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AN INTERVAL-VALUED PROGRAMMING APPROACH TO MATRIX GAMES WITH PAYOFFS OF TRIANGULAR INTUITIONISTIC FUZZY NUMBERS

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Abstract. The purpose of this paper is to develop a methodology for solving a new type of matrix games in which payoffs are expressed with triangular intuitionistic fuzzy numbers (TIFNs). In this methodology, the concept of solutions for matrix games with payoffs of TIFNs is introduced. A pair of auxiliary intuitionistic fuzzy programming models for players are established to determine optimal strategies and the value of the matrix game with payoffs of TIFNs. Based on the cut sets and ranking order relations between TIFNs, the intuitionistic fuzzy programming models are transformed into linear programming models, which are solved using the existing simplex method. Validity and applicability of the proposed methodology are illustrated with a numerical example of the market share problem.

1. Introduction

The traditional game theory assumes that payoffs are known exactly by players. However, it is difficult to assess payoffs exactly in real game situations because of imprecise information and fuzzy understanding of situations by players. In such situations, the fuzzy set [22] is a very useful tool to model the problems as games with fuzzy payoffs [5, 7-8, 10]. Therefore, the fuzzy game theory is an active research field in operational research, management science and systems engineering [5-10, 15, 19, 21]. However, in some situations, players could only know the payoffs with some imprecise degree approximately. It is possible that players are not so sure about them. In other words, there may be a hesitation about the approximate payoffs. The fuzzy set uses only a membership function to indicate the degree of belongingness to a fuzzy set under consideration. the degree of non-belongingness is just automatically the complement to 1. The fuzzy set is no means to incorporate the hesitation degree. In 1983, Atanassov [2] introduced the concept of an intuitionistic fuzzy (IF) set which is characterized by two functions expressing the degree of membership and the degree of non-membership, respectively. The hesitation degree is equal to 1 minus both membership and non-membership degrees. The IF set may express and describe information more abundant and flexible than the fuzzy set when uncertain information is involved. The IF set has been applied to different areas [3,11,13]. It is essential to apply the IF set to game problems [1,18].

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The reason is that, in comparison with the fuzzy set, the IF set seems to suitably express an important factor which should be taken into account in game processes, i.e., players's hesitation degree. As far as we know, however, there exist few studies on application of the IF set to resolving game problems [1, 18]. Thereby, the aim of this paper is to develop an effective and practical methodology for solving matrix games with payoffs of triangular IF numbers (TIFNs).

This paper is organized as follows. Section 2 introduces the concept and operations of TIFNs as well as cut sets. We define the ranking order relations between TIFNs and briefly review the ranking order relations of intervals and interval-valued optimization methods. Section 3 gives the concepts of matrix games with payoffs of TIFNs and solutions. A pair of auxiliary IF programming models for players and interval-valued optimization methods are proposed to solve the matrix games with payoffs of TIFNs. Section 4 gives a numerical example of the market share problem. Short conclusion is given in Section 5.

2. **Definitions and Notations**

Let $\hat{a} = [a_L, a_R]$ be an interval on the real number R and $m(\hat{a}) = (a_L + a_R)/2$ be the mid-point of the interval $\hat{a}$.We briefly review the ranking order relations [14, 16, 17, 20] and interval-valued maximization and minimization problems.

Definition 2.1. [14, 16, 20] Let $\hat{a} = [a_L, a_R]$ and $\hat{b} = [b_L, b_R]$ be two intervals. Then, $\hat{a} \geq \hat{b}$ if and only if $a_L \geq b_L$ and $a_R \geq b_R$. Similarly, $\hat{a} \leq \hat{b}$ if and only if $a_L \leq b_L$ and $a_R \leq b_R$.

Definition 2.2. [14, 16] Let $\hat{a} = [a_L, a_R]$ be an interval.The maximization problem with the interval-valued objective function is expressed as $\max\{\hat{a}|\hat{a} \in \Omega_1\}$,which is equivalent to the bi-objective mathematical programming: $\max\{(a_L, m(\hat{a}))|\hat{a} \in \Omega_1\}$, where Ω_1 is a set of constraint conditions in which the interval-valued variable $\hat{a}$ should satisfy according to requirements in the real situations.

Definition 2.3. [14, 16] Let $\hat{a} = [a_L, a_R]$ be an interval.The minimization problem with the interval-valued objective function is expressed as $\min\{\hat{a}|\hat{a} \in \Omega_1\}$,which is equal to the bi-objective mathematical programming: $\min\{(a_R, m(\hat{a}))|\hat{a} \in \Omega_1\}$.

The concept of a TIFN is of important use to express ill-known quantities such as "approximately 5" and "a sizeable value" in real game situations. In this subsection, TIFNs and their operations are defined as follows.

Definition 2.4. A TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a special IF-set on the real number set R of real numbers, whose membership function and non-membership function are defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} (x-\underline{a})/(a-\underline{a})w_{\tilde{a}} & \text{if } \underline{a} \leq x < a \\ w_{\tilde{a}} & \text{if } x = a \\ (\overline{a}-x)/(\overline{a}-a)w_{\tilde{a}} & \text{if } a < x \leq \overline{a} \\ 0 & \text{if } x < \underline{a} \text{ or } x > \overline{a} \end{cases} \tag{1}$$

and

$$\upsilon_{\tilde{a}}(x) = \begin{cases} [a - x + u_{\tilde{a}}(x - \underline{a})]/(a - \underline{a}) & \text{if } \underline{a} \leq x < a \\ u_{\tilde{a}} & \text{if } x = a \\ [x - a + u_{\tilde{a}}(\overline{a} - x)]/(\overline{a} - a) & \text{if } a < x \leq \overline{a} \\ 1 & \text{if } x < \underline{a} \text{ or } x > \overline{a}, \end{cases} \quad (2)$$

respectively.

