THE URYSOHN AXIOM AND THE COMPLETELY HAUSDORFF AXIOM IN L-TOPOLOGICAL SPACES

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Abstract. In this paper, the Urysohn and completely Hausdorff axioms in general topology are generalized to L-topological spaces so as to be compatible with pointwise metrics. Some properties and characterizations are also derived.

1. Introduction and Preliminaries

There have been diverse studies on separation axioms in L-topology [1, 6, 7, 8, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 42, 44, 45, 46, 47, 48]. Some of them were only motivated by the syntax of separation axioms in general topology; while others were motivated by the relations between separation axioms and compactness, uniformities, metrics, convergence, etc.

The Urysohn and completely Hausdorff axioms are two important separation axioms in general topology. Chen and Wu generalized the Urysohn axiom to topological molecular lattices and called this generalized version the U_2 axiom [1]. Fang also generalized these axioms to L-topological spaces and topological molecular lattices in [6, 7]. However, neither of the L-extended real line, the L-real line, and the L-unit interval satisfy these axioms (see Example 5.10 in [43]).

In [43], the L-Urysohn and L-completely Hausdorff axioms are introduced for L-topological spaces so that they are satisfied by the L-extended real line and its subspaces, as well as pointwise metric spaces.

The class of metric spaces is important in general topology and has been generalized to the class of L- topological spaces by many authors [3, 5, 11, 15, 16, 38, 39, 40, 42]. In these fuzzy metrics, only the pointwise metric [38, 39, 40, 42] can reflect the characteristics of pointwise L- topology, i.e., the relation between a fuzzy point and its Q- neighborhoods (or R-neighborhoods). Moreover, we do not know which fuzzy metric spaces satisfy the U_2 [2], Urysohn and $H(\lambda)$-completely Hausdorff axioms in the sense of [6, 7].

In this paper our aim is to redefine Urysohn and completely Hausdorff axioms in L-topology in such a way that they are compatible with pointwise metrics in L-set theory. For an L- space, we have the following implications:

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$$\begin{array}{ccccccc}
\text{Complete regularity}+T_1 & \Rightarrow & \text{regularity}+T_1 & & & & \\
\Downarrow & & \Downarrow & & & & \\
\text{completely Hausdorff} & \Rightarrow & \text{Urysohn} & \Rightarrow & T_2 & \Rightarrow & T_1 \\
\Downarrow & & \Downarrow & & & & \\
L\text{-completely Hausdorff} & \Rightarrow & L\text{-Urysohn} & & & &
\end{array}$$

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ denotes a completely distributive DeMorgan algebra, i.e., a completely distributive lattice with an order-reversing involution. Also, X denotes a nonempty set and L^X the set of all L-fuzzy sets (L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X. For a crisp subset $A \subseteq X$, we do not distinguish between A and χ_A.

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called a co-prime element if a' is a prime element [9]. The set of all non-unit prime elements in L is denoted by $P(L)$ and the set of all non-zero co-prime elements by $M(L)$. Thus $M(L^X)$ is the set of all non-zero co-prime elements in L^X. It is easy to see that members of $M(L^X)$ are exactly the L-fuzzy points x_λ with height $\lambda \in M(L)$.

The binary relation $\prec$ on L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [4]. In a completely distributive DeMorgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. Following [22, 45], $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b and denoted by $\beta(b)$. Moreover, for $b \in L$, we define the greatest maximal family of b by $\alpha(b) = \{a \in L \mid a' \prec b'\}$.

For $a \in L$ and $A \in L^X$, we use the following notation [41]:

$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A_{[a]} = \{x \in X \mid A(x) \geq a\}.$$

An L-topological space (L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\chi_\emptyset$, χ_X and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Members of $\mathcal{T}$ are called open L-sets and their complements are called closed L- sets. A closed L-set P is called a closed remote-neighborhood (or closed R-neighborhood) of $e \in M(L^X)$ if $e \not\leq P$. An open L-set Q is called an open Q-neighborhood of $e \in M(L^X)$ if Q' is a closed R-neighborhood of e. An open L-set U is called an open neighborhood of $e \in M(L^X)$ if $e \leq U$. The set of all closed R-neighborhoods of e is denoted by $\eta^-(e)$ and the set of all open neighborhood of e is denoted by $\mathcal{N}^\circ(e)$.

Definition 1.1. [22, 45] For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all the lower semi-continuous maps from (X, τ) to L; i.e. $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X. We say that $(X, \omega_L(\tau))$ is topologically generated by (X, τ).

Definition 1.2. [22, 45] An L-space $(X, \mathcal{T})$ is called weakly induced if $\forall a \in L$, $\forall A \in \mathcal{T}$, we have $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

It is obvious that $(X, \omega_L(\tau))$ is weakly induced.

Lemma 1.3. [41] *Let $(X, \mathcal{T})$ be a weakly induced L-space, $a \in L, A \in \mathcal{T}$. Then $A_{(a)}$ is an open L-set in $[\mathcal{T}]$.*

Definition 1.4. [22, 45] An L-space $(X, \mathcal{T})$ is said to be T_1, if for all $a, b \in M(L^X)$ with $a \not\leq b$, there exists a closed L-set $P \in \eta^-(a)$ such that $P \geq b$.

