ROUGH SET OVER DUAL-UNIVERSES IN FUZZY APPROXIMATION SPACE

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Abstract. To tackle the problem with inexact, uncertainty and vague knowledge, constructive method is utilized to formulate lower and upper approximation sets. Rough set model over dual-universes in fuzzy approximation space is constructed. In this paper, we introduce the concept of rough set over dual-universes in fuzzy approximation space by means of cut set. Then, we discuss properties of rough set over dual-universes in fuzzy approximation space from two viewpoints: approximation operators and cut set of fuzzy set. Reduction of attributes and rules extraction of rough set over dual-universes in fuzzy approximation space are presented. Finally, an example of disease diagnoses expert system illustrates the possibility and efficiency of rough set over dual-universes in fuzzy approximation space.

1. Introduction

Rough set [10, 11] introduced by Pawlak in 1982 has received a lot of attention on areas in both of real-life applications and the theory itself. Meanwhile, the real-life applications promote rough set theory research by means of extending structures of rough set.

Pawlak rough set is discussed in approximation space (U, R), which is constituted by universe U, equivalence relation R and approximation space (U, R). The promoted research of rough set is an extension of universe, equivalence relation and approximation space. For equivalence relation, the notions of approximation operators have been generalized by similarity relation [14], tolerance relation [16, 8],dominance relation(or preference relation) [3, 30] and general binary relation on the universe of discourse [27, 28]. In view of approximation space, many research communities have studied rough set in fuzzy approximation space [1, 2], probability approximation space [32, 26] and grey approximation space [21]. In the aspect of universes, rough set models start with the "same" framework (two universes of discourse with a relation) [18, 22, 23, 19, 20, 13, 12, 5, 4, 9, 7, 6, 31, 24, 25, 17, 15]. For rough set extension on universes, it is initially presented by Wong et al. [18] in an attempt to establish a unified framework for representing uncertain information. Later scholars

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have done much research by means of constructive method and axiomatic approach. Wu et al. [22, 23, 19, 20] presented generalized fuzzy rough set model, discussed constructive and axiomatic approaches of fuzzy approximation operators in generalized fuzzy rough set model and characterized various classes of fuzzy rough sets by different sets of axioms in the axiomatic approach. Pei [13, 12] mainly forced on researching algebraic characterization of rough set extension on two universes. Li [5, 4] formulated four types of rough fuzzy approximation operators on a single universe induced from rough set extension on two universes and investigated the generalizations of the rough approximation operators in a generalized fuzzy approximation space. Mi [9] introduced a general framework for the study of T-fuzzy rough approximation operators and discussed an operator-oriented characterization of rough set in the axiomatic approach. A novel uncertainty measure of the generalized fuzzy rough sets was proposed also in [9]. Liu [7, 6] introduced the solitary sets, gave solutions of the simultaneous Boolean equations by means of rough set technology and established the connection between rough set extension on two universes and evidence theory for arbitrary binary relations. Zhang [31] developed a general framework for the study of (I, T) rough set extension on two universes by employing an implicator I and a t-norm T. In [24, 25], Yan et al. discussed properties of rough set extension on two universes by introducing character function and relation matrix, proposed algorithms for obtaining lower and upper approximation of rough set extension on two universes and studied Pawlak rough set induced by rough set extension on two universes. Sun et al. [17] defined fuzzy rough set model on two universes using the fuzzy compatible relation between two different universes and discussed two extended models of the fuzzy rough sets model based on the fuzzy rough set model on two universes. Shen et al. [15] proposed the variable precision rough set model on two universes using the inclusion degree. Results in [15] further extend theory of rough set extension on two universes.

From the analysis of literature review, we know that research on rough set extension on two universes mainly focuses on the following aspects:

(1) Some articles have presented a construction of approximations and studied the relation between rough set extension on two universes and Pawlak rough set.

(2) There is no consensus on definitions and descriptions of rough set extension on two universes. Because lower and upper approximation operators of rough set extension on two universes are structured by different constructive methods.

(3) Research communities are seldom touched upon rough set extension on two universes in fuzzy approximation space from cut set.

In a decision making process, we may face a hybrid uncertain environment where fuzziness and roughness exist at the same time. Both rough set theory and fuzzy set theory [29] can be combined into a more flexible framework for the study of imprecise information. We discussed rough set extension on two universes in fuzzy approximation space utilizing cut set in fuzzy set theory. In this paper, we research rough set extension on two universes in fuzzy approximation space on the basis of Yan [25].

To be in conformity with Yan [25], rough set extension on two universes in fuzzy approximation space is called Rough Set over Dual-universes(RSDU for short) in fuzzy approximation space and denoted as RSDUF for short. The objective of this paper is to construct the model of RSDUF and applies RSDUF to expert system. In this paper, we discussed RSDUF utilizing cut set in fuzzy set theory. The organization of this paper is as follows. In section 2, we describe some basic knowledge of fuzzy set, rough set and RSDU. In section 3, the concept of RSDUF is presented and its properties are discussed. In section 4, we introduce expert system and provide a practical example about medical diagnosis dealt with by RSDUF. Finally, conclusions have been made in section 5.