In Figure 1, we report the relationship between the membership function and the non-membership function of the TIFN.

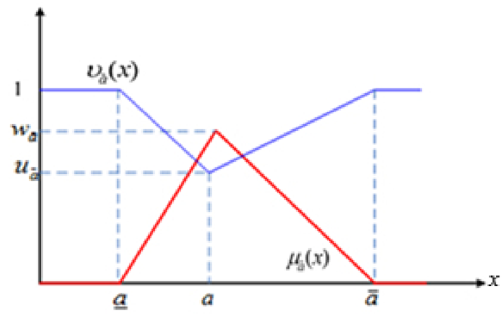

FIGURE 1. Triangular Intuitionistic Fuzzy Number $\tilde{\alpha}$

The values $w_{\tilde{a}}$ and $u_{\tilde{a}}$ respectively represent the maximum membership degree and the minimum non-membership degree which satisfy the conditions, that is, $0 \leq w_{\tilde{a}} \leq 1$, $0 \leq u_{\tilde{a}} \leq 1$ and $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$.

Let$\chi_{\tilde{a}(x)} = 1 - \mu_{\tilde{a}(x)} - \nu_{\tilde{a}(x)}$, which is called as the intuitionistic fuzzy index of an element x in $\tilde{a}$. It is the degree of indeterminacy membership of the element x to $\tilde{a}$.

Obviously, if $w_{\tilde{a}} + u_{\tilde{a}} = 1$, then $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is reduced to $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, 1 - w_{\tilde{a}} >$, which is just a triangular fuzzy number(TFN). Therefore, the concept of the TIFN is a generalization of that of the TFN [12].

Two new parameters $w_{\tilde{a}}$ and $u_{\tilde{a}}$ are introduced to reflect the confidence level and non-confidence level of a TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$, respectively. Compared with a TFN, a TIFNs may express more uncertainty.

Definition 2.5. Let $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ and $\tilde{b} =< (\underline{b}, b, \bar{b}); w_{\tilde{b}}, u_{\tilde{b}} >$ be two TIFNs and γ be a real number. The arithmetic operations are defined as follows:

$$\tilde{a} + \tilde{b} =< (\underline{a} + \underline{b}, a + b, \bar{a} + \bar{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > \quad (3)$$

$$\tilde{a} - \tilde{b} =< (\underline{a} - \bar{b}, a - b, \bar{a} - \underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > \quad (4)$$

$$\tilde{a}\tilde{b} = \begin{cases} < (\underline{a}\underline{b}, ab, \bar{a}\bar{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ < (\underline{a}\bar{b}, ab, \bar{a}\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}\bar{b}, ab, \underline{a}\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases} \quad (5)$$

$$\tilde{a}/\tilde{b} = \begin{cases} < (\underline{a}/\bar{b}, a/b, \bar{a}/\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}/\bar{b}, a/b, \underline{a}/\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}/\underline{b}, a/b, \underline{a}/\bar{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases} \quad (6)$$

$$\gamma\tilde{a} = \begin{cases} <(\gamma\underline{a}, a, \gamma\overline{a}); w_{\tilde{a}}, u_{\tilde{a}}> & \text{if } \gamma > 0 \\ <(\gamma\overline{a}, a, \gamma\underline{a}); w_{\tilde{a}}, u_{\tilde{a}}> & \text{if } \gamma < 0 \end{cases} \quad (7)$$

$$\tilde{a}^{-1} =< (1/\overline{a}, 1/a, 1/\underline{a}); w_{\tilde{a}}, u_{\tilde{a}} > \text{if } \tilde{a} > 0 \quad (8)$$

The cut sets and the ranking order relations of TIFNs are defined as follows.

Definition 2.6. [3, 4] A (α, β)-cut set of a TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a crisp subset of R which is defined as $\tilde{a}_{\alpha,\beta} = \{x|\mu_{\tilde{a}}(x) \geq \alpha, \nu_{\tilde{a}}(x) \leq \beta\}$, where $0 \leq \alpha \leq w_{\tilde{a}}$, $u_{\tilde{a}} \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1$.

Definition 2.7. [3, 4] A α-cut set of a TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a crisp subset of R which is defined as $\tilde{a}_{\alpha} = \{x|\mu_{\tilde{a}}(x) \geq \alpha\}$, where $0 \leq \alpha \leq w_{\tilde{a}}$.

For any $\alpha \in [0, w_{\tilde{a}}]$, it is easily derived from Definitions 2.4 and 2.7 that $\tilde{a}_{\alpha}$ is a closed interval and calculated as $\tilde{a}_{\alpha} = \left[\underline{a} + \frac{\alpha}{w_{\tilde{a}}}(a - \underline{a}), a - \frac{\alpha}{w_{\tilde{a}}}(\bar{a} - a)\right]$.

Definition 2.8. [3, 4] A β-cut set of a TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a crisp subset of R which is defined as $\tilde{a}_{\beta} = \{x|\nu_{\tilde{a}}(x) \leq \beta\}$, where $u_{\tilde{a}} \leq \beta \leq 1$.

For any $\beta \in [u_{\tilde{a}}, 1]$, it is easily derived from Definitions 2.4 and 2.8 that a $\tilde{a}_{\beta}$ is a closed interval and calculated as $\tilde{a}_{\beta} = \left[[(1-\beta)a + (\beta - u_{\tilde{a}})\underline{a}]/(1 - u_{\tilde{a}}), [(1-\beta)a + (\beta - u_{\tilde{a}})\bar{a}]/(1 - u_{\tilde{a}})\right]$.