Definition 1.5. [38, 42] An L-space $(X, \mathcal{T})$ is said to be T_2, if for all $a, b \in M(L^X)$ with $a \not\leq b$, there exists a closed L-set P and an open L-set Q such that $a \not\leq P \geq Q \geq b$.

Definition 1.6. [14] An L-space $(X, \mathcal{T})$ is said to be regular, if each open L-set U is the suprema of some open L-sets V such that $V^- \leq U$.

We have the following characterizations of regularity.

Theorem 1.7. [35] *For an L-space $(X, \mathcal{T})$, the following conditions are equivalent:*
(1) *$(X, \mathcal{T})$ is regular.*
(2) *For all $a \in M(L^X)$ and all $G \in \mathcal{T}$ satisfying $a \prec G$, there exists an open L-set Q such that $a \prec Q \leq Q^- \leq G$.*
(3) *For all $a \in M(L^X)$ and all $G \in \mathcal{T}$ satisfying $a \prec G$, there exists an open L-set Q such that $a \leq Q \leq Q^- \leq G$.*
(4) *For all $a \in M(L^X)$ and all closed L-sets F satisfying $a \not\leq F$, there exists a closed L-set E such that $a \not\leq E \geq E^\circ \geq F$.*

Definition 1.8. [45] A map $f : L^X \to L_1^Y$ is called an order-homomorphism if
(1) f is union-preserving;
(2) $f^{\leftarrow}(B') = (f^{\leftarrow}(B))'$, where for all $B \in L_1^Y$, $f^{\leftarrow}(B) = \bigvee\{A \in L^X \mid f(A) \leq B\}$.

Remark 1.9. In the original definition of Wang, the condition $f(\underline{0}) = \underline{0}$ is assumed. In fact, this condition is contained in (1). We also note that $f^{\leftarrow}$ is a right adjoint of f.

Definition 1.10. [36, 37] An L-space $(X, \mathcal{T})$ is said to be (pointwise) completely regular if, for every $e \in M(L^X)$ and every $B \in \eta^-(e)$, there exists a continuous order-homomorphism $f : L^X \to L^{[0,1](L)}$ such that $e \not\leq f^{\leftarrow}(\mathcal{R}_0)$ and $B \leq f^{\leftarrow}(\mathcal{L}'_1)$.

Remark 1.11. From [37] we know that pointwise complete regularity is equivalent to complete regularity in the sense of Hutton.

Definition 1.12. [13] An L-space $(X, \mathcal{T})$ is said to be normal if, for every closed L-set K and every open L-set U with $K \leq U$, there exists an L-set V such that $K \leq V^\circ \leq V^- \leq U$.

Definition 1.13. [39] A pointwise pseudo-metric (p. metric for short) on L^X is a map $d : M(L^X) \times M(L^X) \to [0, +\infty)$ satisfying:
(M1) $\forall a \in M(L^X), d(a, a) = 0$;
(M2) $\forall a, b, c \in M(L^X), d(a, c) \leq d(a, b) + d(b, c)$;
(M3) $\forall a, b \in M(L^X), d(a, b) = \bigwedge_{c \ll b} d(a, c)$;

(M4) $\forall a, b, c \in M(L^X), a \leq b$ implies $d(a, c) \leq d(b, c)$;
(M5) $\forall \lambda, \mu \in M(L^X)$, $\bigwedge_{a \not\leq \lambda'} d(a, \mu) = \bigwedge_{b \not\leq \mu'} d(b, \lambda)$.

A pointwise p. metric d is said to be a pointwise metric if d satisfies
(M6) $d(a, b) = 0$ if and only if $a \leq b$.

Theorem 1.14. [39] *Let d be a pointwise p. metric on L^X. For all $r \in (0, +\infty)$, define a map $P_r : M(L^X) \to L^X$ by $P_r(a) = \bigvee\{b \in M(L^X) \mid d(a, b) \geq r\}$. Then $\{P_r(a) \mid a \in M(L^X), r \in (0, +\infty)\}$ is a closed base for a co-topology on L^X. This co- topology which corresponds to d is denoted by $\eta(d)$. Moreover, $\eta(d)'$ can also be induced by the interior operator int, where $\forall A \in L^X$, $int(A) = \bigvee_{r>0} \bigwedge_{e \not\leq A} P_r(e)$.*

Theorem 1.15. [39] *If d is a pointwise p.metric in L^X, $A \in L^X$ and $a \in M(L^X)$, then we have*

$$a \leq A^- \Leftrightarrow \forall r > 0,\ A \not\leq P_r(a) \Leftrightarrow d(a, A) = \bigwedge_{c \leq A} d(a, c) = 0.$$

Theorem 1.16. [39] *A pointwise p.metric on L^X is a pointwise metric if and only if its co- topology is T_1.*