Notation: $I = [0, 1]$ is the unit interval.

A^C denotes the complementation of A.

Let p and q be propositions. The conjunction of p and q, denoted as $p \wedge q$, is the proposition p and q. The disjunction of p and q, denoted as $p \vee q$, is the proposition p or q.

2. Preliminaries

2.1. Fuzzy Set and Fuzzy Relation.

Professor L. A. Zadeh put forward fuzzy set theory in 1965 [29] and since then extraordinary progress has been made in its theory and practice, which demonstrates its mighty application values.

Definition 2.1. (Fuzzy set) Let X be a universe of discourse. A fuzzy set A in X is expressed as a set of ordered pairs: $A = \{(x, \mu_A(x))|x \in X\}$, where $\mu_A : X \to [0, 1]$ is a membership function which associates with each x in X. The value $\mu_A(x)$ at x represents the grade of membership of x in A and is interpreted as the degree to which x belongs to A.

When X is discrete, A can be denoted as $A = \sum\limits_{x_i \in X} \mu_A(x_i)/x_i$. When X is continuous, A can be denoted as $A = \int\limits_X \mu_A(x)/x$.

Let X be a universe of discourse. The class of all fuzzy subsets (respectively, subsets) of X will be denoted by $F(X)$(respectively, by $P(X)$).

$\forall A \in F(X), \quad \alpha \in I, \quad A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$ and $A_{\alpha+} = \{x \in X : \mu_A(x) > \alpha\}$ are $\alpha-$cut set and strong $\alpha-$cut set of A which are denoted by A_α and $A_{\alpha+}$ respectively. The normal cut set A_α is a subset of universe X and $x \in A_\alpha$ shows that the degree of x belonging to A is not less than α. $\alpha(\alpha \in I)$ is taken as a threshold value.

Definition 2.2. (Binary fuzzy relation) Let X and Y be two finite and nonempty universes. A binary fuzzy relation $F(X, Y)$ is a fuzzy set in $X \times Y$. $F(X, Y)$ is defined as

$$F(X, Y) = ((x, y), \mu_F(x, y)), (x, y) \in X \times Y \tag{1}$$

where $\mu_F : X \times Y \to [0, 1]$ is a membership function. When X and Y are discrete, F can be denoted as $F(x_i) = \sum\limits_{x_i \in X, y_j \in Y} \mu_F(x_i, y_j)/y_j$ for short.

Definition 2.3. Let X and Y be two finite and nonempty universes. $F(X, Y)$ is a binary fuzzy relation from X to Y. Define α-cut relation of F as

$$F_\alpha(x, y) = \begin{cases} 1, \mu_F(x, y) \geq \alpha \\ 0, \mu_F(x, y) < \alpha \end{cases} \tag{2}$$

Denote $F_\alpha(x) = \{y|y \in Y \wedge \mu_F(x, y) \geq \alpha\}$.

Fuzzy matrices provide convenient representations for fuzzy relations on finite universes. Let X and Y be two finite and nonempty universes. $F(X, Y)$ is a binary fuzzy relation between X and Y. Fuzzy matrix of F is denoted by $M = [m_{ij}]$, where $m_{ij} = \mu_F(x_i, y_j)$, $x_i \in X$ and $y_j \in Y$. α-cut relation F_α is a crisp relation and the matrix of F_α is denoted by $M_\alpha = [n_{ij}]$, where $n_{ij} = F_\alpha(x_i, y_j)$. M_α is defined as α threshold matrix of M.

For the sake of illustration, we consider the following example.

Example 2.4. Let $X = \{x_1, x_2, \cdots, x_6\}$ and $Y = \{y_1, y_2, \cdots, y_7\}$. F is a fuzzy relation from X to Y.$F(x_1) = \{\frac{0.95}{y_1}, \frac{0.1}{y_2}, \frac{0.1}{y_3}, \frac{0.9}{y_4}, \frac{0.2}{y_5}\}$, $F(x_2) = \{\frac{0.9}{y_1}, \frac{0.95}{y_2}, \frac{0.95}{y_3}, \frac{0.9}{y_4}, \frac{0.2}{y_6}\}$, $F(x_3) = \{\frac{0.9}{y_1}, \frac{0.3}{y_2}, \frac{0.1}{y_3}, \frac{0.9}{y_4}, \frac{0.9}{y_5}, \frac{0.9}{y_6}, \frac{1.0}{y_7}\}$, $F(x_4) = \{\frac{0.9}{y_1}, \frac{0.9}{y_2}, \frac{0.9}{y_3}, \frac{0.9}{y_4}, \frac{0.1}{y_5}, \frac{0.1}{y_6}, \frac{0.2}{y_7}\}$, $F(x_5) = \{\frac{0.1}{y_1}, \frac{0.2}{y_4}, \frac{0.9}{y_5}, \frac{0.9}{y_6}, \frac{1.0}{y_7}\}$, $F(x_6) = \{\frac{0.1}{y_1}, \frac{0.2}{y_3}, \frac{0.9}{y_5}, \frac{0.9}{y_6}, \frac{1.0}{y_7}\}$

