NUMERICAL SOLUTIONS OF FUZZY NONLINEAR INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract. In this paper, we use the parametric form of fuzzy numbers, and an iterative approach for obtaining approximate solution for a class of fuzzy nonlinear Fredholm integral equations of the second kind is proposed. This paper presents a method based on Newton-Cotes methods with positive coefficient. Then we obtain approximate solution of the fuzzy nonlinear integral equations by an iterative approach.

1. **Introduction**

The solutions of integral equations have a major role in the field of science and engineering. Since few of these equations can be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [6, 25]. There are several numerical methods for solving linear Volterra integral equation [11, 33] and system of nonlinear Volterra integral equations [7]. Kauthen in [23] used a collocation method to solve the Volterra- Fredholm integral equation numerically. Borzabadi and Fard in [9] obtained a numerical solution of nonlinear Fredholm integral equations of the second kind.

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [35], Dubois and Prade [13]. We refer the reader to [21] for more information on fuzzy numbers and fuzzy arithmetic. The topics of fuzzy integral [2, 3] and fuzzy integral equations (FIE) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. The fuzzy mapping function was introduced by Chang and Zadeh [10]. Later, Dubois and Prade [14] presented an elementary fuzzy calculus based on the extension principle also the concept of integration of fuzzy functions was first introduced by Dubois and Prade [14]. Babolian et al. and Abbasbandy et al. in [1, 5] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind, while finding an approximate solution for the fuzzy nonlinear kinds

$$X(s) = y(s) + \int_a^b k(s,t,X(t))dt,$$

is more difficult and a numerical method in this case can be found in [8].

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In this paper, we present a novel and very simple numerical method based upon iterative methods for solving fuzzy nonlinear Fredholm integral equations of the second kind.

2. Preliminaries

In this section the basic notations used in fuzzy calculus are introduced. We start by defining the fuzzy number.

Definition 2.1. A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \longrightarrow I = [0, 1]$ such that (see [24])

i. u is upper semi-continuous;

ii. $u(x) = 0$ outside some interval $[a, d]$;

iii. There are real numbers b and c, $a \leq b \leq c \leq d$, for which

1. $u(x)$ is monotonically increasing on $[a, b]$,
2. $u(x)$ is monotonically decreasing on $[c, d]$,
3. $u(x) = 1, b \leq x \leq c$.

The set of all fuzzy numbers (as given in Definition 2.1) is denoted by E^1. An alternative definition which yields the same E^1 is given by Kaleva [22, 26].

Definition 2.2. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

i. $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0, 1]$ and right continuous at 0;

ii. $\overline{u}(r)$ is a bounded monotonically decreasing, left continuous function on $(0, 1]$ and right continuous at 0;

iii. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number r is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \ \leq \alpha \leq 1$. The set of all fuzzy numbers is denoted by E^1. This fuzzy number space as shown in [12], can be embedded into the Banach space $B = \overline{C}[0, 1] \times \overline{C}[0, 1]$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$ we define addition and multiplication by k as

$$\underline{(u + v)}(r) = (\underline{u}(r) + \underline{v}(r)),$$

$$\overline{(u + v)}(r) = (\overline{u}(r) + \overline{v}(r)),$$

$$\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\overline{u}(r),\ if\ k \geq 0,$$

$$\underline{ku}(r) = k\overline{u}(r), \overline{ku}(r) = k\underline{u}(r),\ if\ k < 0.$$

Definition 2.3. For arbitrary fuzzy numbers u, v, we use the distance (see [19])

$$D(u, v) = sup_{0 \leq r \leq 1} max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}$$

and it is shown that (E^1, D) is a complete metric space [31].

Definition 2.4. Let $f : [a, b] \longrightarrow E^1$, for each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i], 1 \leq i \leq n$ suppose

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

$$\Delta := max\{|t_i - t_{i-1}|, i = 1, 2, \ldots, n\}.$$

The definite integral of $f(t)$ over $[a,b]$ is

$$\int_a^b f(t)dt = lim_{\Delta\longrightarrow 0} R_p$$

provided that this limit exists in the metric D (see [17, 19]).