Definition 1.17. [30] The L-extended real line $\mathbb{E}(L)$ is the set of all equivalence classes $[\lambda]$ of antitone maps $\lambda : \mathbb{R} \to L$, where the equivalence identifies two such maps λ, μ iff $\forall t \in \mathbb{R}$, $\lambda(t+) = \mu(t+)$. The canonical L-topology is generated by the subbase $\{\mathcal{L}_t, \mathcal{R}_t \mid t \in \mathbb{R}\}$, where

$$\mathcal{L}_t : \mathbb{E}(L) \to L \ \text{ by } \ \mathcal{L}_t(\lambda) = \lambda(t-)'$$
$$\mathcal{R}_t : \mathbb{E}(L) \to L \ \text{ by } \ \mathcal{R}_t(\lambda) = \lambda(t+).$$

The L-dual extended real line is $\mathbb{R}$ furnished with the canonical L-topology *co*-$\tau(L)$ generated by the subbase $\{\mathcal{L}_\lambda, \mathcal{R}_\lambda \mid [\lambda] \in \mathbb{R}(L)\}$, where

$$\mathcal{L}_\lambda : \mathbb{R} \to L \ \text{ by } \ \mathcal{L}_\lambda(t) = \lambda(t-)'$$
$$\mathcal{R}_\lambda : \mathbb{R} \to L \ \text{ by } \ \mathcal{R}_\lambda(t) = \lambda(t+).$$

2. The Urysohn Axiom in L-topological Spaces

Definition 2.1. An L-space $(X, \mathcal{T})$ is said to be Urysohn if, for all $a, b \in M(L^X)$ with $a \not\leq b$, there exists $P \in \eta^-(a)$ and $Q \in \mathcal{N}^\circ(b)$ such that $P^\circ \geq Q^-$.

The following theorem is obvious.

Theorem 2.2. *A Urysohn L-space is a T_2 L-space, hence it is also a T_1 L-space.*

Remark 2.3. By the results of [45] we know that the L-real line is not a T_1 L-space, hence it is not a completely Hausdorff L-space.

Let (X, d) be any metric space and let $L = [0, 1]$. If we define

$$d(x_\lambda, y_\mu) = \max\{\lambda - \mu, 0\} + d(x, y),$$

then it is easy to check that d is a pointwise metric on I^X. By Theorem 3.14 in section 3 we know that $(X, \mathcal{T}_d)$ is a Urysohn I- space, where $\mathcal{T}_d$ is the I-topology induced by d.

Theorem 2.4. *If $(X,\mathcal{T})$ is an induced L-space, then $(X,\mathcal{T})$ is a Urysohn L- space if and only if $(X,[\mathcal{T}])$ is a Urysohn space.*

Proof. ($\Leftarrow$). Let $(X,[\mathcal{T}])$ be a Urysohn space, and let $x_\lambda, y_\mu \in M(L^X)$ satisfying $x_\lambda \not\leq y_\mu$. Then $x \neq y$ or $\lambda \not\leq \mu$.

If $x \neq y$, then there exists an open neighborhood U of x and an open neighborhood V of y in $[\mathcal{T}]$ such that $U^- \bigcap V^- = \emptyset$ (or $V^- \subseteq U'^\circ$). Take $P = U'$ and $Q = V$. Obviously P is a closed R- neighborhood of x_λ, Q is an open neighborhood of y_μ in L-topology $\mathcal{T}$ and $P^\circ \geq Q^-$.

If $x = y$, then $\lambda \not\leq \mu$. Take $P = Q = \underline{\mu}$, where $\underline{\mu}$ denotes the constant L-set taking value μ. Obviously $x_\lambda \not\leq P^\circ = \underline{\mu} = Q^- \geq y_\mu$.

Therefore $(X,\mathcal{T})$ is a Urysohn L-space.

($\Rightarrow$). Let $(X,\mathcal{T})$ be a Urysohn L-space. To prove that $(X,[\mathcal{T}])$ is a Urysohn space, take $x, y \in X$ satisfying $x \neq y$. Let $\gamma \in M(L), \mu \in \beta^*(\gamma)$ and $\lambda \in \beta^*(\mu)$. Then $x_\lambda \not\leq y_\gamma$. Thus there exist $P \in \eta^-(x_\lambda)$ and $Q \in \mathcal{N}^\circ(y_\gamma)$ in $\mathcal{T}$ such that $P^\circ \geq Q^-$. Hence $(P^\circ)_{(\lambda)} \supseteq (Q^-)_{(\lambda)}$. So we obtain

$$x \notin P_{[\lambda]} \supseteq (P^\circ)_{[\lambda]} \supseteq (P^\circ)_{(\lambda)} \supseteq (Q^-)_{(\lambda)} \supseteq (Q^-)_{[\mu]} \supseteq Q_{(\mu)} \supseteq Q_{[\gamma]} \ni y.$$

It is easy to see that $P_{[\lambda]}, (Q^-)_{[\mu]}$ are closed sets in $(X,[\mathcal{T}])$ and from Lemma 1.3 we know that $(P^\circ)_{(\lambda)}, Q_{(\mu)}$ are open sets in $(X,[\mathcal{T}])$. Thus we have $(P_{[\lambda]})^\circ \supseteq (Q_{(\mu)})^-$. This shows that $(X,[\mathcal{T}])$ is a Urysohn space. $\square$

It is easy to prove the following two theorems.