The fuzzy matrix M of F is denoted as

$$M = \begin{bmatrix} 0.95 & 0.1 & 0.1 & 0.9 & 0.2 & 0 & 0 \\ 0.9 & 0.95 & 0.95 & 0.9 & 0 & 0.2 & 0 \\ 0.9 & 0.3 & 0.1 & 0.9 & 0.9 & 0.9 & 1 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.2 \\ 0.1 & 0 & 0 & 0.2 & 0.9 & 0.9 & 1 \\ 0.1 & 0 & 0.2 & 0 & 0.9 & 0.9 & 1 \end{bmatrix}.$$

Let $\alpha = 0.9$. The matrix of $F_\alpha$is $M_\alpha = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

2.2. **Rough Set and RSDU.** Rough set theory is an extension of set theory. In rough set theory, a subset of a universe is described as a pair of ordinary sets called the lower approximation and upper approximation.

Definition 2.5. (Rough Set) Let U be a non-empty set and R an equivalence relation on U. The pair (U, R) is called an approximation space. $\forall X \subseteq U$, we define operations $\underline{R}$ and $\overline{R}$ as:

$$\underline{R}X = \{x|[x]_R \subseteq X\}, \overline{R}X = \{x|[x]_R \cap X \neq \emptyset\} \tag{3}$$

If $\underline{R}X = \overline{R}X$, then X is a crisp set on U about R. If $\underline{R}X \neq \overline{R}X$, then X is a rough set on U about R. $\underline{R}X$ is called the lower approximation of X and $\overline{R}X$ is the upper approximation of X.

Definition 2.6. (Inverse Relation) If $R \subseteq X \times Y$ is a relation from X to Y, then R', the inverse relation of R, is the relation defined so that $yR'x$ if and only if xRy, where $x \in X$, $y \in Y$.

In another way, $R' = \{(y,x) \in Y \times X | (x,y) \in R\}$.

The inverse relation of a binary relation R is the relation that occurs when you switch the order of the elements in the relation R. For example, the inverse of the relation 'child of' is the relation 'parent of'.

Definition 2.7. (RSDU) Let U and V be the universe of discourse. $R \subseteq U \times V$ and R' the inverse relation of R. The triple (U, V, R) is called a generalized approximation space. $\forall Y \subseteq V$, the lower approximation and upper approximation of Y over dual-universes with respect to R are defined as

$$\underline{R'}Y = \{x | x \in U, R(x) \subseteq Y \wedge R(x) \neq \emptyset\}, \overline{R'}Y = \{x | x \in U, R(x) \cap Y \neq \emptyset \vee R(x) = \emptyset\} \quad (4)$$

If $\underline{R'}Y = \overline{R'}Y$, then Y is a crisp set on V over dual-universes. If $\underline{R'}Y \neq \overline{R'}Y$, then Y is called a rough set over dual-universes on V denoted by RSDU in this paper.

Operators $\underline{R'}, \overline{R'} : P(V) \to P(U)$ are referred to as approximate operators from $P(V)$ to $P(U)$ where $P(V)$ and $P(U)$ are power sets of U and V.

In order to discuss the relation from U to V distinctly, we use matrix to describe the relation under the help of characteristic function.

Definition 2.8. Let U and V be the universes of discourse. $R \subseteq U \times V$ and $R' \subseteq V \times U$ be the inverse relation of R. $\forall x \in U$ and $\forall y \in V$, the characteristic function of R and the characteristic function of R' are defined respectively as

$$\chi_R(x,y) = \begin{cases} 1 & (x,y) \in R \\ 0 & (x,y) \notin R \end{cases}, \quad \chi_{R'}(y,x) = \begin{cases} 1 & (x,y) \in R \\ 0 & (x,y) \notin R \end{cases}$$

Using characteristic function, we define relation matrix of R denoted by $A = [a_{ij}]_{m \times n}$, where $a_{ij} = \begin{cases} 1 & \chi_R(x_i, y_j) = 1 \\ 0 & \chi_R(x_i, y_j) = 0 \end{cases}$. While the relation matrix of R' denoted by A' is the transpose of matrix A.

If A is a matrix with none row filled with zeros, then A is called an information matrix.

For the sake of illustration, we consider the following example.