Theorem 2.9 is easily derived from Definitions 2.6-2.8.

Theorem 2.9. *Let $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ be any TIFN. For any $\alpha \in [0, w_{\tilde{a}}]$ and $\beta \in [u_{\tilde{a}}, 1]$, where $0 \leq \alpha + \beta \leq 1$, the following equality is valid $\tilde{a}_{\alpha,\beta} = \tilde{a}_{\alpha} \bigcap \tilde{a}_{\beta}$.*

The ranking order of TIFNs is a difficult problem. In this paper, a new ranking order relation of TIFNs is defined as in the following Definition 2.10.

Definition 2.10. Let $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ and $\tilde{b} =< (\underline{b}, b, \bar{b}); w_{\tilde{b}}, u_{\tilde{b}} >$ be two TIFNs. $\tilde{a}_{\alpha}$ and $\tilde{b}_{\alpha}$ are any α-cut sets of $\tilde{a}$ and $\tilde{b}$, $\tilde{a}_{\beta}$ and $\tilde{b}_{\beta}$ are any β-cut sets of $\tilde{a}$ and $\tilde{b}$, respectively. Then, we stipulate the following relations.

(1) $\tilde{a} \tilde{\leq} \tilde{b}$ if and only if $\tilde{a}_{\alpha} \leq \tilde{b}_{\alpha}$ and $\tilde{a}_{\beta} \leq \tilde{b}_{\beta}$ for any $\alpha \in [0, min\{w_{\tilde{a}}, w_{\tilde{b}}\}]$ and $\beta \in [max\{u_{\tilde{a}}, u_{\tilde{b}}\}, 1]$, where $0 \leq \alpha + \beta \leq 1$;

(2) $\tilde{a} \tilde{\geq} \tilde{b}$ if and only if $\tilde{a}_{\alpha} \geq \tilde{b}_{\alpha}$ and $\tilde{a}_{\beta} \geq \tilde{b}_{\beta}$ for any $\alpha \in [0, min\{w_{\tilde{a}}, w_{\tilde{b}}\}]$ and $\beta \in [max\{u_{\tilde{a}}, u_{\tilde{b}}\}, 1]$, where $0 \leq \alpha + \beta \leq 1$.

The symbols "$\tilde{\leq}$" and "$\tilde{\geq}$" are intuitionistic fuzzy versions of the order relations "$\leq$" and "$\geq$" on the set of real numbers, which have the linguistic interpretation "approximately less than or equal to" and "approximately greater than or equal to", respectively.

3. **Solutions of Matrix Games with Payoffs of TIFNs**

Let us consider the matrix games with payoffs of TIFNs. Assume that $S_1 = \{\delta_1, \delta_2, \cdots, \delta_m\}$ and $S_2 = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ are sets of pure strategies for two players I and II, respectively. The vectors $\mathbf{y} = (y_1, y_2, \cdots, y_m)^T$ and $\mathbf{z} = (z_1, z_2, \cdots, z_n)^T$ are probabilities in which players I and II choose their pure strategies $\delta_i \in S_1$ and $\sigma_j \in S_2$, respectively. Sets of all mixed strategies for players I and II are denoted by $\mathbf{Y}=\left\{\mathbf{y}\middle|\sum_{i=1}^m y_i=1, y_i\geq 0(i=1,2,\cdots,m)\right\}$, and $\mathbf{Z}=\left\{\mathbf{z}\middle|\sum_{j=1}^n z_j=1, z_j\geq 0(j=1,2,\cdots,n)\right\}$. respectively.

Without loss of generality, assume that player I's payoff matrix is given as $\tilde{\mathbf{A}}=(\tilde{a}_{ij})_{m\times n}$, whose elements are TIFNs $\tilde{a}_{ij}=<(\underline{a}_{ij},a_{ij},\bar{a}_{ij});w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}}>(i=1,2,\cdots,m;j=1,2,\cdots,n)$ defined as above. Thus, such a two-person zero-sum matrix game with payoffs of TIFNs is usually called the matrix game $\tilde{\mathbf{A}}$ with payoffs of TIFNs for short. Now, the concept of a solution of any matrix game $\tilde{\mathbf{A}}$ with payoffs of TIFNs is defined as follows.

Definition 3.1. Let $\tilde{v}=<(\underline{v},v,\bar{v});w_{\tilde{v}},u_{\tilde{v}}>$ and $\tilde{\omega}=<(\underline{\omega},\omega,\bar{\omega});w_{\tilde{\omega}},u_{\tilde{\omega}}>$ be TIFNs. Assume that there exist $\mathbf{y}^*\in\mathbf{Y}$ and $\mathbf{z}^*\in\mathbf{Z}$ such that they satisfy both the conditions $\mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z}\tilde{\geq}\tilde{v}$ and $\mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z}^*\tilde{\leq}\tilde{\omega}$ for any strategies $\mathbf{y}\in\mathbf{Y}$ and $\mathbf{z}\in\mathbf{Z}$. Then, $(\mathbf{y}^*,\mathbf{z}^*,\tilde{v},\tilde{\omega})$ is called a reasonable solution of a matrix game $\tilde{\mathbf{A}}$ with payoffs of TIFNs.

If $(\mathbf{y}^*,\mathbf{z}^*,\tilde{v},\tilde{\omega})$ is a reasonable solution of a matrix game $\tilde{\mathbf{A}}$ with payoffs of TIFNs, then $\tilde{v}$ and $\tilde{\omega}$ are called reasonable values of the players I and II,$\mathbf{y}^*$ and $\mathbf{z}^*$ are called reasonable strategies for players I and II, respectively. Let V and W denote the sets of all reasonable values $\tilde{v}$ and $\tilde{\omega}$ for players I and II, respectively.