Theorem 2.5. *If $(Y,\mathcal{T}|Y)$ a subspaceof a Urysohn L-space $(X,\mathcal{T})$, then $(Y,\mathcal{T}|Y)$ is also a Urysohn L-space.*

Theorem 2.6. *If $f : (X,\mathcal{T}) \to (Y,\mu)$ is a homeomorphism order-homomorphism and $(X,\mathcal{T})$ is a Urysohn L-space, then (L^Y,μ) is also a Urysohn L-space.*

Theorem 2.7. *Suppose that $(X,\mathcal{T})$ is the product space of $\{(X(t),\mathcal{T}(t)) \mid t \in T\}$. If for all $t \in T$, $(X(t),\mathcal{T}(t))$ is a Urysohn L-space, then so is $(X,\mathcal{T})$. Conversely, if $(X,\mathcal{T})$ is a Urysohn L-space and for some $t \in T$, $(X(t),\mathcal{T}(t))$ is fully stratified, then $(X(t),\mathcal{T}(t))$ is also a Urysohn L-space.*

Proof. Suppose that for all $t \in T$, $(X(t),\mathcal{T}(t))$ is a Urysohn L-space and $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, where $x = \{x(t) \mid x(t) \in X(t), t \in T\}, y = \{y(t) \mid y(t) \in Y(t), t \in T\}$. Then there exists $t \in T$ such that $x(t)_\lambda \not\leq y(t)_\mu$. Hence there exist $A(t) \in \eta^-(x(t)_\lambda)$ and $B(t) \in \mathcal{N}^\circ(y(t)_\mu)$ such that $A(t)^\circ \geq B(t)^-$. Thus

$$(P_t^\leftarrow(A(t)))^\circ \geq P_t^\leftarrow(A(t)^\circ) \geq P_t^\leftarrow\left(B(t)^-\right) \geq (P_t^\leftarrow(B(t)))^-,$$

where P_t denotes the project from $(X,\mathcal{T})$ to $(X(t),\mathcal{T}(t))$. Obviously $P_t^\leftarrow(A(t)) \in \eta^-(x_\lambda)$ and $P_t^\leftarrow(B(t)) \in \mathcal{N}^\circ(y_\mu)$. Therefore $(X,\mathcal{T})$ is a Urysohn L-space.

Conversely, suppose that $(X,\mathcal{T})$ is a Urysohn L-space and, for some $t \in T$, $(X(t),\mathcal{T}(t))$ is fully stratified, then $(X(t),\mathcal{T}(t))$ is homeomorphic to a subspace of $(X,\mathcal{T})$. Therefore $(X(t),\mathcal{T}(t))$ is also a Urysohn L-space. $\square$

Theorem 2.8. *If $(X,\mathcal{T})$ is regular and T_1 L-space, then it is also a Urysohn L-space.*

Proof. Let $a, b \in M(L^X)$ satisfying $a \not\leq b$. Since $(X, \mathcal{T})$ is a T_1 L-space, we have $b \in \eta^-(a)$. By the regularity of $(X, \mathcal{T})$ and Theorem 1.7, we know that there exists a closed L-set E such that $a \not\leq E \geq E^\circ \geq b$. Moreover there exists a closed L-set D such that $a \not\leq D \geq D^\circ \geq E$. Therefore $a \not\leq D \geq D^\circ \geq (E^\circ)^- \geq E^\circ \geq b$. It follows that $(X, \mathcal{T})$ is a Urysohn L-space. □

3. Completely Hausdorff Axiom in L-topological Spaces

Definition 3.1. An L-space $(X, \mathcal{T})$ is called a completely Hausdorff L-space if, for all $a, b \in M(L^X)$ with $a \not\leq b$, there exists a family of L-sets $\{A(t) \mid t \in (0,1)\} \subseteq L^X$ such that for all $r, s \in (0,1)$ with $s < r$, $a \not\leq A(s) \geq A(s)^\circ \geq A(r)^- \geq A(r) \geq b$.

The following two theorems are obvious.

Theorem 3.2. *A completely Hausdorff L-space is a Urysohn L-space, hence it is also a T_2 and T_1 L-space.*

Theorem 3.3. *$(X, \mathcal{T})$ is a completely Hausdorff L-space if and only if for all $a, b \in M(L^X)$ with $a \not\leq b$, there exists a decreasing mapping $F : (0,1) \to L^X$ such that $a \not\leq F(0+), b \leq F(1-)$, $F(t-) \in \mathcal{T}'$ for all $t \in (0,1]$ and $F(t+) \in \mathcal{T}$ for all $t \in [0,1)$.*

Remark 3.4. By the results of [45] it follows that the L-real line is not a T_1 L-space, hence it is not a completely Hausdorff L-space. However, $(X, \mathcal{T}_d)$ in Remark 2.3 is a completely Hausdorff I-space.

