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ON COMPACTNESS AND G-COMPLETENESS IN FUZZY METRIC SPACES

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Abstract. In [Fuzzy Sets and Systems 27 (1988) 385-389], M. Grabiec introduced a notion of completeness for fuzzy metric spaces (in the sense of Kramosil and Michalek) that successfully used to obtain a fuzzy version of Banach´s contraction principle. According to the classical case, one can expect that a compact fuzzy metric space be complete in Grabiec´s sense. We show here that this is not the case, for which we present an example of a compact fuzzy metric space that is not complete in Grabiec´s sense. On the other hand, Grabiec used a notion of compactness to obtain a fuzzy version of Edelstein ´s contraction principle. We present here a generalized version of Grabiec´s version of the Edelstein fixed point theorem and different interesting facts on the topology of fuzzy metric spaces.

1. **Introduction**

By using the notion of a fuzzy metric space, in the sense of Kramosil and Michalek ([5]), Grabiec proved in [3] fuzzy versions of the celebrated Banach fixed point theorem and Edelstein fixed point theorem respectively. To this end Grabiec introduced a notion of a complete fuzzy metric space and of a compact fuzzy metric space respectively.

Later on, George and Veeramani [2] introduced an interesting modification of Kramosil and Michalek´s notion of a fuzzy metric space to introduce a Hausdorff topology. George and Veeramani showed in [2] that with Grabiec´s notion of completeness, even $\mathbb{R}$ fails to be complete. Then they introduced another notion of completeness for which $\mathbb{R}$, with the standard fuzzy metric induced by the Euclidean metric, is complete.

On the other hand, it is well known (see, for instance [4]) that any fuzzy metric space is a metrizable topological space and that any compact fuzzy metric space is complete in George and Veeramani´s sense. In the light of these facts a natural question arises in this context: is a compact fuzzy metric space a complete one in Grabiec´s sense? We answer this question in the negative.

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2. **Preliminaries**

Definition 2.1. [11] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0,1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Paradigmatic examples of continuous t-norm are the product, that we will denote by $\cdot$ and the Lukasievicz t-norm that we will denote by $*_L$. Recall that $a *_L b = \max\{a + b - 1, 0\}$, for all $a, b \in [0,1]$.

Definition 2.2. [5] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that for all $x, y, z \in X$:

$M(x, y, 0) = 0$,
$M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
$M(x, y, t) = M(y, x, t)$,
$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, for all $t, s \geq 0$,
$M(x, y, \_) : [0, \infty) \to [0,1]$ is left continuous.

Given a fuzzy metric space $(X, M, *)$ we define the open ball $B_M(x, r, t)$, for $x \in X$, $0 < r < 1$, and $t > 0$, as the set $B_M(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}$. Obviously, $x \in B_M(x, r, t)$. For each $x \in X$, $0 < r_1 \leq r_2 < 0$ and $0 < t_1 \leq t_2$, we have $B_M(x, r_1, t_1) \subseteq B_M(x, r_2, t_2)$. Consequently, we may define a topology τ_M on X as

$$\tau_M := \{A \subseteq X : \text{for each } x \in A \text{ there are } r \in (0,1) \text{ and } t > 0, \text{ with } B_M(x, r, t) \subseteq A\}.$$

Grabiec gave in [3] a notion of a Cauchy sequence and of a complete fuzzy metric space. In the following a Cauchy sequence and a complete fuzzy metric space in Grabiec´s sense will be called a G-Cauchy sequence and a G-complete fuzzy metric space respectively.

Definition 2.3. [3] A sequence $\{x_n\}_n$ in a fuzzy metric space $(X, M, *)$ is G-Cauchy if $\lim_n M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$.

Note that $\{x_n\}$ is a G-Cauchy sequence if and only if for each $p \in \mathbb{N}$, $\varepsilon > 0$, and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+p}, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A sequence $\{x_n\}_n$ in X is convergent to $x \in X$ if $\lim M(x_n, x, t) = 1$ for each $t > 0$.