It is worth noticing that Definition 3.1 only gives the notion of the reasonable solution rather than the notion of an optimal solution.

Definition 3.2. Assume that there exist $\tilde{v}^*\in V$ and $\tilde{\omega}^*\in W$. If there do not exist any $\tilde{v}\in V(\tilde{v}\neq\tilde{v}^*)$ and $\tilde{\omega}\in W(\tilde{\omega}\neq\tilde{\omega}^*)$ such that they satisfy the following conditions: $\tilde{v}\tilde{\leq}\tilde{v}^*$ and $\tilde{\omega}\tilde{\geq}\tilde{\omega}^*$,then $(\mathbf{y}^*,\mathbf{z}^*,\tilde{v}^*,\tilde{\omega}^*)$ is called a solution of the matrix game $\tilde{\mathbf{A}}$ with payoffs of TIFNs. $\mathbf{y}^*\in\mathbf{Y}$ is called player I' maximin strategy and $\mathbf{z}^*\in\mathbf{Z}$ is called player II's minimax strategy. $\tilde{v}^*$ and $\tilde{\omega}^*$ are called player I's gain-floor and player II's loss-ceiling, respectively.

According to Definitions 3.1 and 3.2, player I's maximin strategy $\mathbf{y}^*\in\mathbf{Y}$ and player II's minimax strategy $\mathbf{z}^*\in\mathbf{Z}$ can be obtained through solving the pair of IF mathematical programming models as follows:

$$\max\{\tilde{v}\}$$
$$\text{s. t.}\begin{cases}\sum_{i=1}^m\tilde{a}_{ij}y_iz_j\tilde{\geq}\tilde{v}\ (j=1,2,\cdots,n)\text{ for any }\mathbf{z}\in\mathbf{Z}\\ \sum_{i=1}^m y_i=1\\ y_i\geq 0\ (i=1,2,\cdots,m)\end{cases}\tag{9}$$

and

$$\min\{\tilde{\omega}\}$$
$$\text{s. t.}\begin{cases}\sum_{j=1}^n\tilde{a}_{ij}y_iz_j\tilde{\leq}\tilde{\omega}\ (i=1,2,\cdots,m)\text{ for any }\mathbf{y}\in\mathbf{Y}\\ \sum_{j=1}^n z_j=1\\ z_j\geq 0\ (j=1,2,\cdots,n),\end{cases}\tag{10}$$

respectively,where $\tilde{v}$ and $\tilde{\omega}$ TIFN variables.

It makes sense to consider only the extreme points of the sets **Y** and **Z** in the constraints of equations (9) and (10) since "$\tilde{\leq}$" and "$\tilde{\geq}$" preserve the ranking order relations when TIFNs are multiplied by positive numbers. The equations (9) and (10) can be changed into the intuitionistic fuzzy mathematical programming models,

$$\max\{\tilde{v}\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{m}\tilde{a}_{ij}y_i\tilde{\geq}\tilde{v}\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1\\ y_i\geq 0\ (i=1,2,\cdots,m)\end{cases}\tag{11}$$

and

$$\min\{\tilde{\omega}\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n}\tilde{a}_{ij}z_j\tilde{\leq}\tilde{\omega}\ (i=1,2,\cdots,m)\\ \sum_{j=1}^{n}z_j=1\\ z_j\geq 0\ (j=1,2,\cdots,n)\end{cases}\tag{12}$$

In this study, the above IF optimization problems are made in the sense of Definitions 2.1-2.3 and 2.9. In the sequel, we will focus on studying the solution method of equations (11) and (12). For any values $\alpha\in\Omega\equiv[0,\min\{\omega_{\tilde{a}_{ij}}|i=1,2,\cdots,m;j=1,2,\cdots,n\}]$ and $\beta\in\Pi\equiv[\max\{u_{\tilde{a}_{ij}}|i=1,2,\cdots,m;j=1,2,\cdots,n\},1]$ using Definition 2.9, equation(11) is transformed into the interval-valued bi-objective mathematical programming as follows:

$$\max\{\tilde{v}_\alpha,\tilde{v}_\beta\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{m}(\tilde{a}_{ij}y_i)_\alpha\geq\tilde{v}_\alpha,\quad\sum_{i=1}^{m}(\tilde{a}_{ij}y_i)_\beta\geq\tilde{v}_\beta\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1,y_i\geq 0\ (i=1,2,\cdots,m)\end{cases}\tag{13}$$

According to Definition 2.1, equation (13) can be rewritten as follows:

$$\max\{[\tilde{v}_\alpha^L,\tilde{v}_\alpha^R],[\tilde{v}_\beta^L,\tilde{v}_\beta^R]\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^L y_i\geq\tilde{v}_\alpha^L,\quad\sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^R y_i\geq\tilde{v}_\alpha^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^L y_i\geq\tilde{v}_\beta^L,\quad\sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^R y_i\geq\tilde{v}_\beta^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1,y_i\geq 0\ (i=1,2,\cdots,m)\end{cases}\tag{14}$$

The two objective functions in equation (14) may be regarded as equal importance, i.e., their weights are the same.