Example 2.9. Let U and V denote the universe of discourse. Let $U = \{x_1, x_2, x_3, x_4\}$, $V = \{y_1, y_2, y_3\}$. $R \subseteq U \times V$. $R(x_1) = \{y_1, y_2\}$, $R(x_2) = \{y_2, y_3\}$, $R(x_3) = \{y_3\}$, $R(x_4) = \{y_1, y_3\}$. Then the relation matrix of R is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Suppose $Y = \{y_1, y_3\}$, we have $\underline{R'}Y = \{x_3, x_4\}$, $\overline{R'}Y = \{x_1, x_2, x_3, x_4\}$ from Definition 2.7.

3. RSDUF: RSDU in Fuzzy Approximation Space

Definition 3.1. Let U and V be the universes of discourse and F be a fuzzy relation from U to V. The triple (U, V, F) is called a generalized fuzzy approximation space.

Definition 3.2. (Inverse Fuzzy Relation) If F is a relation from X to Y, then F', the inverse relation of F, is the fuzzy relation defined so that $((y,x),\mu_F(x,y))$ if and only if $((x,y),\mu_F(x,y))$, where $x \in X$, $y \in Y$ and μ_F is the membership function of F.

In another way, $F'(Y,X) = ((y,x),\mu_F(x,y)),(y,x) \in Y \times X$.

The inverse relation of a fuzzy relation F is the fuzzy relation that occurs when you switch the order of the elements in the fuzzy relation F.

Definition 3.3. Let U and V be the universes of discourse, F a fuzzy relation from U to V and F' the inverse relation of F. (U,V,F,F') constructed by U, V, F and F' is called a generalized fuzzy information system.

Definition 3.4. Let (U,V,F,F') be a generalized fuzzy information system. $I = [0,1]$ is the unit interval and $\alpha \in I$. $\forall Y \subseteq V$, the lower approximation and upper approximation of Y over dual-universes about F under α are defined as

$$\begin{aligned} \underline{F'_\alpha} Y &= \{x | x \in U, F_\alpha(x) \subseteq Y \wedge F_\alpha(x) \neq \emptyset\}, \\ \overline{F'_\alpha} Y &= \{x | x \in U, F_\alpha(x) \cap Y \neq \emptyset \vee F_\alpha(x) = \emptyset\} \end{aligned} \tag{5}$$

If $\underline{F'_\alpha} Y = \overline{F'_\alpha} Y$, then Y is a crisp set on V over dual-universes under α in (U,V,F,F'). If $\underline{F'_\alpha} Y \neq \overline{F'_\alpha} Y$, then Y is called a rough set over dual-universes on V under α in (U,V,F,F') denoted as RSDUF in this paper.

Operators $\underline{F'_\alpha}, \overline{F'_\alpha} : P(V) \to P(U)$ are referred to as approximate operators from $P(V)$ to $P(U)$ where $P(V)$ and $P(U)$ are power sets of U and V.

In this paper, we use (U,V,F,α) to denote information system of generalized fuzzy approximation space (U,V,F,F') under α.

The proposed model (Definition 3.4) is similar to Wong[15]. However, there is a minor but strange difference: the relation from Definition 3.4 in this paper has no limit while it is compatible relation in [15]. The application scope of RSDUF is broad. An example is advanced as follows to explain the sense.

Example 3.5. In information system (U,V,F,α), $U = \{x_1,x_2,x_3,x_4,x_5\}$ and $V = \{y_1,y_2,y_3,y_4\}$. The fuzzy relation matrix of F is M. $M = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.6 \\ 0.8 & 0.8 & 0.7 & 1 \\ 1 & 0.5 & 0.8 & 0.5 \\ 0.5 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.9 \end{bmatrix}$.

Let $\alpha = 0.7$. We have $M_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The information matrix M_α has a row whose elements equal to zero. That is the relation between U and V is not a compatibility relation. Definition in [15] cannot be utilized to solve the problem in Example 3.5 for definition about rough set extension on two universes in [15] is based on compatible relation.

Let us further analysis Definition 3.4 from another aspect. From Definition 3.4, POS(Y)=$\underline{F'_\alpha}Y$ is the positive region of Y which can be ruled in as members of the target set Y. $NEG(Y) = U - \overline{F'_\alpha}Y$ is the negative region of Y which can be definitely ruled out as members of the target set. BND(Y)=$\overline{F'_\alpha}Y - \underline{F'_\alpha}Y$ is the boundary region of Y which can neither be ruled in nor ruled out as members of the target set Y. Thus, objects in boundary region of Y could be further discussed to make decision with caution when necessary. Moreover, some information may be loss when we transform fuzzy information system to crisp information system by cut set. In order to further extract useful information from upper approximation set, we weak condition for obtaining upper approximation set by considering objects such that $F_\alpha(x) = \emptyset$.

By above knowable, Definition 3.4 in this paper is reasonable.

Analogously, we discuss rough set over dual-universes on U in (U, V, F, F').