Definition 2.4. [3] A fuzzy metric space in which every G-Cauchy sequence is convergent is called G-complete.

Grabiec used the following notion of compactness to obtain a fuzzy version of Edelstein ´s contraction principle:

Definition 2.5. [3] A fuzzy metric space $(X, M, *)$ is called compact if every sequence has a convergent subsequence.

In order to introduce a Hausdorff topology, George and Veeramani gave in [2] the following definition of a fuzzy metric space.

Definition 2.6. [2] The 3-tuple $(X, M, *)$ is said to be a GV-fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$:

$M(x, y, t) > 0$,
$M(x, y, t) = 1$ if and only if $x = y$,
$M(x, y, t) = M(y, x, t)$,
$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
$M(x, y, \_) : (0, \infty) \to [0, 1]$ is continuous.

Each GV-fuzzy metric space $(X, M, *)$ can be considered as a fuzzy metric space by defining $M(x, y, 0) = 0$ for all $x, y \in X$.

Remark 2.7. [2] Let (X, d) be a metric space. Define

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M_d, \cdot)$ is a GV-fuzzy metric space, which is called the standard GV-fuzzy metric space induced by d.

Proposition 2.8. [2] *Let (X, d) be a metric space. Let $(X, M_d, \cdot)$ be the standard fuzzy metric space induced by d. Then the topology τ_d induced by the metric d and the topology τ_{M_d} are the same.*

George and Veeramani gave in [2] the following example which shows that with Grabiec´s notion of completeness, even $\mathbb{R}$ fails to be complete.

Example 2.9. [2] Let $X = \mathbb{R}$, be the set of all real numbers. For $x, y \in X$, $t > 0$, define

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

where d is the Euclidean metric on $\mathbb{R}$. Then $(\mathbb{R}, M_d, \cdot)$ is the standard GV-fuzzy metric space induced by d on $\mathbb{R}$.

Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, for $n \in \mathbb{N}$.

Then

$$\lim_{n\to\infty} M(s_{n+p}, s_n, t) = \lim_{n\to\infty} \frac{t}{t+1/(n+1)+\cdots+1/(n+p)} = 1$$

for all $p \in \mathbb{N}$. Thus $\{s_n\}_n$ is a G-Cauchy sequence in $(\mathbb{R}, M_d, \cdot)$. It is well known that $\{s_n\}_n$ is not convergent, therefore $\mathbb{R}$ fails to be G-complete.

3. On Compactness and G-completeness in Fuzzy Metric Spaces

Because of Example 2.9, George and Veeramani modified the definition of Cauchy sequence in the following way:

Definition 3.1. [2] A sequence $\{x_n\}_n$ in a (GV-)fuzzy metric space $(X, M, *)$ is a Cauchy sequence if for any $\varepsilon > 0, t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$

Definition 3.2. [2] A (GV-)fuzzy metric space is said to be complete if every Cauchy sequence is convergent.

We present another example to show that Grabiec´s notion of completeness is very strong. To this end we first give the corresponding notions of G-Cauchyness and G-completeness for metric spaces.

Definition 3.3. A sequence $\{x_n\}_n$ in a metric space (X, d) is called G-Cauchy sequence if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for each $p \in \mathbb{N}$.

Definition 3.4. A metric space (X, d) is called G-complete if every G-Cauchy sequence in X is convergent with respect to τ_d. In this case, d is called a G-complete metric on X.