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [19] and also,

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t;r)dt,$$

$$\overline{(\int_a^b f(t;r)dt)} = \int_a^b \overline{f}(t;r)dt.$$

3. **Fuzzy Integral Equation**

In this section, we consider the nonlinear Fredholm integral equations of the second kind (see [20])

$$X(s) = y(s) + \int_a^b k(s,t,X(t))dt, \tag{1}$$

where k is an arbitrary given kernel function and $y(s)$ is a given function of $s \in [a,b]$. If $y(s)$ is a crisp function, then the solution of above equation is crisp as well. However, if $y(s)$ is a fuzzy function this equation may only possess fuzzy solution. Sufficient condition for the existence of the solution of the equation of the second kind, where $y(s)$ is a fuzzy function, are given in [8].

For solving equation (1) we may replace equation (1) by the equivalent system

$$\underline{X}(s) = \underline{y}(s) + \int_a^b \underline{k}(s,t,X(t))dt = \underline{y}(s) + \int_a^b F(s,t,\underline{X},\overline{X}))dt,$$
$$\overline{X}(s) = \overline{y}(s) + \int_a^b \overline{k}(s,t,X(t))dt = \overline{y}(s) + \int_a^b G(s,t,\underline{X},\overline{X}))dt, \tag{2}$$

which possesses a unique solution $(\underline{X},\overline{X}) \in B$, which is a fuzzy function, i.e. for each s, the pair $(\underline{X}(s;r),\overline{X}(s;r))$ is a fuzzy number.

The parametric form of the equation (2) is given by

$$\underline{X}(s;r) = \underline{y}(s;r) + \int_a^b F(s,t,\underline{X}(t;r),\overline{X}(t;r))dt,$$
$$\overline{X}(s;r) = \overline{y}(s;r) + \int_a^b G(s,t,\underline{X}(t;r),\overline{X}(t;r))dt \tag{3}$$

for $r \in [0,1]$. In most cases, however, analytical solution to equation (3) may not be found and a numerical approach must be considered.

4. **The Newton-Cotes Method**

We replace the interval $[a,b]$ by a set of discrete equally spaced grid points

$$a = s_0 < s_1 < \ldots < s_n = b$$

at which the exact solution $(\underline{X}(s;r),\overline{X}(s;r))$ is approximated by some $(\underline{x}(s;r),\overline{x}(s;r))$. The exact and approximate solutions at s_i, $0 \leq i \leq n$ are denoted by $X_i(r) = (\underline{X}_i(r),\overline{X}_i(r))$ and $x_i(r) = (\underline{x}_i(r),\overline{x}_i(r))$, respectively. The grid points at which the solution is calculated are

$$s_i = s_0 + ih, \quad h = (b-a)/n; \quad 1 \leq i \leq n.$$

Then for the grid points on $[a, b]$, we have

$$\underline{X}_i(r) = \underline{y}_i(r) + \int_a^b F(s_i, t, \underline{X}(t;r), \overline{X}(t;r))dt,$$
$$\overline{X}_i(r) = \overline{y}_i(r) + \int_a^b G(s_i, t, \underline{X}(t;r), \overline{X}(t;r))dt, \ i = 0, 1, \ldots, n. \quad (4)$$

The Newton-Cotes method is given by (see [4])

$$\int_a^b Z(t)dt = \sum_{j=0}^{n} w_j Z(t_j) + O(h^{\nu}) \quad (5)$$

where Z is F and G alternatively and ν depends upon the used method of Newton-Cotes with positive coefficient for estimating of the integral in equation (5). By virtue of equation (5) we obtain

$$\underline{X}_i(r) \approx \underline{y}_i(r) + \sum_{j=0}^{n} w_j F(s_i, t_j, \underline{X}_j(r), \overline{X}_j(r)),$$
$$\overline{X}_i(r) \approx \overline{y}_i(r) + \sum_{j=0}^{n} w_j G(s_i, t_j, \underline{X}_j(r), \overline{X}_j(r)), \ i = 0, 1, \ldots, n. \quad (6)$$

Following equation (6) we define

$$\underline{x}_i(r) = \underline{y}_i(r) + \sum_{j=0}^{n} w_j F(s_i, t_j, \underline{x}_j(r), \overline{x}_j(r)),$$
$$\overline{x}_i(r) = \overline{y}_i(r) + \sum_{j=0}^{n} w_j G(s_i, t_j, \underline{x}_j(r), \overline{x}_j(r)), \ i = 0, 1, \ldots, n. \quad (7)$$

