vector Spaces**

Definition 5.1. [5] Let (V_1,μ) and (V_2,λ) be two fuzzy vector spaces over the field F and $V_1 \cap V_2 = 0$. Define direct sum $\widetilde{V_1} \oplus \widetilde{V_2}$ to be $\widetilde{V_1} \oplus \widetilde{V_2} = (V_1 \oplus V_2, \mu_\oplus)$ for any $v \in V_1 \oplus V_2$,

$$\mu_\oplus(v) = \mu(v_1) \wedge \lambda(v_2), \qquad v = v_1 + v_2.$$

Theorem 5.2. *Let (V_1,μ) and (V_2,λ) be two fuzzy vector spaces over the field F and $V_1 \cap V_2 = 0$. Then $(V_1 \oplus V_2, \mu_\oplus)$ is a fuzzy vector space.*

Proof. For any $a,b \in K$ and $x,y \in V_1 \oplus V_2$, suppose that $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1,y_1 \in V_1$, $x_2,y_2 \in V_2$. Since (V_1,μ) and (V_2,λ) are fuzzy vector spaces, and $V_1 \oplus V_2$ is a vector space, we have

$$\begin{aligned}\mu_\oplus(ax+by) &= \mu_\oplus(ax_1 + ax_2 + by_1 + by_2)\\ &= \mu_\oplus(ax_1 + by_1 + ax_2 + by_2)\\ &= \mu(ax_1 + by_1) \wedge \lambda(ax_2 + by_2)\\ &\geq \mu(x_1) \wedge \mu(y_1) \wedge \lambda(x_2) \wedge \lambda(y_2)\\ &= \mu_\oplus(x) \wedge \mu_\oplus(y).\end{aligned}$$

Hence, $(V_1 \oplus V_2, \mu_\oplus)$ is a fuzzy vector space. □

Theorem 5.3. *Let (V_1, μ) and (V_2, λ) be two fuzzy vector spaces over field F and $V_1 \cap V_2 = 0$. Then for any $a \in [0, 1)$, $(\mu_\oplus)_{(a)} = \mu_{(a)} \oplus \lambda_{(a)}$.*

Proof. For any $a \in [0, 1)$, we have

$$\begin{aligned}\forall v \in (\mu_\oplus)_{(a)} \quad &\Leftrightarrow \mu_\oplus(v) > a \\ &\Leftrightarrow \mu(v_1) \wedge \lambda(v_2) > a, \quad v = v_1 + v_2 \in V_1 \oplus V_2 \\ &\Leftrightarrow \mu(v_1) > a \text{ and } \lambda(v_2) > a \\ &\Leftrightarrow v_1 \in \mu_{(a)} \text{ and } v_2 \in \lambda_{(a)} \\ &\Leftrightarrow v = v_1 + v_2 \in \mu_{(a)} \oplus \lambda_{(a)}, \\ &\quad \text{since} \quad \mu_{(a)} \cap \lambda_{(a)} \subseteq V_1 \cap V_2 = \emptyset.\end{aligned}$$

□

Theorem 5.4. *Let (V_1, μ) and (V_2, λ) be two fuzzy vector spaces and $\widetilde{V} = (V_1 \oplus V_2, \mu_\oplus)$. Then $\dim(\mu_\oplus) = \dim(\mu) + \dim(\lambda)$.*

Proof. We only need to prove that for any $a \in [0, 1)$, the formulas $(\dim \mu_\oplus)_{(a)} = (\dim(\mu) + \dim(\lambda))_{(a)}$ holds. By Theorem 4.4 and Theroem 2.3, we need to prove that $\dim\big((\mu_\oplus)_{(a)}\big) = \dim(\mu_{(a)}) + \dim(\lambda_{(a)})$ holds.

By Theorem 5.3, $(\mu_\oplus)_{(a)} = \mu_{(a)} \oplus \lambda_{(a)}$. By the property of direct sum for vector spaces, $\dim\big((\mu_\oplus)_{(a)}\big) = \dim(\mu_{(a)}) + \dim(\lambda_{(a)})$. □

By the above theorem, it is easy to obtain the following.

Corollary 5.5. *Let V be the direct sum of vector spaces $V_1, V_2, \cdots, V_k$ and (V_i, μ_i) $(1 \leq i \leq k)$ be fuzzy vector spaces. Then $\dim(\mu_\oplus) = \sum_{i=1}^{k} \dim(\mu_i)$.*

Example 5.6. Let $V = \{(a, 0) : a \in \mathbb{R}\}$ and $U = \{(0, b) : b \in \mathbb{R}\}$. We define the fuzzy sets μ and λ of V and U as follows, respectively,

$$\mu(x) = \begin{cases} 1, & \text{if } x = (0, 0), \\ 0.2, & \text{otherwise}, \end{cases}$$
$$\lambda(x) = \begin{cases} 0.8, & \text{if } x = (0, 0), \\ 0.4, & \text{otherwise}. \end{cases}$$

It is easy to check that $\widetilde{V} = (V, \mu)$ and $\widetilde{U} = (U, \lambda)$ are fuzzy vector spaces, and calculate their fuzzy dimensions as follows

$$\dim(\widetilde{V})(n) = \begin{cases} 1, & n = 0, \\ 0.2, & n = 1, \\ 0, & n \geq 2, \end{cases} \qquad \dim(\widetilde{U})(n) = \begin{cases} 1, & n = 0, \\ 0.4, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Obviously, $\widetilde{V} \oplus \widetilde{U} = (V \oplus U, \mu_\oplus) = (\mathbb{R}^2, \mu_\oplus)$ is a fuzzy vector space, where

$$\mu_\oplus(x) = \begin{cases} 0.8, & \text{if } x = (0, 0), \\ 0.4, & \text{if } x \in \{U - (0, 0)\} \\ 0.2, & \text{otherwise}. \end{cases}$$

By Definition 2.2 and Definition 2.4, We can obtain

$$\dim(\mu_{\oplus})(n) = \dim(\mu) + \dim(\lambda) = \begin{cases} 1, & \text{if } n = 0, \\ 0.4, & \text{if } n = 1, \\ 0.2, & \text{if } n = 2, \\ 0, & \text{otherwise}, \end{cases}$$

Thus $\dim(\widetilde{V} \oplus \widetilde{U}) = \dim(\widetilde{V}) + \dim(\widetilde{U})$.

6. Conclusions

We study the relation between bases of a fuzzy vector space in P. Lubczonok sense and in Shi and Huang sense, and prove that the two different definitions about bases of a fuzzy vector space are equivalent. We show that two fuzzy vector spaces are isomorphic if and only if they have the same fuzzy dimension, and prove that if two fuzzy vector spaces have the same fuzzy dimensions, then they have the same dimension. For a finite number of fuzzy vector spaces, we prove that the fuzzy dimension of their direct sum is equal to the sum of fuzzy dimensions of fuzzy vector spaces.

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