Theorem 3.5 gives another example of a completely Hausdorff L-space.

Theorem 3.5. *The L-dual extended real line is a completely Hausdorff L-space and hence it is also a Urysohn L-space.*

Proof. For each $x \in \mathbb{R}$ and each $a \in L$, define

$$(a \wedge \lambda_x)(t) = a, \text{ when } t < x \text{ and } (a \wedge \lambda_x)(t) = 0 \text{ when } t > x.$$

$$(a' \vee \lambda_x)(t) = 1, \text{ when } t < x \text{ and } (a' \vee \lambda_x)(t) = a' \text{ when } t > x.$$

Then $[a \wedge \lambda_x], [a' \vee \lambda_x] \in \mathbb{R}(L)$. It is easy to prove that

$$\mathcal{L}'_{a \wedge \lambda_x} \wedge \mathcal{R}'_{a' \vee \lambda_x} = x_a$$

Now we suppose that $x_a, y_b \in M(L^{\mathbb{R}})$ with $y_b \not\leq x_a$. If $x = y$, then $b \not\leq a$. Let C_a denote the constant L-set in $L^{\mathbb{R}}$ with value a. It is easy to see that $\forall t \in \mathbb{R}$, $\mathcal{L}'_{C_a}(t) = \mathcal{R}_{C_a}(t) = a$. Now, $\forall s \in (0,1)$, define $A(s) = \mathcal{R}_{C_a}$. Then for all $r, s \in (0,1)$ with $s < r$,

$$y_b \not\leq A(s) \geq A(s)^\circ \geq A(r)^- \geq A(r) \geq x_a.$$

If $x \neq y$, let $x < y, \epsilon = y - x$ and $A(s) = \mathcal{L}_{a' \vee \lambda_{x-(1-s)\epsilon}} \wedge \mathcal{R}_{a \wedge \lambda_{x+(1-s)\epsilon}}$ for any $s \in (0,1)$. Then, for all $r, s \in (0,1)$ with $s < r$, we have

$$\begin{aligned} y_b \not\leq A(s) \geq \;& A(s)^\circ = \mathcal{L}_{a' \vee \lambda_{x-(1-s)\epsilon}} \wedge \mathcal{R}_{a \wedge \lambda_{x+(1-s)\epsilon}} \\ \geq \;& \mathcal{R}'_{a' \vee \lambda_{x-(1-r)\epsilon}} \wedge \mathcal{L}'_{a \wedge \lambda_{x+(1-r)\epsilon}} \geq A(r)^- \\ \geq \;& A(r) = \mathcal{L}_{a' \vee \lambda_{x-(1-r)\epsilon}} \wedge \mathcal{R}_{a \wedge \lambda_{x+(1-r)\epsilon}} \\ \geq \;& \mathcal{R}'_{a' \vee \lambda_x} \wedge \mathcal{L}'_{a \wedge \lambda_x} = x_a. \end{aligned}$$

The proof is completed. □

Theorem 3.6. *$(X,\mathcal{T})$ is a completely Hausdorff L-space if and only if for all $a,b\in M(L^X)$ with $a\not\leq b$, there exists a continuous order-homomorphism $f:L^X\to L^{I(L)}$ such that $a\not\leq f^{\leftarrow}(\mathcal{R}_0)$ and $b\leq f^{\leftarrow}(\mathcal{L}'_1)$.*

Proof. Suppose that for all $a,b\in M(L^X)$ with $a\not\leq b$, there exists a continuous order-homomorphism $f:L^X\to L^{I(L)}$ such that $a\not\leq f^{\leftarrow}(\mathcal{R}_0)$ and $b\leq f^{\leftarrow}(\mathcal{L}'_1)$. For each $t\in(0,1)$, let $A(t)=f^{\leftarrow}(\mathcal{R}_t)$. Obviously, for all $r,s\in(0,1)$ with $s<r$, we have

$$a\not\leq A(s)\geq A(s)^{\circ}\geq A(r)^{-}\geq A(r)\geq b.$$

Thus $(X,\mathcal{T})$ is a completely Hausdorff L- space.

Conversely, let $(X,\mathcal{T})$ be a completely Hausdorff L-space. Take $a,b\in M(L^X)$ with $a\not\leq b$. Then there exists $c\in\beta^*(a)$ such that $c\not\leq b$. By the definition of completely Hausdorff L-space we know that there exists a family of L-sets $\{A(t)\mid t\in(0,1)\}$ such that for all $r,s\in(0,1)$ with $s<r$,

$$c\not\leq A(s)\geq A(s)^{\circ}\geq A(r)^{-}\geq A(r)\geq b.$$

Now we define a set mapping $f:X\to I(L)$ as follows:

$$\forall x\in X, f(x)(t)=A(t)(x),\ \text{where } A(t)=\begin{cases}\chi_X, & t\leq 0;\\ \chi_{\emptyset}, & t\geq 1.\end{cases}$$