The polygon curves

$$\underline{x}(s;h;r) \triangleq \{[s_0, \underline{x}_0(r)], [s_1, \underline{x}_1(r)], \}, \ldots, [s_n, \underline{x}_n(r)]\},$$
$$\overline{x}(s;h;r) \triangleq \{[s_0, \overline{x}_0(r)], [s_1, \overline{x}_1(r)], \}, \ldots, [s_n, \overline{x}_n(r)]\} \quad (8)$$

are the approximates to $\underline{X}(s;r)$ and $\overline{X}(s;r)$, respectively, over the interval $s_0 \leq s \leq s_n$.

Let $F(s, t, u, v)$ and $G(s, t, u, v)$ be the functions F and G of equation (2) where u and v are constants and $u \leq v$. In other words $F(s, t, u, v)$ and $G(s, t, u, v)$ are obtained by substituting $X = (u, v)$ in equation (2). The domain where F and G are defined is therefore

$$M = \{(s, t, u, v) | a \leq s, t \leq b, -\infty < v < +\infty, -\infty < u \leq v\}.$$

Theorem 4.1. *Let $F(s, t, u, v)$ and $G(s, t, u, v)$ belong to $C^1(M)$ and let the partial derivatives of F, G be bounded over M by $L > 0$ and $L < \frac{1}{2(b-a)}$. Then, for arbitrary fixed $r : 0 \leq r \leq 1$,*

$$max_{0 \leq i \leq n}\{D(X_i, x_i)\} = D(X_p, x_p) \leq \frac{O(h^{\nu})}{1 - 2L(b-a)}.$$

Proof. Let

$$\underline{X}_p(r) = \underline{y}_p(r) + \sum_{j=0}^{n} w_j F(s_p, t_j, \underline{X}_j(r), \overline{X}_j(r)) + O(h^{\nu}),$$
$$\overline{X}_p(r) = \overline{y}_p(r) + \sum_{j=0}^{n} w_j G(s_p, t_j, \underline{X}_j(r), \overline{X}_j(r)) + O(h^{\nu}), \ i = 0, 1, \ldots, n \quad (9)$$

and

$$\underline{x}_p(r) = \underline{y}_p(r) + \sum_{j=0}^{n} w_j F(s_p, t_j, \underline{x}_j(r), \overline{x}_j(r)),$$
$$\overline{x}_p(r) = \overline{y}_p(r) + \sum_{j=0}^{n} w_j G(s_p, t_j, \underline{x}_j(r), \overline{x}_j(r)), \ i = 0, 1, \ldots, n. \quad (10)$$

Consequently

$$\underline{X}_p(r)-\underline{x}_p(r)=\sum_{j=0}^{n}w_j[F(s_p,t_j,\underline{X}_j(r),\overline{X}_j(r))$$
$$-F(s_p,t_j,\underline{x}_j(r),\overline{x}_j(r))]+O(h^{\nu}),$$
$$\overline{X}_p(r)-\overline{x}_p(r)=\sum_{j=0}^{n}w_j[G(s_p,t_j,\underline{X}_j(r),\overline{X}_j(r))$$
$$-G(s_p,t_j,\underline{x}_j(r),\overline{x}_j(r))]+O(h^{\nu}),\ i=0,1,\ldots,n.$$

Then

$$|\underline{X}_p(r)-\underline{x}_p(r)|\leq 2L\sum_{j=0}^{n}w_j max\{|\underline{X}_j(r)-\underline{x}_j(r)|,$$
$$|\overline{X}_j(r)-\overline{x}_j(r)|\}+O(h^{\nu})\leq 2Lmax\{|\underline{X}_p(r)-\underline{x}_p(r)|,$$
$$|\overline{X}_p(r)-\overline{x}_p(r)|\}\sum_{j=0}^{n}w_j+O(h^{\nu}),$$
$$|\overline{X}_p(r)-\overline{x}_p(r)|\leq 2L\sum_{j=0}^{n}w_j max\{|\underline{X}_j(r)-\underline{x}_j(r)|,$$
$$|\overline{X}_j(r)-\overline{x}_j(r)|\}+O(h^{\nu})\leq 2Lmax\{|\underline{X}_p(r)-\underline{x}_p(r)|,$$
$$|\overline{X}_p(r)-\overline{x}_p(r)|\}\sum_{j=0}^{n}w_j+O(h^{\nu}),\ i=0,1,\ldots,n.$$