Therefore, equation(14) can be aggregated into the interval-valued mathematical programming as follows:

$$\max\{[(\tilde{v}_\alpha^L+\tilde{v}_\beta^L)/2,(\tilde{v}_\alpha^R+\tilde{v}_\beta^R)/2]\}$$
$$\text{s. t.}\begin{cases}\sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^L y_i\ge\tilde{v}_\alpha^L,\quad \sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^R y_i\ge\tilde{v}_\alpha^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^L y_i\ge\tilde{v}_\beta^L,\quad \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^R y_i\ge\tilde{v}_\beta^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1,y_i\ge 0\ (i=1,2,\cdots,m)\end{cases}\tag{15}$$

According to Definition 2.2, equation(15) can be further converted into the bi-objective linear programming as follows:

$$\max\{(\tilde{v}_\alpha^L+\tilde{v}_\beta^L)/2,(\tilde{v}_\alpha^L+\tilde{v}_\beta^L+\tilde{v}_\alpha^R+\tilde{v}_\beta^R)/4\}$$
$$\text{s. t.}\begin{cases}\sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^L y_i\ge\tilde{v}_\alpha^L,\quad \sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^R y_i\ge\tilde{v}_\alpha^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^L y_i\ge\tilde{v}_\beta^L,\quad \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^R y_i\ge\tilde{v}_\beta^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1,y_i\ge 0\ (i=1,2,\cdots,m)\end{cases}\tag{16}$$

There are few standard ways of defining a solution of multiobjective programming such as equation (16). Normally, the concept of Pareto optimal/efficient solutions is commonly-used. There exist several solution methods for them. However, in this study we focus on a weighted average approach to solve equation(16) in the sense of Pareto optimality.

According to equation (16), the linear programming model can be easily constructed as follows:

$$\max\{\lambda(\tilde{v}_\alpha^L+\tilde{v}_\beta^L)/2+(1-\lambda)(\tilde{v}_\alpha^L+\tilde{v}_\beta^L+\tilde{v}_\alpha^R+\tilde{v}_\beta^R)/4\}$$
$$\text{s. t.}\begin{cases}\sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^L y_i\ge\tilde{v}_\alpha^L,\quad \sum_{i=1}^{m}(\tilde{a}_{ij})_\alpha^R y_i\ge\tilde{v}_\alpha^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^L y_i\ge\tilde{v}_\beta^L,\quad \sum_{i=1}^{m}(\tilde{a}_{ij})_\beta^R y_i\ge\tilde{v}_\beta^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1,y_i\ge 0\ (i=1,2,\cdots,m)\end{cases}\tag{17}$$

where $\lambda\in[0,1]$ is the weight determined by players a priori according to situations.

According to Definitions 2.7 and 2.8, equation(17) can be explicitly written as below

$$\max\{\lambda(\tilde{v}_\alpha^L+\tilde{v}_\beta^L)/2+(1-\lambda)(\tilde{v}_\alpha^L+\tilde{v}_\beta^L+\tilde{v}_\alpha^R+\tilde{v}_\beta^R)/4\}$$

$$\text{s. t.}\begin{cases}\sum_{i=1}^m[\underline{a}_{ij}+\alpha(a_{ij}-\underline{a}_{ij})/w_{\tilde{a}_{ij}}]y_i\geq\tilde{v}_\alpha^L\ (j=1,2,\cdots,n)\\ \sum_{i=1}^m[\overline{a}_{ij}-\alpha(\overline{a}_{ij}-a_{ij})/w_{\tilde{a}_{ij}}]y_i\geq\tilde{v}_\alpha^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^m\{[(1-\beta)a_{ij}+(\beta-u_{\tilde{a}_{ij}})\underline{a}_{ij}]/(1-u_{\tilde{a}_{ij}})\}y_i\geq\tilde{v}_\beta^L\ (j=1,2,\cdots,n)\\ \sum_{i=1}^m\{[(1-\beta)a_{ij}+(\beta-u_{\tilde{a}_{ij}})\overline{a}_{ij}]/(1-u_{\tilde{a}_{ij}})\}y_i\geq\tilde{v}_\beta^R\ (j=1,2,\cdots,n)\\ \sum_{i=1}^m y_i=1,\ y_i\geq 0\ (i=1,2,\cdots,m)\end{cases}\tag{18}$$

where,$\tilde{v}_\alpha^L$, $\tilde{v}_\alpha^R$, $\tilde{v}_\beta^L$, $\tilde{v}_\beta^R$ and y_i $(i=1,2,\cdots,m)$ are decision variables. For given parameters $\lambda\in[0,1]$, $\alpha\in\Omega$ and $\beta\in\Pi$, using the simplex method for linear programming, an optimal solution of equation (18) is obtained, denoted by $(\mathbf{y}^*,\tilde{v}_\alpha^{L*},\tilde{v}_\alpha^{R*},\tilde{v}_\beta^{L*},\tilde{v}_\beta^{R*})$. The equation (18) can be used to calculate player I's maximin strategy $\mathbf{y}^*$ and the corresponding upper and lower bounds of α-cut sets and β-cut sets of the gain-floor $\tilde{v}^*$.

According to Theorem 2.9, a (α,β)-cut set of player I's gain-floor $\tilde{v}^*$ can be obtained, denoted by $\tilde{v}^*_{\alpha,\beta}$. The value of $\tilde{v}^*_{\alpha,\beta}$ represents that player I's gain-floor $\tilde{v}^*$ may appear in the associated range at the level (α,β). For two levels (α_1,β_1) and (α_2,β_2) such that they satisfy $\alpha_1\geq\alpha_2$ and $\beta_1\leq\beta_2$, we have $\tilde{v}^*_{\alpha_1,\beta_1}\leq\tilde{v}^*_{\alpha_2,\beta_2}$. Different values of $\tilde{v}^*_{\alpha,\beta}$ represent different intervals and the uncertainty levels of player I's gain-floor $\tilde{v}^*$. Specifically, if $\alpha=0$ and $\beta=1$, then $\tilde{v}^*_{\alpha,\beta}$ has the widest interval whereas the lowest possibility, which indicates that the value of player I's gain-floor $\tilde{v}^*$ will never fall outside this range. At other extreme case of $\alpha\in\Omega$ and $\beta\in\Pi$, the interval $\tilde{v}^*_{\alpha,\beta}$ is the most likely value of player I's gain-floor $\tilde{v}^*$. $\tilde{v}^*_{\alpha,\beta}$ may be used to approximate the value of player I's gain-floor $\tilde{v}^*$ at any level (α,β).