Definition 3.6. Let (U, V, F, F') be a generalized fuzzy information system. $I = [0, 1]$ is the unit interval and $\alpha \in I$. $\forall X \subseteq U$, the lower approximation and upper approximation of X over dual-universes about F under α are defined as

$$\underline{F_\alpha}X = \{y | y \in V, F'_\alpha(y) \subseteq X \wedge F'_\alpha(y) \neq \emptyset\},$$

$$\overline{F_\alpha}X = \{y | y \in V, F'_\alpha(y) \cap X \neq \emptyset \vee F'_\alpha(y) = \emptyset\} \tag{6}$$

If $\underline{F_\alpha}X = \overline{F_\alpha}X$, then X is a crisp set on U over dual-universes under α. If $\underline{F_\alpha}X \neq \overline{F_\alpha}X$, then X is called a rough set over dual-universes on U under α denoted by RSDUF in this paper.

POS(X)=$\underline{F_\alpha}X$ is positive region of X. BNA$(X) = \overline{F_\alpha}X - \underline{F_\alpha}X$ is boundary region of X. NEG(X)=$V - \overline{F_\alpha}X$ is negative region of X.

Definitions 3.4 and 3.6 are relative and are adopted depending on objectives researched. In the following, the properties based on Definition 3.4 are just studied. The properties based on Definition 3.6 can be obtained by analogy.

Remark 3.7. When $\alpha = 1$, RSDUF is RSDU in [29].

Theorem 3.8. *In information system (U, V, F, α), $Y, Y_1, Y_2 \subseteq V$, we have*

(1) $\underline{F'_\alpha}Y \subseteq \overline{F'_\alpha}Y$;

(2) $Y_1 \subseteq Y_2 \Rightarrow \underline{F'_\alpha}Y_1 \subseteq \underline{F'_\alpha}Y_2, \overline{F'_\alpha}Y_1 \subseteq \overline{F'_\alpha}Y_2$;

(3) $\underline{F'_\alpha}(Y_1 \cap Y_2) = \underline{F'_\alpha}Y_1 \cap \underline{F'_\alpha}Y_2$;

(4) $\overline{F'_\alpha}(Y_1 \cup Y_2) = \overline{F'_\alpha}Y_1 \cup \overline{F'_\alpha}Y_2$;

(5) $(\underline{F'_\alpha}Y)^C = \overline{F'_\alpha}(Y^C)$, $\underline{F'_\alpha}(Y^C) = (\overline{F'_\alpha}Y)^C$.

Proof. (1) The proof is straightforward from Definition 3.4

(2) The proof is straightforward from Definition 3.4

(3)$x \in \underline{F'_\alpha} Y_1 \cap \underline{F'_\alpha} Y_2 \Leftrightarrow F_\alpha(x) \in Y_1 \wedge F_\alpha(x) \in Y_2 \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow F_\alpha(x) \in Y_1 \cap Y_2 \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow x \in \underline{F'_\alpha}(Y_1 \cap Y_2)$. Then $\underline{F'_\alpha}(Y_1 \cap Y_2) = \underline{F'_\alpha} Y_1 \cap \underline{F'_\alpha} Y_2$.

(4) $x \in \overline{F'_\alpha}(Y_1 \cup Y_2) \Leftrightarrow F_\alpha(x) \cap (Y_1 \cup Y_2) \neq \emptyset \vee F_\alpha(x) = \emptyset \Leftrightarrow (F_\alpha(x) \cap Y_1) \cup (F_\alpha(x) \cap Y_2) \neq \emptyset \vee F_\alpha(x) = \emptyset \Leftrightarrow F_\alpha(x) \cap Y_1 \neq \emptyset \vee F_\alpha(x) \cap Y_2 \neq \emptyset \vee F_\alpha(x) = \emptyset \Leftrightarrow x \in \overline{F'_\alpha} Y_1 \vee x \in \overline{F'_\alpha} Y_2 \Leftrightarrow x \in \overline{F'_\alpha} Y_1 \cup \overline{F'_\alpha} Y_2$. Then $\overline{F'_\alpha}(Y_1 \cup Y_2) = \overline{F'_\alpha} Y_1 \cup \overline{F'_\alpha} Y_2$.

(5) $x \in \underline{F'_\alpha} Y \Leftrightarrow F_\alpha(x) \subseteq Y \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow F_\alpha(x) \cap Y^C = \emptyset \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow x \notin \overline{F'_\alpha}(Y^C) \Leftrightarrow y \in (\overline{F'_\alpha}(Y^C))^C$. That is $(\underline{F'_\alpha} Y)^C = \overline{F'_\alpha}(Y^C)$. $x \in \underline{F'_\alpha}(Y^C) \Leftrightarrow F_\alpha(x) \subseteq Y^C \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow F_\alpha(x) \cap Y = \emptyset \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow y \notin \overline{F'_\alpha} Y \Leftrightarrow y \in (\overline{F'_\alpha} Y)^C$. Then $\underline{F'_\alpha}(Y^C) = (\overline{F'_\alpha} Y)^C$.