Since in every Newton-Cotes formula $\sum_{j=0}^{n}w_j=b-a$,

$$D(X_p,x_p)\leq\frac{O(h^{\nu})}{1-2L(b-a)}.$$

□

Corollary 4.2. *$D(X_p,x_p)$ vanishes when $h\longrightarrow 0$.*

So far, we came to the nonlinear equation system (7) with a special form that let us offer a numerical approach for obtaining the approximate solution.

5. The Numerical Approach

Iterative methods are widely used for finding approximate solution of nonlinear equations systems (see [34]). The nonlinear equations system (7) also has a structure that permits to approximate its solution by an iterative method. For this purpose, we apply a successive substitution, similar to Jacobi method of solving linear equations systems and thereby define an iterative process leading to the sequence of vectors $\underline{x}^k$ and $\overline{x}^k$, where the components of the vectors satisfy the iteration formulas,

$$\underline{x}_i^{(k+1)}(r)=\underline{y}_i(r)+\sum_{j=0}^{n}w_jF(s_i,t_j,\underline{x}_j^{(k)}(r),\overline{x}_j^{(k)}(r)),$$
$$\overline{x}_i^{(k+1)}(r)=\overline{y}_i(r)+\sum_{j=0}^{n}w_jG(s_i,t_j,\underline{x}_j^{(k)}(r),\overline{x}_j^{(k)}(r)),$$
$$i=0,\ldots,n,k=0,1,\ldots. \qquad (11)$$

However, we should first study the conditions that guarantee the convergence of the approximate solution.

Theorem 5.1. *Considering assumptions of Theorem 4.1, the produced sequence $x^{(k)}$ from the iteration process (11) tends to the exact solution of (7), say x^*, for any arbitrary fuzzy initial vector $x^{(0)}$.*

Proof. By (7) and (11) we have,

$$\underline{x}_i^{(k+1)}(r) - \underline{x}_i^*(r) = \sum_{j=0}^n w_j[F(s_i, t_j, \underline{x}_j^{(k)}(r), \overline{x}_j^{(k)}(r)) - F(s_p, t_j, \underline{x}_j^*(r), \overline{x}_j^*(r))],$$
$$\overline{x}_i^{(k+1)}(r) - \overline{x}_i^*(r) = \sum_{j=0}^n w_j[G(s_i, t_j, \underline{x}_j^{(k)}(r), \overline{x}_j^{(k)}(r)) - G(s_i, t_j, \underline{x}_j^*(r), \overline{x}_j^*(r))],$$
$$i = 0, 1, \ldots, n,$$

and according to the conditions of Theorem 4.1,

$$|\underline{x}_i^{(k+1)}(r) - \underline{x}_i^*(r)| \le 2LD(x_p^{(k)}, x_p^*) \sum_{j=0}^n w_j,$$
$$|\overline{x}_i^{(k+1)}(r) - \overline{x}_i^*(r)| \le 2LD(x_p^{(k)}, x_p^*) \sum_{j=0}^n w_j,$$
$$i = 0, 1, \ldots, n.$$

By setting $\lambda = 2L(b-a)$ we conclude that

$$max_{0\le i\le n}\{D(x_i^{(k+1)}, x_i^*)\} \le \lambda D(x_p^{(k)}, x_p^*).$$

By induction on k, we get

$$max_{0\le i\le n}\{D(x_i^{(k+1)}, x_i^*)\} \le \lambda^k D(x_p^{(0)}, x_p^*),$$

for each $k = 0, 1, \ldots$ where $x_p^{(0)} = (\underline{x}_p^{(0)}(r), \overline{x}_p^{(0)}(r))$ and $x_p^* = (x_p^*(r), x_p^*(r))$. Since $0 < \lambda < 1, k \longrightarrow +\infty$ implies that $max_{0\le i\le n}\{D(x_i^{(k+1)}(r), x_i^*(r))\}$ vanishes. □

6. Algorithm of the Approach

In this section, we try to propose an algorithm on the basis of the above discussions and suppose that we face with the nonlinear fuzzy integral equations (1), where its kernel satisfies the conditions of Theorem 4.1. This algorithm is presented in two stages, initialization step and main steps.