Clearly, $f_L^{\rightarrow}:L^X\to L^{I(L)}$ is an order-homomorphism. Then for all $t\in[0,1]$, we have

$$f_L^{\leftarrow}(\mathcal{R}_t)=\bigvee_{s>t}A(s),\ \ f_L^{\leftarrow}(\mathcal{L}'_t)=\bigwedge_{s<t}A(s).$$

In fact, for each $x\in X$, from the following equality

$$f_L^{\leftarrow}(\mathcal{R}_t)(x)=\mathcal{R}_t(f(x))=f(x)(t+)=\bigvee_{s>t}f(x)(s)=\left(\bigvee_{s>t}A(s)\right)(x)$$

we obtain that $f_L^{\leftarrow}(\mathcal{R}_t)=\bigvee_{s>t}A(s)$. Similarly, $f^{\leftarrow}(\mathcal{L}'_t)=\bigwedge_{s<t}A(s)$. This shows that $f_L^{\leftarrow}(\mathcal{R}_t)$ is open and $f_L^{\leftarrow}(\mathcal{L}'_t)$ is closed. Therefore $f_L^{\rightarrow}:L^X\to L^{I(L)}$ is continuous. It is obvious that $a\not\leq f_L^{\leftarrow}(\mathcal{R}_0)$ and $b\leq f_L^{\leftarrow}(\mathcal{L}'_1)$. □

The next theorem follows from the proof of Theorem 3.6.

Theorem 3.7. *$(X,\mathcal{T})$ is a completely Hausdorff L-space if and only if for all $a,b\in M(L^X)$ with $a\not\leq b$, there exists a set mapping $f:X\to I(L)$ such that $f_L^{\rightarrow}:L^X\to L^{I(L)}$ is continuous, $a\not\leq f_L^{\leftarrow}(\mathcal{R}_0)$ and $b\leq f_L^{\leftarrow}(\mathcal{L}'_1)$.*

Theorem 3.8. *Let $(X,\mathcal{T})$ be an induced L-space. If $(X,[\mathcal{T}])$ is a completely Hausdorff space, then $(X,\mathcal{T})$ is a completely Hausdorff L-space.*

Proof. Let $(X,\mathcal{T})$ be an induced L-space, $(X,[\mathcal{T}])$ a completely Hausdorff space and $x_\lambda,y_\mu\in M(L^X)$ such that $x_\lambda\not\leq y_\mu$. Then $x\neq y$ or $\lambda\not\leq\mu$.

If $x\neq y$, then there exists a family of sets $\{A(t)\mid t\in(0,1)\}\subseteq 2^X$ such that for all $r,s\in(0,1)$ with $s<r$, $x\notin A(s)\supseteq A(s)^{\circ}\supseteq A(r)^{-}\supseteq A(r)\ni y$, where $A(s)^{\circ}$ denotes

the interior of $A(s)$ in $[\mathcal{T}]$ and $A(r)^-$ denotes the closure of $A(r)$ in $[\mathcal{T}]$. Clearly, for all $r, s \in (0,1)$ with $s < r$, we have $x_\lambda \not\leq A(s) \geq int(A(s)) \geq cl(A(r)) \geq A(r) \geq y_\mu$, where $int(A(s))$ denotes the interior of $A(s)$ in $\mathcal{T}$ and $cl(A(r))$ denotes the closure of $A(r)$ in $\mathcal{T}$.

If $x = y$, then we have $\lambda \not\leq \mu$. In this case, for all $t \in (0,1)$, let $A(t) = \mu$. Then, obviously, for all $r, s \in (0,1)$ with $s < r$, we have that $x_\lambda \not\leq A(s) \geq int(A(s)) \geq cl(A(r)) \geq A(r) \geq y_\mu$.

Therefore, $(X, \mathcal{T})$ is a completely Hausdorff L-space. □

Theorem 3.9. *Let $(X, \mathcal{T})$ be an induced $[0,1]$-space. If $(X, \mathcal{T})$ is a completely Hausdorff $[0,1]$- space, then $(X, [\mathcal{T}])$ is a completely Hausdorff space.*

Proof. Let $(X, \mathcal{T})$ be a completely Hausdorff $[0,1]$-space. To prove that $(X, [\mathcal{T}])$ is a complete Hausdorff space, take $x, y \in X$ satisfying $x \neq y$. Then we have $x_{\frac{1}{2}} \not\leq y_1$. Since $(X, \mathcal{T})$ is a completely Hausdorff I-space, there exists a set mapping $f : X \to [0,1](L)$ such that

$$x_{\frac{1}{2}} \not\leq f_L^{\leftarrow}(\mathcal{R}_0) \geq f_L^{\leftarrow}(\mathcal{L}'_1) \geq y_1.$$

For all $t \in (0,1)$, let $A(t) = f_L^{\leftarrow}(\mathcal{L}'_t)_{[\frac{1+t}{2}]}$. Then $A(t)$ is closed in $[\mathcal{T}]$. Moreover, for all $r, s \in (0,1)$ with $s < r$, we have the following:

$$A(s) \supseteq f_L^{\leftarrow}(L'_s)_{(\frac{1+s}{2})} \supseteq f_L^{\leftarrow}(R_s)_{(\frac{1+s}{2})} \supseteq f_L^{\leftarrow}(L'_r)_{(\frac{1+s}{2})} \supseteq f_L^{\leftarrow}(L'_r)_{[\frac{1+r}{2}]} = A(r) \ni y.$$