We can obtain $\overline{F'_\alpha}(Y_1 \cap Y_2) \subseteq \overline{F'_\alpha} Y_1 \cap \overline{F'_\alpha} Y_2$ from Theorem 3.8(2). However, $\overline{F'_\alpha}(Y_1 \cap Y_2) = \overline{F'_\alpha} Y_1 \cap \overline{F'_\alpha} Y_2$ is not always correct. As $x \in \overline{F'_\alpha} Y_1 \cap \overline{F'_\alpha} Y_2 \Leftrightarrow x \in \overline{F'_\alpha} Y_1 \wedge x \in \overline{F'_\alpha} Y_2 \vee F_\alpha(x) = \emptyset \Leftrightarrow F_\alpha(x) \cap Y_1 \neq \emptyset \wedge F_\alpha(x) \cap Y_2 \neq \emptyset \vee F_\alpha(x) = \emptyset \Leftarrow (F_\alpha(x) \cap Y_1) \cap (F_\alpha(x) \cap Y_2) \neq \emptyset \vee F_\alpha(x) = \emptyset \Leftrightarrow F_\alpha(x) \cap (Y_1 \cap Y_2) \neq \emptyset \vee F_\alpha(x) = \emptyset \Leftrightarrow x \in \overline{F_\alpha}'(Y_1 \cap Y_2)$. We can also obtain $\underline{F'_\alpha}(Y_1 \cup Y_2) \supseteq \underline{F'_\alpha} Y_1 \cup \underline{F'_\alpha} Y_2$ from Theorem 3.8(2). Whereas, $\underline{F'_\alpha}(Y_1 \cup Y_2) = \underline{F'_\alpha} Y_1 \cup \underline{R'} Y_2$ is not always correct. Since $x \in \underline{F'_\alpha}(Y_1 \cup Y_2) \Leftrightarrow F_\alpha(x) \subseteq Y_1 \cup Y_2 \wedge F_\alpha(x) \neq \emptyset \Leftarrow F_\alpha(x) \subseteq Y_1 \vee F_\alpha(x) \subseteq Y_2 \wedge F_\alpha(x) \neq \emptyset \Leftrightarrow x \in \underline{F'_\alpha} Y_1 \vee x \in \underline{F'_\alpha} Y_2 \Leftrightarrow x \in \underline{F'_\alpha} Y_1 \cup \underline{F'_\alpha} Y_2$. □

Proposition 3.9. *In information system (U, V, F, α), $Y \subseteq V$. If the matrix of F_α is an information matrix, then $\underline{F_\alpha}'\emptyset = \overline{F_\alpha}'\emptyset = \emptyset, \underline{F_\alpha}'V = \overline{F_\alpha}'V = U$.*

Theorem 3.10. *Let $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$. In information system (U, V, F, α_1) and (U, V, F, α_2), if $\alpha_1 < \alpha_2$, then*

(1)$\underline{F_{\alpha_1}} Y \supseteq \underline{F_{\alpha_2}} Y$;

(2)$\overline{F_{\alpha_1}} Y \subseteq \overline{F_{\alpha_2}} Y$.

Proof. (1) $\forall x \in \underline{F_{\alpha_2}} Y \Rightarrow F_{\alpha_2}(x) \subseteq Y \wedge F_{\alpha_2}(x) \neq \emptyset \Rightarrow F_{\alpha_1}(x) \subseteq Y \wedge F_{\alpha_1}(x) \neq \emptyset \Rightarrow x \in \underline{F_{\alpha_1}} Y$, we have $\underline{F_{\alpha_1}} Y \supseteq \underline{F_{\alpha_2}} Y$.

(2) For $(\underline{F'_\alpha} Y)^C = \overline{F'_\alpha}(Y^C)$ from Theorem 3.8(5) and Theorem 3.10(1), we have $\alpha_1 < \alpha_2 \Rightarrow \underline{F_{\alpha_1}} Y \supseteq \underline{F_{\alpha_2}} Y \Rightarrow (\underline{F_{\alpha_1}} Y)^C \subseteq \underline{F_{\alpha_2}} Y)^C \Leftrightarrow (\overline{F_{\alpha_1}}(Y^C) \subseteq \overline{F_{\alpha_2}}(Y^C)$. For the arbitrariness of Y, we have $\overline{F_{\alpha_1}} Y \subseteq \overline{F_{\alpha_2}} Y$. □

Remark 3.11. In information system (U, V, F, α), $\forall Y \subseteq V$. When $\alpha = 0$, $\underline{F'_\alpha} Y = U$.