Let (X, d) be a metric space and let $(X, M_d, \cdot)$ be the standard GV-fuzzy metric space. Recall (Proposition 2.8) that the topology τ_d, generated by d, coincides with the topology τ_{M_d}. Furthermore a sequence $\{x_n\}_n$ in X is a Cauchy sequence in (X, d) if and only if $\{x_n\}_n$ is a Cauchy sequence in $(X, M_d, \cdot)$. Indeed, let $\{x_n\}_n$ be a Cauchy sequence in (X, d). Fix $t > 0$. Let $\varepsilon \in (0, 1)$ such that $t > 1-\varepsilon$. There exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $m, n \geq n_0$. Therefore $M(x_n, x_m, t) = t/(t + d(x_n, x_m)) > t/(t + \varepsilon) > 1 - \varepsilon$, for all $m, n \geq n_0$. So $\{x_n\}_n$ is a Cauchy sequence in $(X, M_d, \cdot)$. Conversely, if $\{x_n\}_n$ is a Cauchy sequence in $(X, M_d, \cdot)$, given $\varepsilon \in (0, 1/2)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, 1) > 1 - \varepsilon$, for all $m, n \geq n_0$. So $1/(1 + d(x_n, x_m)) > 1 - \varepsilon$, for all $m, n \geq n_0$, thus $d(x_n, x_m) < \varepsilon/(1 - \varepsilon) < 2\varepsilon$, for all $m, n \geq n_0$. We conclude that $\{x_n\}_n$ is a Cauchy sequence in (X, d). Therefore $(X, M_d, \cdot)$ is a complete GV-fuzzy metric space if and only if (X, d) is a complete metric space.

In the same way we obtain that a sequence $\{x_n\}_n$ in X is a G-Cauchy sequence in (X, d) if and only if $\{x_n\}_n$ is a G-Cauchy sequence in $(X, M_d, \cdot)$ and that $(X, M_d, \cdot)$ is a G-complete GV-fuzzy metric space if and only if (X, d) is a G-complete metric space.

Example 3.5. Let $\mathcal{C}([0, 1])$ be the set of all continuous functions from $[0, 1]$ into itself and let d_s be the metric on $\mathcal{C}([0, 1])$ given by $d_s(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. It is known that $(\mathcal{C}([0, 1]), d_s)$ is a complete metric space, therefore $(\mathcal{C}([0, 1]), M_{d_s}, \cdot)$ is a complete GV-fuzzy metric space.

Let $\{f_n\}_n$ be the sequence in $(\mathcal{C}([0, 1]), M_{d_s}, \cdot)$ given by $f_n(x) = x^n$. It is well known that $\{f_n\}_n$ is a non-convergent sequence in $(\mathcal{C}([0, 1]), d_s)$, therefore $\{f_n\}_n$ is a non-convergent sequence in $(\mathcal{C}([0, 1]), M_{d_s}, \cdot)$. We will show that $\{f_n\}_n$ is a G-Cauchy sequence in $(\mathcal{C}([0, 1]), d_s)$, and hence it is a non-convergent G-Cauchy sequence in $(\mathcal{C}([0, 1]), M_{d_s}, \cdot)$, so that $(\mathcal{C}([0, 1]), M_{d_s}, *)$ is not G-complete.
Indeed, let $p \in \mathbb{N}$, then:

$$d(f_n, f_{n+p}) = \sup_{x\in[0,1]} \left|x^n - x^{n+p}\right| = \max_{x\in[0,1]} \left|x^n - x^{n+p}\right|,$$

An easy computation, based on optimization, shows that this maximum is obtained in $x = (\frac{n}{n+p})^{1/p}$, thus:

$$\lim_{n\to\infty} d(f_n, f_{n+p}) = \lim_{n\to\infty} \max_{x\in[0,1]} \left|x^n - x^{n+p}\right| = \lim_{n\to\infty} (\frac{n}{n+p})^{n/p}(1 - \frac{n}{n+p}),$$

so

$$\lim_{n\to\infty} d(f_n, f_{n+p}) = 0.$$

Thus $\{f_n\}_n$ is a G-Cauchy sequence in $(\mathcal{C}([0,1]), d_s)$. We conclude that $(\mathcal{C}([0,1]), M_{d_s}, \cdot)$ is not G-complete.