According to Definition 2.1, the equation (12) can be similarly changed into the interval-valued mathematical programming,

$$\min\{\tilde{\omega}_\alpha,\tilde{\omega}_\beta\}$$

$$\text{s. t.}\begin{cases}\sum_{j=1}^n(\tilde{a}_{ij}z_j)_\alpha\leq\tilde{\omega}_\alpha,\ \sum_{j=1}^n(\tilde{a}_{ij}z_j)_\beta\leq\tilde{\omega}_\beta\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n z_j=1,\\ z_j\geq 0\ (j=1,2,\cdots,n)\end{cases}\tag{19}$$

which infers that

$$\min\{[\tilde{\omega}_\alpha^L,\tilde{\omega}_\alpha^R],[\tilde{\omega}_\beta^L,\tilde{\omega}_\beta^R]\}$$

$$\text{s. t.}\begin{cases}\sum_{j=1}^n(\tilde{a}_{ij})_\alpha^L z_j\leq\tilde{\omega}_\alpha^L,\ \ \sum_{j=1}^n(\tilde{a}_{ij})_\alpha^R z_j\leq\tilde{\omega}_\alpha^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n(\tilde{a}_{ij})_\beta^L z_j\leq\tilde{\omega}_\beta^L,\ \ \sum_{j=1}^n(\tilde{a}_{ij})_\beta^R z_j\leq\tilde{\omega}_\beta^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n z_j=1,\ z_j\geq 0\ (j=1,2,\cdots,n)\end{cases}\tag{20}$$

The equation (20) is changed into the follow mathematical programming,

$$\min\{[(\tilde{\omega}_\alpha^L+\tilde{\omega}_\beta^L)/2,(\tilde{\omega}_\alpha^R+\tilde{\omega}_\beta^R)/2]\}$$
$$\text{s. t.}\begin{cases}\sum_{j=1}^n(\tilde{a}_{ij})_\alpha^L z_j\le\tilde{\omega}_\alpha^L,\ \ \sum_{j=1}^n(\tilde{a}_{ij})_\alpha^R z_j\le\tilde{\omega}_\alpha^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n(\tilde{a}_{ij})_\beta^L z_j\le\tilde{\omega}_\beta^L,\ \ \sum_{j=1}^n(\tilde{a}_{ij})_\beta^R z_j\le\tilde{\omega}_\beta^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n z_j=1,z_j\ge0\ (j=1,2,\cdots,n)\end{cases}\tag{21}$$

Equation (21) can be changed into the following bi-objective linear programming,

$$\min\{(\tilde{\omega}_\alpha^L+\tilde{\omega}_\beta^L)/2,(\tilde{\omega}_\alpha^L+\tilde{\omega}_\beta^L+\tilde{\omega}_\alpha^R+\tilde{\omega}_\beta^R)/4\}$$
$$\text{s. t.}\begin{cases}\sum_{j=1}^n(\tilde{a}_{ij})_\alpha^L z_j\le\tilde{\omega}_\alpha^L,\ \ \sum_{j=1}^n(\tilde{a}_{ij})_\alpha^R z_j\le\tilde{\omega}_\alpha^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n(\tilde{a}_{ij})_\beta^L z_j\le\tilde{\omega}_\beta^L,\ \ \sum_{j=1}^n(\tilde{a}_{ij})_\beta^R z_j\le\tilde{\omega}_\beta^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n z_j=1,z_j\ge0\ (j=1,2,\cdots,n)\end{cases}\tag{22}$$

which is easily constructed as follows:

$$\min\{\lambda(\tilde{\omega}_\alpha^L+\tilde{\omega}_\beta^L)/2+(1-\lambda)(\tilde{\omega}_\alpha^L+\tilde{\omega}_\beta^L+\tilde{\omega}_\alpha^R+\tilde{\omega}_\beta^R)/4\}$$
$$\text{s. t.}\begin{cases}\sum_{j=1}^n[\underline{a}_{ij}+\alpha(a_{ij}-\underline{a}_{ij})/w_{\tilde{a}_{ij}}]z_j\le\tilde{\omega}_\alpha^L\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n[\overline{a}_{ij}-\alpha(\overline{a}_{ij}-a_{ij})/w_{\tilde{a}_{ij}}]z_j\le\tilde{\omega}_\alpha^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n\{[(1-\beta)a_{ij}+(\beta-u_{\tilde{a}_{ij}})\underline{a}_{ij}]/(1-u_{\tilde{a}_{ij}})\}z_j\le\tilde{\omega}_\beta^L\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n\{[(1-\beta)a_{ij}+(\beta-u_{\tilde{a}_{ij}})\overline{a}_{ij}]/(1-u_{\tilde{a}_{ij}})\}z_j\le\tilde{\omega}_\beta^R\ (i=1,2,\cdots,m)\\ \sum_{j=1}^n z_j=1,z_j\ge0\ (j=1,2,\cdots,n)\end{cases}\tag{23}$$

Equation (23) can be solved by the simplex method for linear programming. The optimal solution of equation (23) is denoted by $(\mathbf{z}^*,\tilde{\omega}_\alpha^{L*},\tilde{\omega}_\alpha^{R*},\tilde{\omega}_\beta^{L*},\tilde{\omega}_\beta^{R*})$.