Initialization Step:

Choose $\epsilon > 0$, an equidistance partition $a = s_0 < s_1 < \ldots < s_n = b$ on $[a, b]$ with the step size $h = s_{i+1} - s_i,\ i = 0, 1, ..., n-1$ and a fuzzy initial vector $x^{(0)}$. Set $k = 0$ and go to the main steps.

Main Steps:

Step 1: Compute $x_i = (\underline{x}_i^{(k+1)}(r), \overline{x}_i^{(k+1)}(r))$ and go to step 2.

Step 2: Compute $max_{0\le i\le n}\{D(x_i^{(k+1)}(r), x_i^{(k)}(r))\}$ and go to step 3.

Step 3: If $max_{0\le i\le n}\{D(x_i^{(k+1)}(r), x_i^{(k)}(r))\} < \epsilon$, stop; otherwise, set $k = k+1$ and go to step 1.

7. Comparison with Other Methods

In this subsection, the shortcomings of the existing methods in [1, 5, 8, 32, 16, 18, 29, 30] for solving fuzzy integral equations are pointed out.

Abbasbandy *et al.* in [1, 5], Solaymani Fard *et al.* in [32] and Parandin *et al.* in [29, 30] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind of the form

$$X(s) = y(s) + \int_a^b k(s,t)X(t)dt, \quad a \le s \le b$$

where, $k(s,t)$ is an arbitrary crisp kernel function over the square $a \le s,t \le b, \lambda \ge 0$. Then for solving the above equation, they considered the following relations:

$$\underline{X}(s;r) = \underline{y}(s;r) + \int_a^b \underline{k(s,t)X(t;r)}dt, \quad 0 \le r \le 1,$$
$$\overline{X}(s;r) = \overline{y}(s;r) + \int_a^b \overline{k(s,t)X(t;r)}dt, \quad 0 \le r \le 1,$$

where

$$\underline{k(s,t)X(t;r)} = \begin{cases} k(s,t)\underline{X}(t;r), & k(s,t) \ge 0, \\ k(s,t)\overline{X}(t;r), & k(s,t) < 0, \end{cases}$$

$$\overline{k(s,t)X(t;r)} = \begin{cases} k(s,t)\overline{X}(t;r), & k(s,t) \ge 0, \\ k(s,t)\underline{X}(t;r), & k(s,t) < 0. \end{cases}$$

Then, the authors in [1, 5, 32, 29, 30] used the numerical methods such as Nystrom, successive approximation, Taylor-successive approximation, finite and divided differences methods for solving a parametric of the linear fuzzy Fredholm integral equations of the second kind. Fariborzi Araghi and Parandin in [16] presented the Lagrange interpolation based on the extension principle for approximating solution of the linear Fredholm fuzzy integral equations also Friedman et al. in [18] presented the quadrature formula like an iterative method based on trapezoidal rule for solving linear fuzzy Fredholm integral equations. Also, Molabahrami *et al.* in [27] obtained analytical solution of the linear Fredholm fuzzy integral equations of the second kind.

Bernstein polynomials have been used recently to solve linear fuzzy Fredholm integral equations (see [15, 28]). But in this paper we consider more general form of the fuzzy nonlinear integral equations of the second kind (1).

The existing methods [1, 5, 32, 16, 18, 29, 30] are applicable only to linear Fredholm fuzzy integral equations, eg., they are not possible to find the nonlinear fuzzy integral equations, chosen in Examples 8.1 and 8.2.

Comparison of the results of this paper and [8] shows the efficiency of this algorithm in this paper more clearly. This result is intuitive, since the results of that algorithm depend explicitly on the selection of the starting point which decreases the speed of convergence substantially, while, in this algorithm the rate of convergence is completely independent of the starting point.

8. Numerical Examples

To illustrate the technique proposed in this paper, consider the following examples.