Recall [10] that a fuzzy metric space $(X, M, *)$ such that:

$$M(x, y, t) \geq \min\{M(x, z, t), M(z, y, t)\} \textit{ for all } x, y, z \in X, t > 0$$

is called a non-Archimedean fuzzy metric space. (A more general notion of non-Archimedean fuzzy metric space can be seen in [1]). In [8] it is proved that each complete non-Archimedean fuzzy metric space is G-complete, so the notion of G-completeness is very interesting in the context of non-Archimedean fuzzy metric spaces. In [8] we can find some fixed point theorems in the context of G-complete fuzzy metric spaces. Other results on G-completeness and fixed point theorems can be found in [7].

Following [4], a (GV-)fuzzy metric space $(X, M, *)$ is called compact if (X, τ_M) is a compact topological space. It is known ([4]) that (X, τ_M) is a metrizable topological space, therefore a (GV-)fuzzy metric space $(X, M, *)$ is compact if and only if every sequence in X has a convergent subsequence. Thus Grabiec´s notion of compactness agrees with the fact that a fuzzy metric space $(X, M, *)$ is compact if (X, τ_M) is a compact topological space.

Gregori and Romaguera obtained the following theorem:

Theorem 3.6. [4] *A (GV-)fuzzy metric space is compact if and only if it is precompact and complete.*

Hence, every compact (GV-)fuzzy metric space is complete in George and Veeramani´s sense. Next we present an example of a compact fuzzy metric space which is not G-complete.

Example 3.7. Let $X = [-1, 1]$. Obviously (X, d) is a compact metric space, where d is the Euclidean metric. Therefore, $(X, M_d, \cdot)$ is a compact fuzzy metric space. Let $\{x_n\}_n$ be a sequence in $(X, M_d, \cdot)$ given by $x_n = \sin\sqrt{n}$, for $n \in \mathbb{N}$. It is easy to show that $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for any $t > 0$ and $p \in \mathbb{N}$. So $\{x_n\}_n$ is a non-convergent G-Cauchy sequence, therefore $(X, M_d, \cdot)$ is a compact fuzzy metric space that is not G-complete.

Note that in the case of non-Archimedean fuzzy metric spaces we have that each compact non-Archimedean fuzzy metric space is G-complete.

On the other hand, George and Veeramani proved in [2] that every closed ball is a closed set in a GV-fuzzy metric space. Nevertheless it is not true in a fuzzy metric space as the following example shows.

Recall that given a fuzzy metric space $(X, M, *)$ we define the closed ball $\overline{B}_M(x, r, t)$, for $x \in X$, $0 < r < 1$, and $t > 0$, as the set $\overline{B}_M(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\}$. Obviously, $x \in \overline{B}_M(x, r, t)$.

Example 3.8. Let (X, d) be a metric space where $X = [0, 1]$ and d is the Euclidean metric on X. In [6] it is shown that $(X, M, *)$ is a fuzzy metric space, where $*$ is any continuous t-norm and M is the fuzzy set in $X \times X \times [0, +\infty)$ given by the following way:

$$M(x, y, t) = 1, \text{ if } d(x, y) < t$$
$$M(x, y, t) = 0, \text{ if } d(x, y) \geq t.$$

We will show that there exists a closed ball $\overline{B}_M(x, r, t)$, and a point $z \in X$ such that $z \in \overline{\overline{B}_M(x, r, t)} \backslash \overline{B}_M(x, r, t)$.

Let $\overline{B}_M(0, 1/2, 1) = \{y : M(0, y, 1) \geqslant 1/2\}$. According to the definition of $M(x, y, t)$ we deduce that $\overline{B}_M(0, 1/2, 1) = \{y : M(0, y, 1) = 1\} = \{y : d(0, y) < 1\} = [0, 1)$.

Let $\{x_n\}_n$ be the sequence in $(X, M, *)$, where $x_n = 1 - \frac{1}{n}$, for all $n \in \mathbb{N}$. Obviously this sequence converges to $z = 1$. On the other hand we have that $x_n \in \overline{B}_M(0, 1/2, 1)$ for all $n \in \mathbb{N}$. So, for $z = 1$ we have that $z \in \overline{\overline{B}_M(0, 1/2, 1)}$, and nevertheless $z \notin \overline{B}_M(0, 1/2, 1)$.