By Lemma 1.3 we know that $f_L^{\leftarrow}(R_s)_{(\frac{1+s}{2})}$ is open in $[\mathcal{T}]$. Therefore

$$x \notin A(s) \supseteq A(s)^\circ \supseteq f_L^{\leftarrow}(R_s)_{(\frac{1+s}{2})} \supseteq f_L^{\leftarrow}(L'_r)_{[\frac{1+r}{2}]} = A(r)^- = A(r) \ni y.$$

It follows that $(X, [\mathcal{T}])$ is a completely Hausdorff space. □

Remark 3.10. If $L \neq [0,1]$, is Theorem 3.9 still true? This is an open problem.

Theorem 3.11 follows easily from Definition 1.4, Definition 1.10 and Theorem 3.6.

Theorem 3.11. *If $(X, \mathcal{T})$ is a completely regular and T_1 L-space, then it is also a completely Hausdorff L-space.*

The following two theorems can also be easily proved.

Theorem 3.12. *If $(X, \mathcal{T})$ is a completely Hausdorff L-space and $(Y, \mathcal{T}|Y)$ is its subspace, then $(Y, \mathcal{T}|Y)$ is also a completely Hausdorff L-space.*

Theorem 3.13. *Let $(X, \mathcal{T})$ and (Y, μ) be two L-spaces. If $f : L^X \to L^Y$ is a homeomorphism order- homomorphism and $(X, \mathcal{T})$ is a completely Hausdorff L-space, then (Y, μ) is also a completely Hausdorff L-space.*

Theorem 3.14. *Suppose that $(X, \mathcal{T})$ is the product space of $\{(X(t), \mathcal{T}(t)) \mid t \in T\}$. If for all $t \in T$, $(X(t), \mathcal{T}(t))$ is a completely Hausdorff L-space, then so is $(X, \mathcal{T})$. Conversely, if $(X, \mathcal{T})$ is a completely Hausdorff L-space and for some $t \in T$, $(X(t), \mathcal{T}(t))$ is fully stratified, then $(X(t), \mathcal{T}(t))$ is also a completely Hausdorff L-space.*

Proof. Suppose that for all $t \in T$, $(X(t), \mathcal{T}(t))$ is a completely Hausdorff L-space and $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, where $x = \{x(t) \mid x(t) \in X(t), t \in T\}, y = \{y(t) \mid y(t) \in Y(t), t \in T\}$. Then there exists $t \in T$ such that $x(t)_\lambda \not\leq y(t)_\mu$. By Theorem 3.6, there exists a set map $f : X(t) \to I(L)$ such that $f_L^{\rightarrow} : L^{X(t)} \to L^{I(L)}$ is continuous, $x(t)_\lambda \not\leq f_L^{\leftarrow}(\mathcal{R}_0)$ and $y(t)_\mu \leq f_L^{\leftarrow}(\mathcal{L}'_1)$. In this case $(f \circ P_t)_L^{\rightarrow} : L^X \to L^{I(L)}$ is continuous, $x_\lambda \not\leq (f \circ P_t)_L^{\leftarrow}(\mathcal{R}_0)$ and $y_\mu \leq (f \circ P_t)_L^{\leftarrow}(\mathcal{L}'_1)$. Therefore $(X, \mathcal{T})$ is a completely Hausdorff L-space.

Conversely, suppose that $(X, \mathcal{T})$ is a completely Hausdorff L-space and for some $t \in T$, $(X(t), \mathcal{T}(t))$ is fully stratified, then $(X(t), \mathcal{T}(t))$ is homeomorphic to a subspace of $(X, \mathcal{T})$. Therefore $(X(t), \mathcal{T}(t))$ is also a completely Hausdorff L-space. □

Theorem 3.15. *A pointwise metric space is a completely Hausdorff L-space and hence it is also a Urysohn L-space.*

Proof. Let d be a pointwise metric on L^X, $a, b \in M(L^X)$ and $a \not\leq b$. Then there exists $c \prec a$ such that $c \not\leq b$. Hence $d(c, b) = r \neq 0$. Now let $F : \mathbb{R} \to L^X$ such that

$$F(t) = \begin{cases} \chi_X \ , & t \leq 0; \\ P_{rt}(c) \ , & t \in (0,1); \\ \chi_\emptyset \ , & t \geq 1. \end{cases}$$