Corollary 3.12. *Let $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$. In fuzzy approximation space (U, V, F), we have*

(1)$\underline{F'_\alpha} Y = \bigcup_{\lambda \geq \alpha} \underline{F'_\lambda} Y$;

(2)$\overline{F'_\alpha} Y = \bigcup_{\lambda \leq \alpha} \overline{F'_\lambda} Y$.

Corollary 3.13. *Let* $\lambda \in [0,1]$. *In fuzzy approximation space* (U, V, F)*, we have* $\underline{F'} = \bigcup\limits_{\lambda \in [0,1]} \lambda \underline{F'_\lambda}$.

Definition 3.14. In information system (U, V, F, α), $Y_1 \subseteq Y \subseteq V$. If $\underline{F'_{\alpha'_1}} = \underline{F'_{\alpha'}}$ and $\overline{F'_{\alpha'_1}} = \overline{F'_{\alpha'}}$, then Y_1 is called coordinated set of Y. $\forall y \in Y_1$, if $Y_1 - \{y\}$ is not coordinated set of Y_1, then Y_1 is called a reduction of Y.

Example 3.15. In information system (U, V, F, α), $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Set threshold value α and suppose the matrix of α-cut relation from U to V is $M_\alpha = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $Y = \{y_1, y_2, y_3, y_4, y_6\}$. Then $\underline{F'_{\alpha'}} = \{x_2, x_4, x_5\}$ and $\overline{F'_{\alpha'}} = \{x_1, x_2, x_3, x_4, x_5\}$ from Definition 3.4. Let $Y_1 = \{y_1, y_2, y_3, y_4\}$. $Y_1 \subseteq Y \subseteq V$. $\underline{F'_{\alpha'_1}} = \{x_2, x_4, x_5\} = \underline{F'_{\alpha'}}$ and $\overline{F'_{\alpha'_1}} = \{x_1, x_2, x_3, x_4, x_5\} = \overline{F'_{\alpha'}}$. Y_1 is coordinated set of Y from Definition 3.14. $\forall y_i \in Y_1$, $Y_1 - \{y\}$ is not coordinated set of Y_1, so Y_1 is a reduction of Y.

After acquiring knowledge reduction, we often do decision rule acquisition on information system. In RSDUF, lower approximation, upper approximation, boundary region and negative region are utilized to deal with uncertainty problems in fuzzy environment. Rules acquisition based on RSDUF are described in terms of lower approximation, upper approximation, boundary region and negative region. Lower approximation and negative region form definite rules. Upper approximation and boundary region form probable rules. See Example 3.15 below for detail description about rules acquisition.

4. A Disease Diagnosis System Based on RSDUF

RSDUF is an extended model of Pawlak rough set model. RSDUF has important academicals value, but also has important practical significance. In this paper, we take an expert system in disease diagnosis as example to explain the application of RSDUF.

Expert system is comprised of Experts in systems, Users, Man-machine interface, Knowledge database, Reasoning machine. Let us take disease diagnosis as an example.

(1) Experts in systems: Senior doctors.

(2) Users: Patients or other people.

(3) Man-machine interface: There are input interface and output interface. Input interface has two ports, which are for symptoms and threshold respectively. Output interface has four ports, which are for definite and probable rules respectively.

(4) Knowledge database: Knowledge database is built depending on the experiences from experts. Knowledge database corresponds to the relation between diseases and symptoms.

(5) Reasoning machine: According to the input information, calculate the lower and upper approximation utilizing RSDUF theory. Then acquire rules from lower approximation, upper approximation, boundary region and negative domain.

The technical scheme of reasoning machine based on RSDUF is as follows. Firstly, the expert information is stored in a database using the form of fuzzy information matrix. Given $\alpha \in [0,1]$, we can obtain information matrix under α using the definition of α-cut relation. After knowledge reduction, expert knowledge database is obtained. Then what a user has to do is typing in his request and set threshold value α. After further steps by reasoning machine based on RSDUF, significant results have been thereby derived which meet the requirements of the user.

Example 4.1. Let U be the set of diseases, V the set of symptoms, where $U = \{x_1, x_2, \cdots, x_9\}$={common cold, influenza, myocarditis, tuberculosis, viral hepatitis, acute bronchitis, meningitis, pneumonia, acute tonsillitis} and $V = \{y_1, y_2, \cdots, y_{19}\}$ = {diaphoresis, panicky, night sweat, nausea, headache, tired, general malaise, chills, anorexia, a sore throat, dry cough, sneeze, a stuffed-up nose, low grade fever, high fever, afternoon fever, phlegm, heart beat fast, abdominal pain}. F' is the fuzzy relation from V to U. The relation matrix from V to U is denoted by $M = [m_{ij}]$, where $m_{ij} = \mu_F(x_i, y_j)$, which means the degree that the symptom x_i belongs to disease y_j equals to m_{ij}.