Similarly, player II's minimax strategy $\mathbf{z}^*$ and different (α,β)-cut sets $\tilde{\omega}^*_{\alpha,\beta}$ of the loss-ceiling $\tilde{\omega}^*$ can be obtained. Using all different possibility levels, $\tilde{\omega}^*_{\alpha,\beta}$ can approximate the value of player II's loss-ceiling $\tilde{\omega}^*$.

4. **An Application to the Market Share Problem**

Suppose that there are two companies p_1 and p_2 aiming at enhance the market share of a product in a targeted market under the circumstance that the demand amount of the product in the targeted market is fixed basically. In other words, the market share of one company increases while the market share of another company decreases. The two companies considering about two strategies to increase the market share: $\delta_1=\sigma_1$ (advertisement), $\delta_2=\sigma_2$(reduce the price). The above problem may be regarded as a matrix game. Namely, the companies p_1 and p_2 may be respectively regarded as players I and II who use pure strategies δ_1(i.e.,σ_1) and δ_2(i.e.,σ_2). Due to a lack of information or imprecision of the available information, the managers of the two companies usually are not able to forecast the exactly sales amount of the companies. They estimate the sales amount with a certain confidence degree, but it is possible that they are not so sure about it. Thus, there may be a hesitation about the estimation of the sales amount. In order to handle the uncertain situation, TIFNs are used to express the sales amount of the product. The payoff matrix $\tilde{\mathbf{A}}$ for the company p_1(i.e., player I) is given as follows:

$$\tilde{A} = \begin{pmatrix} <(175,180,190);0.6,0.2> & <(150,156,158);0.6,0.1> \\ <(80,90,100);0.9,0.1> & <(175,180,190);0.6,0.2> \end{pmatrix}$$

where $<(175,180,190);0.6,0.2>$ is a TIFN which indicates that the company p_1's sales amount is about 180 when the companies p_1 and p_2 use the strategy δ_1 (i.e., advertisement) simultaneously. The maximum confidence degree 0.6 and the minimum non-confidence degree 0.2. In other words his hesitation degree is 0.2. Other TIFNs in the payoff matrix $\tilde{\mathbf{A}}$ can be explained similarly.

According to equations (18) and (23) , the linear programming models are obtained as follows:

$$\max\{\lambda(\tilde{v}_\alpha^L+\tilde{v}_\beta^L)/2+(1-\lambda)(\tilde{v}_\alpha^L+\tilde{v}_\beta^L+\tilde{v}_\alpha^R+\tilde{v}_\beta^R)/4\}$$

$$\text{s.t.}\begin{cases}(175+5\alpha/0.6)y_1+(80+10\alpha/0.9)y_2\geq\tilde{v}_\alpha^L\\(150+6\alpha/0.6)y_1+(175+5\alpha/0.6)y_2\geq\tilde{v}_\alpha^L\\(190-10\alpha/0.6)y_1+(100-10\alpha/0.9)y_2\geq\tilde{v}_\alpha^R\\(158-2\alpha/0.6)y_1+(190-10\alpha/0.6)y_2\geq\tilde{v}_\alpha^R\\{[180(1-\beta)+175(\beta-0.2)]}y_1/0.8+[90(1-\beta)+80(\beta-0.1)]y_2/0.9\geq\tilde{v}_\beta^L\\{[156(1-\beta)+150(\beta-0.2)]}y_1/0.9+[180(1-\beta)+175(\beta-0.1)]y_2/0.8\geq\tilde{v}_\beta^L\\{[180(1-\beta)+190(\beta-0.2)]}y_1/0.8+[190(1-\beta)+100(\beta-0.2)]y_2/0.9\geq\tilde{v}_\beta^R\\{[156(1-\beta)+158(\beta-0.1)]}y_1/0.9+[180(1-\beta)+190(\beta-0.2)]y_2/0.8\geq\tilde{v}_\beta^R\\y_1+y_2=1\\y_1\geq0,y_2\geq0\end{cases}\tag{24}$$

and

$$\min\{\lambda(\tilde{\omega}_\alpha^R+\tilde{\omega}_\beta^R)/2+(1-\lambda)(\tilde{\omega}_\alpha^L+\tilde{\omega}_\beta^L+\tilde{\omega}_\alpha^R+\tilde{\omega}_\beta^R)/4\}$$

$$\text{s.t.}\begin{cases}(175+5\alpha/0.6)z_1+(150+6\alpha/0.6)z_2\leq\tilde{\omega}_\alpha^L\\(80+10\alpha/0.9)z_1+(175+5\alpha/0.6)z_2\leq\tilde{\omega}_\alpha^L\\(190-10\alpha/0.6)z_1+(158-2\alpha/0.6)z_2\leq\tilde{\omega}_\alpha^R\\(100-10\alpha/0.9)z_1+(190-10\alpha/0.6)z_2\leq\tilde{\omega}_\alpha^R\\{[180(1-\beta)+175(\beta-0.2)]}z_1/0.8+[156(1-\beta)+150(\beta-0.1)]z_2/0.9\leq\tilde{\omega}_\beta^L\\{[90(1-\beta)+80(\beta-0.1)]}z_1/0.9+[180(1-\beta)+175(\beta-0.2)]z_2/0.8\leq\tilde{\omega}_\beta^L\\{[180(1-\beta)+190(\beta-0.2)]}z_1/0.8+[156(1-\beta)+158(\beta-0.1)]z_2/0.9\leq\tilde{\omega}_\beta^R\\{[90(1-\beta)+100(\beta-0.1)]}z_1/0.9+[180(1-\beta)+190(\beta-0.2)]z_2/0.8\leq\tilde{\omega}_\beta^R\\z_1+z_2=1\\z_1\geq0,z_2\geq0,\end{cases}\tag{25}$$

Solving equations (24) and (25), the upper and lower bounds of α-cut sets and β-cut sets of player I's gain-floor $\tilde{v}^*$ and player II's loss-ceiling $\tilde{\omega}^*$ can be obtained, respectively. According to Theorem 2.9, we can obtain (α, β)-cut sets $\tilde{v}^*_{\alpha,\beta}$ and $\tilde{\omega}^*_{\alpha,\beta}$ of player I's gain-floor $\tilde{v}^*$ and player II's loss-ceiling $\tilde{\omega}^*$ as well as corresponding maximin strategies $\mathbf{y}^*$ and minimax strategies $\mathbf{z}^*$, which are listed as in Table 1.