Obviously, $F : \mathbb{R} \to L^X$ is a decreasing map and for all $t \in (0, 1]$, $F(t-) = \bigwedge_{s<t} P_{rs}(c) = P_{rt}(c) \in \eta(d)$. Moreover, it is easy to see that $a \not\leq F(0+)$, $b \leq P_r(c) = F(1-)$, $F(0-) = \chi_X \in \eta(d)$ and $F(1+) = \chi_\emptyset \in \eta(d)'$. Now we shall prove that for all $t \in [0, 1)$, $F(t+) = \bigvee_{s>t} P_{rs}(c) \in \eta(d)'$. By Theorem 1.14, it is enough to prove that

$$int\left(\bigvee_{s>t} P_{rs}(c)\right) = \bigvee_{u>0} \bigwedge_{e \not\leq \bigvee_{s>t} P_{rs}(c)} P_u(e) \geq \bigvee_{s>t} P_{rs}(c).$$

Let $x \in M(L^X)$ and $x \not\leq int\left(\bigvee_{s>t} P_{rs}(c)\right)$. Then there exists a point $y \prec x$ such that $y \not\leq int\left(\bigvee_{s>t} P_{rs}(c)\right)$. Thus for all $u > 0$, there exists a point $e \not\leq \bigvee_{s>t} P_{rs}(c)$ such that $y \not\leq P_u(e)$. Obviously, for all $s > t$, $e \not\leq P_{rs}(c)$. This implies that for all $s > t$, $y \not\leq (P_u \odot P_{rs})(c) \geq P_{u+rs}(c) = P_{r(s+\frac{u}{r})}(c)$. Therefore $x \not\leq \bigvee_{s>t} P_{rs}(c)$. It follows that $int\left(\bigvee_{s>t} P_{rs}(c)\right) \geq \bigvee_{s>t} P_{rs}(c)$. We complete the proof using Theorem 3.3. □

4. **A Comparison of Different Axioms**

In [2], the Urysohn axiom was generalized to topological molecular lattices by Chen and Wu, where it was called the U_2 axiom. It was also generalized to topological molecular lattices by Fang and Yue in a different manner [7]. Now we consider the relation between the Urysohn axiom in the sense of Fang, the U_2 axiom and our generalization. Since an L-topological space can be regarded as a topological molecular lattice, we have the following definition:

Definition 4.1. [2] An L-space $(X, \mathcal{T})$ is said to be U_2 if $\forall a, b \in M(L^X)$ with $a \wedge b = \underline{0}$, there exist $P \in \eta^-(a)$ and $Q \in \eta^-(b)$ such that $P^\circ \vee Q^\circ = \chi_X$.

Definition 4.2. [7] An L-space $(X, \mathcal{T})$ is said to be Fang Urysohn if $\forall a, b \in M(L^X)$ with $a \wedge b = \underline{0}$, there exist $P \in \eta^-(a)$ and $Q \in \eta^-(b)$ such that $P^\neg \wedge Q^\neg = \chi_\emptyset$, where $P^\neg = \bigwedge\{Q \in \mathcal{T}' \mid Q \vee P = \chi_X\}$.

Obviously a topological space, i.e., a $\{0,1\}$-space is U_2 if and only if it is Fang Urysohn and if and only if it is Urysohn in our sense. But for an L-space, the U_2 and Fang Urysohn axioms are not equivalent to our Urysohn axiom. This is illustrated by following example.

Example 4.3. Let $X = L = [0, 1]$ and $\mathcal{T} = \{\chi_E \mid E \subset X\}$, where χ_E is the characteristic function of E. Then $\mathcal{T}$ is a $[0, 1]$-topology on X. It is easy to check that $(X, \mathcal{T})$ is U_2 and Fang Urysohn. But it is not Urysohn in our sense. In fact, for any $x \in X$ and for any $P \in \eta^-(x_1)$, it follows that $P^\circ(x) = 0$. But there is no $Q \in \mathcal{N}^\circ(x_{0.5})$ such that $P^\circ \geq Q^-$.

In [6], Fang generalized the completely Hausdorff axiom to L-topological spaces as follows:

Definition 4.4. $(X, \mathcal{T})$ is said to be $H(\lambda)$-completely Hausdorff L-space if for each pair $x_a, y_b \in M(L^X)$ with $x \neq y$, there exists a continuous set function $f : X \to \widetilde{I}(L)$ such that $x_a \not\leq f^\leftarrow(\mathcal{R}_0)$ and $y_b \leq f^\leftarrow(\mathcal{L}'_1)$.

Obviously a $\{0,1\}$-space is $H(\lambda)$-completely Hausdorff if and only if it is completely Hausdorff in our sense. However, as the following example shows, for an L-space, the $H(\lambda)$-completely Hausdorff axiom is not equivalent to our completely Hausdorff axiom.

Example 4.5. Let $(X, \mathcal{T})$ the L-space as in Example 4.3. It is easy to check that $(X, \mathcal{T})$ is $H(\lambda)$-complete Hausdorff. By Example 4.3 we know that $(X, \mathcal{T})$ is not Urysohn and hence, by Theorem 3.2, it is not completely Hausdorff in our sense.

The following result is obvious.

Theorem 4.6. *A completely Hausdorff L-space is an L-completely Hausdorff L-space, and a Urysohn L-space is an L-Urysohn L-space.*

By Corollary 5.9 in [42] we know that the converse of the statements of Theorem 4.6 are not generally true.

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