Example 8.1. Consider the following fuzzy nonlinear Fredholm integral equation

$$X(s) = (r - \frac{r^2}{40}, \frac{76 - 36r - r^2}{40})s + \int_0^1 \frac{st}{10} X^2(t)dt, \quad 0 \le s,t \le 1, 0 \le r \le 1.$$

The exact solution in this case is given by $X(s;r) = (r, 2-r)s$.
According to equation (2) we have the following two crisp nonlinear Fredholm integral equations

$$\underline{X}(s;r) = (r - \tfrac{r^2}{40})s + \int_0^1 \tfrac{st}{10}[min\{(\underline{X}(t;r))^2, \underline{X}(t;r)\overline{X}(t;r), (\overline{X}(t;r))^2\}]dt,$$

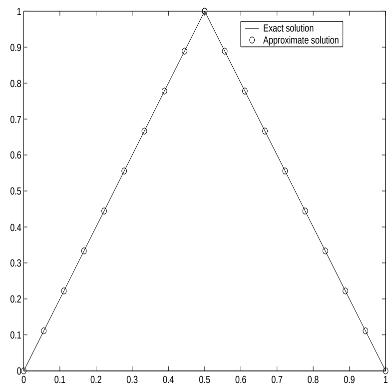

FIGURE 1. Compares the Exact Solution and Obtained Solution at $s = 0.5$ with Newton-Cotes Method

$$\overline{X}(s;r) = (\tfrac{76-36r-r^2}{40})s + \int_0^1 \tfrac{st}{10}[max\{(\underline{X}(t;r))^2, \underline{X}(t;r)\overline{X}(t;r), (\overline{X}(t;r))^2\}]dt,$$

$$0 \leq s, t \leq 1, 0 \leq r \leq 1.$$

It can be seen from the conditions of Theorem 4.1 where for convergence of the produced sequence, the kernel of the integral equation must satisfy the following inequalities.
Without any loss of generality, assume that $0 \leq \underline{X}(t;r) \leq \overline{X}(t;r)$ for $0 \leq t \leq 1$ and $0 \leq r \leq 1$, we have $F(s,t,\underline{X},\overline{X}) = \frac{st}{10}\underline{X}^2(t;r)$ and $G(s,t,\underline{X},\overline{X}) = \frac{st}{10}\overline{X}^2(t;r)$. Therefore

$$|\tfrac{\partial F(s,t,\underline{X},\overline{X})}{\partial \underline{X}}| = \tfrac{st}{5}|\underline{X}(t;r)| < \tfrac{1}{2} \Rightarrow |\underline{X}(t;r)| < \tfrac{5}{2},$$
$$|\tfrac{\partial G(s,t,\underline{X},\overline{X})}{\partial \overline{X}}| = \tfrac{st}{5}|\overline{X}(t;r)| < \tfrac{1}{2} \Rightarrow |\overline{X}(t;r)| < \tfrac{5}{2}.$$

Assume that $\underline{X}(t;r) \leq \overline{X}(t;r) \leq 0$, therefore we have $F(s,t,\underline{X},\overline{X}) = \frac{st}{10}\overline{X}^2(t;r)$, $G(s,t,\underline{X},\overline{X}) = \frac{st}{10}\underline{X}^2(t;r)$ and

$$|\tfrac{\partial F(s,t,\underline{X},\overline{X})}{\partial \overline{X}}| = \tfrac{st}{5}|\overline{X}(t;r)| < \tfrac{1}{2} \Rightarrow |\overline{X}(t;r)| < \tfrac{5}{2},$$
$$|\tfrac{\partial G(s,t,\underline{X},\overline{X})}{\partial \underline{X}}| = \tfrac{st}{5}|\underline{X}(t;r)| < \tfrac{1}{2} \Rightarrow |\underline{X}(t;r)| < \tfrac{5}{2}.$$

We can also obtain similar relations for $\underline{X}(t;r) \leq 0, 0 \leq \overline{X}(t;r)$ and other cases.

We take $max_{0\leq i\leq n}\{D(x_i^{(k+1)}, x_i^{(k)})\} < 10^{-4}$ and the initial vector $x_i^{(0)} = (r - \frac{r^2}{40}, \frac{76-36r-r^2}{40})s_i$ is considered for starting. The exact and obtained solution of fuzzy nonlinear Fredholm integral equation in this example at $s = 0.5$ are shown in Figure 1 and Table 1.