(α, β)	$\mathbf{y}^{*T}$	$\tilde{v}^*_{\alpha,\beta}$	$\mathbf{z}^{*T}$	$\tilde{\omega}^*_{\alpha,\beta}$
(0,1)	(0.792,0.208)	[155.208,164.667]	(0.262,0.738)	[156.557,166.393]
(0.3,0.6)	(0.794,0.206)	[158.058,162.781]	(0.238,0.762)	[158.823,163.74]
(0.4,0.5)	(0.794,0.206)	[159.009,162.155]	(0.231,0.769)	[159.627,163.037]
(0.5,0.3)	(0.795,0.205)	[159.959,161.531]	(0.218,0.782)	[160.266,161.850]
(0.6,0.2)	(0.796,0.204)	160.909	(0.213,0.787)	[161.113,161.288]

Table 1. Cut Sets of the Gain-Floor and Loss-Ceiling and Maximin Strategies and Minimax Strategies for Players

It is easily seen from Table 1 that the larger the value α and the smaller the value β values the lower the degree of uncertainty. Specifically, when $\alpha = 0$ and $\beta = 1$, the cut sets of player I's gain-floor and player II's loss-ceiling are respectively the intervals [155.208, 164.667] and [156.557, 166.393], which are the widest. At $\alpha = 0.6$ and $\beta = 0.2$, the cut sets of player I's gain-floor and player II's loss-ceiling are the most likely values. In this example, it is impossible that the value of player I's gain-floor falls outside of the interval [155.208, 164.667]. The most likely value is 160.909 for player I (i.e., the company p_1). Similarly, the value of player II's loss-ceiling never fall outside of the interval [156.557, 166.393] and the most likely value falls in the interval [161.113, 161.288]. The approximate values of player I's gain-floor and player II's loss-ceiling can be obtained as $\tilde{v}^* =< (155.208, 160.909, 164.667); 0.6, 0.2 >$, and $\tilde{\omega}^* =< (156.557, 161.113, 161.288, 166.393); 0.6, 0.2 >$, respectively, depicted as in Figures. 2 and 3.

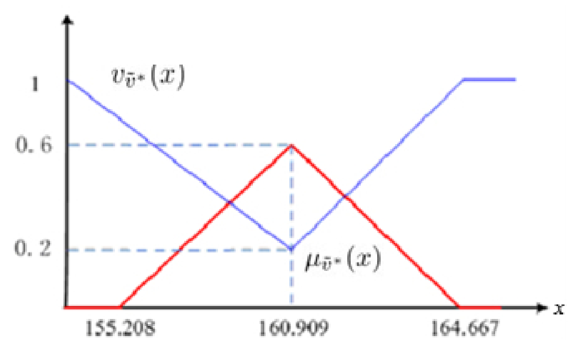

Figure 2. Player I's Gain-Floor $\tilde{v}^*$

It is easily seen that the proposed models and method may provide more information, which is not only the values (i.e., player I's gain-floor and player II's loss-ceiling) of the matrix game with payoffs of TIFNs but also the maximum degree of satisfaction and the minimum degree of non-satisfaction. It is noted from Figure. 3 that the value $\tilde{\omega}^*$ is not a TIFN. In fact, $\tilde{\omega}^*$ is a trapezoidal IF number, which is explored in the near future.

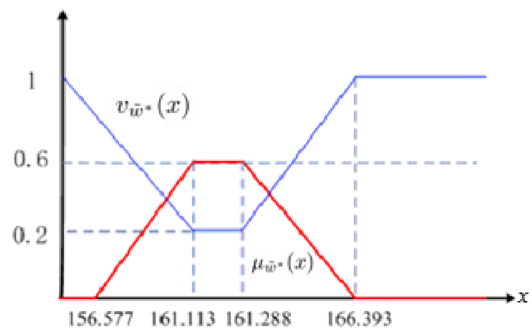

FIGURE 3. Player II's Loss-Ceiling $\tilde{w}^*$

5. Conclusion

Matrix games with payoffs of TIFNs are formulated and their solutions are firstly introduced. New auxiliary IF mathematical programming models are established. In this methodology, interval-valued mathematical programming models and multi-objective linear programming models are employed to determine the upper and lower bounds of α-cut sets and β-cut sets of player I's gain-floor and player II's loss-ceiling for any possible values α and β. Thus, player I's gain-floor and player II's loss-ceiling can be obtained when all possible values α and β are taken. It is easily seen that matrix games with payoffs of TIFNs are a generalization of matrix games with payoffs of TFNs.

In this paper, a pair of linear programming problems need to be solved for a given level (α, β). Therefore, the computation amount for determining player I's gain-floor and player II's loss-ceiling is larger. More effective methods for solving matrix games with payoffs of TIFNs will be investigated in the near future.

Although the models and method proposed in this paper is illustrated with the market share problem, it can also be applied to competitive decision problems using IF sets in many real fields such as management, business, military and politics as well as environment.

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