Remark 3.9. Fix $\varepsilon \in (0, 1)$. Given $t \in (0, \varepsilon)$ there exists $n \in \mathbb{N}$ such that $1/n < t$, so $B_M(x, r, 1/n) \subseteq B_M(x, r, t)$ for all $r \in (0, 1)$. On the other hand given $n \in \mathbb{N}$ there exists $t \in (0, \varepsilon)$ such that $B_M(x, r, t) \subseteq B_M(x, r, 1/n)$. Therefore the collection $\{B_M(x, r, t) : t \in (0, \varepsilon)\}$ is a base for the topology τ_M.

By using the notion of a fuzzy metric space, Grabiec proved in [3] the following fuzzy version of the celebrated Edelstein fixed point theorem:

Theorem 3.10. [3] *Let $(X, M, *)$ be a compact fuzzy metric space and let $T : X \to X$ be a self-map satisfying:*

$$M(Tx, Ty, t) > M(x, y, t)$$

for all $x, y \in X$ such that $x \neq y$, and $t > 0$. Then T has a unique fixed point.

We consider the above observation to generalize the fuzzy metric version of fixed point theorem of Edelstein given by Grabiec. Next we show an example where Theorem 3.10 can not be applied.

Example 3.11. Let (X, d) be the metric space where $X = [0, 1]$ and d the Euclidean metric on X. Let $*_L$ be the Lukasiewicz continuous t-norm. We define a fuzzy set M in $X \times X \times [0, +\infty)$ given by the following way:

$$\begin{aligned} M(x, y, 0) &= 0, \\ M(x, y, t) &= 1 - d(x, y), \text{ if } 0 < t \leqslant 1, \\ M(x, y, t) &= 1, \text{ if } t > 1. \end{aligned}$$

It is clear that $(X, M, *_L)$ is a compact fuzzy metric space.

Let $T : X \to X$ be given by $Tx = x/2$ for all $x \in X$ obviously T has a unique fixed point $x = 0$. Nevertheless the conditions of Theorem 3.10 are not satisfied because $M(Tx, Ty, t) = M(x, y, t) = 1$ for all $t > 1$ and for all $x, y \in X$.

Since the topology τ_M can be defined by means of open balls with $t \in (0, \varepsilon), \varepsilon \in (0, 1)$, and for each $x, y \in X$, $M(x, y, \_)$ is nondecreasing, following the proof of Edelstein´s theorem given in [3] we can generalize the fuzzy metric version of the fixed point theorem of Edelstein given by Grabiec as follows:

Theorem 3.12. *Let $(X, M, *)$ be a compact fuzzy metric space and let $T : X \to X$ be a self-map satisfying:*

$$M(Tx, Ty, t) > M(x, y, t)$$

for all $x, y \in X$ such that $x \neq y$, and for all $t \in (0, \varepsilon)$, with $\varepsilon > 0$. Then T has a unique fixed point.

Observe that the conditions of Theorem 3.12 are satisfied in Example 3.11.

We conclude the paper by establishing the corresponding version of Theorem 3.12 for GV-fuzzy metric spaces.

Theorem 3.13. *Let $(X, M, *)$ be a compact GV-fuzzy metric space and let $T : X \to X$ be a self-map satisfying:*

$$M(Tx, Ty, t) > M(x, y, t)$$

for all $x, y \in X$ such that $x \neq y$, and for all $t \in (0, \varepsilon)$, with $\varepsilon > 0$. Then T has a unique fixed point.

Proof. Let M' be the fuzzy set on $X^2 \times [0, \infty)$ defined as $M'(x, y, 0) = 0$ and $M'(x, y, t) = M(x, y, t)$ for all $x, y \in X$ and $t > 0$. Then $(X, M', *)$ is a fuzzy metric space such that $\tau_{M'} = \tau_M$ (see the comment below Definition 2.6). Hence $(X, M', *)$ is a compact fuzzy metric space which satisfies the conditions of Theorem 3.12, so T has a unique fixed point. □

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