Example 8.2. Consider the following fuzzy nonlinear Fredholm integral equation

$$X(s) = y(s) + \int_0^{\frac{\Pi}{2}} (0.1s).(cost).e^{-X(t)}dt,$$

with

$$\underline{y}(s;r) = (1 + r + r^2)sins + \frac{0.1s}{5 - r^3 - r}(e^{r^3+r-5} - 1),$$

r	$\underline{X}(0.5;r)$	$\underline{x}^{(19)}(0.5;r)$	$\overline{X}(0.5;r)$	$\overline{x}^{(19)}(0.5;r)$
0	0	0.2031×10^{-4}	1	1.0009
0.1	0.05	0.05006	0.95	0.95007
0.2	0.1	0.10007	0.9	0.90005
0.3	0.15	0.15006	0.85	0.85006
0.4	0.2	0.20004	0.8	0.80006
0.5	0.25	0.25006	0.75	0.75006
0.6	0.3	0.30001	0.7	0.70003
0.7	0.35	0.35008	0.65	0.65007
0.8	0.4	0.40005	0.6	0.60006
0.9	0.45	0.45004	0.55	0.55003
1	0.5	0.50003	0.5	0.50003

TABLE 1. Compares the Exact Solution and Obtained Solution

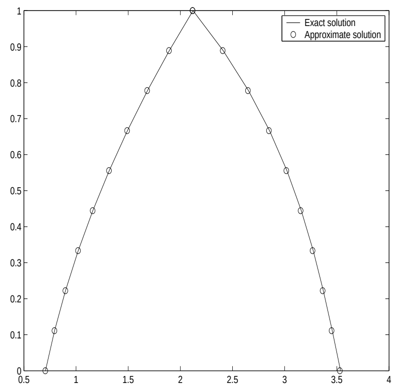

FIGURE 2. Compares the Exact Solution and Obtained Solution With Newton-Cotes Method

$$\overline{y}(s;r) = (5 - r - r^3)sins + \frac{0.1s}{1 + r^2 + r}(e^{-1-r-r^2} - 1),\ 0 \le r \le 1.$$

The exact solution in this case is given by $X(s;r) = (1 + r + r^2, 5 - r - r^3)sins$. According to equation (2) we have the following two crisp nonlinear Fredholm integral equations

$$\underline{X}(s;r) = \underline{y}(s;r) + \int_0^{\frac{\Pi}{2}} (0.1s).(cost).e^{-\overline{X}(t;r)}dt,$$

$$\overline{X}(s;r) = \overline{y}(s;r) + \int_0^{\frac{\Pi}{2}} (0.1s).(cost).e^{-\underline{X}(t;r)}dt,\ 0 \le r \le 1.$$

It can be seen from the conditions of Theorem 4.1 where for convergence of the produced sequence, the kernel of the integral equation must satisfy the following inequalities.

In this case, we have $F(s,t,\underline{X},\overline{X}) = (0.1s).(cost).e^{-\overline{X}(t;r)}$ and $G(s,t,\underline{X},\overline{X}) = (0.1s).(cost).e^{-\underline{X}(t;r)}$. Therefore

$$|\frac{\partial F(s,t,\underline{X},\overline{X})}{\partial \overline{X}}| = (0.1s)|(cost).e^{-\overline{X}(t;r)}| < \frac{1}{\Pi} \Rightarrow \overline{X}(t;r) > ln\frac{\Pi^2}{20},$$
$$|\frac{\partial G(s,t,\underline{X},\overline{X})}{\partial \underline{X}}| = (0.1s)|(cost).e^{-\underline{X}(t;r)}| < \frac{1}{\Pi} \Rightarrow \underline{X}(t;r) > ln\frac{\Pi^2}{20}.$$

We take $max_{0\leq i\leq n}\{D(x_i^{(k+1)}, x_i^{(k)})\} < 10^{-4}$ and the initial vector $x_i^{(0)} = (1 + r + r^2, 5 - r - r^3)s_i$ is considered for starting. The exact and obtained solution of fuzzy nonlinear Fredholm integral equation in this example at $s = \frac{\Pi}{4}$ are shown in Figure 2.

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