Suppose the matrix depending on the fuzzy relation F' from V to U is M. The transpose of M is as follows.

$$M' = \begin{bmatrix}
1 & 0.1 & 0 & 0 & 0.3 & 0 & 0.4 & 0 & 0.2 \\
0.2 & 0 & 1 & 0 & 0 & 0.2 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 1 & 0.3 & 0 & 0 & 0 & 0 \\
0 & 0.3 & 0 & 0.2 & 1 & 0 & 0.95 & 0 & 0 \\
0.95 & 1 & 0.2 & 0.1 & 0 & 0.9 & 1 & 0.3 & 0.3 \\
0.1 & 1 & 0 & 0.95 & 1 & 0.95 & 1 & 0 & 0 \\
0 & 1 & 0 & 0.2 & 1 & 0 & 0 & 0.2 & 0 \\
0.9 & 1 & 0 & 0 & 0.1 & 1 & 0 & 0 & 1 \\
0 & 0.1 & 0 & 1 & 0.9 & 0 & 1 & 0.2 & 0 \\
0 & 0.6 & 0 & 0 & 0.3 & 0 & 0.1 & 0 & 1 \\
0 & 1 & 0.2 & 0.9 & 0 & 1 & 0.3 & 0.9 & 0.1 \\
1 & 0.3 & 0 & 0 & 0.2 & 0 & 0.1 & 0 & 0.1 \\
1 & 0.9 & 0 & 0.2 & 0 & 0.3 & 0 & 0 & 0.1 \\
0.95 & 0 & 0.1 & 0 & 1 & 0 & 0.3 & 0.2 & 0 \\
0 & 1 & 0 & 0.3 & 0 & 1 & 0.95 & 1 & 1 \\
0.1 & 0.2 & 0 & 1 & 0 & 0.3 & 0 & 0.3 & 0 \\
1 & 0.6 & 0 & 0.8 & 0 & 1 & 0 & 1 & 0.1 \\
0.1 & 0.4 & 1 & 0.5 & 0 & 0.8 & 0 & 0.2 & 0 \\
0.7 & 0 & 0.2 & 0 & 1 & 0.2 & 0 & 0.6 & 0
\end{bmatrix}$$

Let $Y = \{y_5, y_{11}, y_{15}, y_{17}\}$and $\alpha = 0.9$. α may be a trust level in expert system. The matrix of $F'_{0.9}$ is denoted as $M_{0.9}$. The transpose of $M_{0.9}$ is as follows.

$$M'_{0.9}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&1&0&0\\1&1&0&0&0&1&1&0&0\\0&1&0&1&1&1&1&0&0\\0&1&0&0&1&0&0&0&0\\1&1&0&0&0&1&0&0&1\\0&0&0&1&1&0&1&0&0\\0&0&0&0&0&0&0&0&1\\0&1&0&1&0&1&0&1&0\\1&0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0&0\\0&1&0&0&0&1&1&1&1\\0&0&0&1&0&0&0&0&0\\1&0&0&0&0&1&0&1&0\\0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\end{bmatrix}$$

Then $\underline{F'_{0.9}}Y=\{x_8\}$ and $\overline{F'_{0.9}}Y=\{x_1,x_2,x_4,x_6,x_7,x_8,x_9\}$. NEG(Y)=$\{x_3,x_5\}$ and BNA$(Y)=\{x_1,x_2,x_4,x_6,x_7,x_9\}$. The explanations are as follows.

(1) If a person has symptoms headache, dry cough, high fever and phlegm, then he suffer disease pneumonia.

This is a definite rule from lower approximation.

(2) If a person just has symptoms headache, dry cough, high fever and phlegm, then he does not suffer disease myocarditis and viral hepatitis.

This is a definite rule from negative region.

(3) If a person has symptoms headache, dry cough, high fever and phlegm, then he might suffer disease common cold, influenza, tuberculosis, acute bronchitis, meningitis, or acute tonsillitis except pneumonia.

This is a probable rule from upper approximation.

(4) If a person has symptoms headache, dry cough, high fever and phlegm, then he might suffer disease common cold, influenza, tuberculosis, acute bronchitis, meningitis, or acute tonsillitis.

This is a probable rule from boundary region.

5. **Conclusions and Future Work**

In this paper, we have developed a general framework for the study of rough sets over dual-universes in fuzzy approximation space. By means of αcut set and constructive method, we have weaken the constriction and constructed lower and upper approximation operators. The main contributions of the paper are:

(1) We utilized cut set to transform fuzzy relation to crisp relation and extended RSDU to fuzzy approximation space.

(2) We discussed properties of RSDUF from approximation operators and presented the operations between RSDUF.

(3) We researched properties of RSDUF from the view of α cut set and presented the relation of approximations under different threshold value.

However, RSDUF research is only in start and is still being worked up. Future work will be performed in researching the reduction of attributes and the extraction of the rules from RSDUF.

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