A DUALITY BETWEEN FUZZY DOMAINS AND STRONGLY COMPLETELY DISTRIBUTIVE L-ORDERED SETS

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Abstract. The aim of this paper is to establish a fuzzy version of the duality between domains and completely distributive lattices. All values are taken in a fixed frame L. A definition of (strongly) completely distributive L-ordered sets is introduced. The main result in this paper is that the category of fuzzy domains is dually equivalent to the category of strongly completely distributive L-ordered sets. The results in this paper establish close connections among fuzzy-set approach of quantitative domains and fuzzy topology with modified L-sober spaces and spatial L-frames as links. In addition, some mistakes in [K.R. Wagner, Liminf convergence in Ω-categories, Theoretical Computer Science 184 (1997) 61–104] are pointed out.

1. **Introduction**

Domains introduced by Scott [34] and independently by Ershov [6] are structure modeling of the notion of approximation and computation. A computation performed using an algorithm proceeds in discrete steps. After each step there is more information available about the result of the computation. In this way the result obtained after each step can be seen as an approximation of the final result.

Quantitative domain theory has undergone active research in the past three decades which models concurrent systems and is in the hope of arriving at semantics that allow not only qualitative results but also taking into account complexity, runtime, etc [16].

On one hand, unlike the analytical mathematics, where natural metrics are at hand to measure the grade of an approximation, the theory of approximation based on domains was mainly of a qualitative nature. The situation started to change when Smyth [35] discovered that there is a notion of distance in domains, but it is necessarily not symmetric. The corresponding structure is called generalized metrics. Similarly, Matthews [26, 27] found that canonical metrics defined for the maximal elements of certain domains can be extended to the whole domain by allowing that points may have a positive self-distance, which is considered as the weight of that point. In subsequent research [29, 33, 40], weights turned out to be a powerful tool for the introduction of partial metrics. In 1996, Rutten [32] carried out a fundamental study on domain theory by the means of generalized

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ultrametrics. The idea of using certain kinds of metrics leads to an approach to quantitative domain theory including above mentioned papers and [8, 9, 19], etc.

On the other hand, certain kinds of posets are mathematical models for domain theory. In a poset, for two elements x and y, we have $x \leq y$ or $x \not\leq y$. In theoretical computer science, the less-than-or-equal-to relation between elements can be interpreted as the amount of computable information. If $x \leq y$, then there is more computable information of y than that of x. Otherwise, the amount of computable information of y is not larger than that of x. While in real life the following situation maybe occur: y contains a part of computable information of x, while we don't know how much does y contain. Thus we don't know which one is more complex when we compute x and y. In other words, the order relation in classical posets only gives us some qualitative information and has no quantitative information for computing. In order to being quantitative, we need to assign each pair of elements to a truth value. For explicit, a quantitative poset is a classical poset such that there is an assignment that each pair of elements corresponding to an element in a truth valued table Ω. This is now what we call a category enriched over Ω and the Ω-category leads to another approach to quantitative domain including [18, 20, 21, 22, 37, 38, 39, 41].

In fact, both approaches of the generalized (ultra)-metrics and Ω-categories go back to Lawvere [25]. Besides, in a narrow setting, authors would like to study quantitative domain theory via fuzzy sets [7, 42, 44, 46]. In fact in their approach, an L-ordered set or a fuzzy poset is just a special Ω-category for some special Ω. Thus fuzzy set approach to quantitative domains can be considered as a case of Ω-categories.

The Stone duality and Stone representation come from the classical Stone representation of Boolean algebras [36], and lead to locale theory as *pointfree topology* [2]. Abramsky related the important application of Stone duality in theoretical computer science, particularly in domain theory of denotational semantics of computer programming languages [1]. It provides the right framework for understanding the relationship between denotational semantics and program logic. Study of dualities between categories of certain domains were originated by Hofmann, Mislove and Stralka [13] and Lawson [24]. Therein, one of the most famous dualities in domain theory maybe is the duality between the category of domains (i.e., continuous dcpos) and the category of completely distributive lattices.

This time we shall establish a quantitative version of the duality of domains and completely distributive lattices by means of the fuzzy Scott topology in [44] and L-frame homomorphisms in [43].

This paper is organized as follows. In Section 2, we recall some basic definitions and results related to category theory, lattices, L-topology, L-order and fuzzy domains. In Section 3, we study completely distributive L-ordered sets and strongly completely distributive L-ordered sets. We also show that every (resp., strongly) completely distributive L-ordered set is a (resp., spatial) L-frame and every completely distributive L-ordered set is also a fuzzy domain. In Section 4, we show that fuzzy domains and strongly completely distributive L-ordered sets can be mutually induced by each other. The fuzzy Scott topology on a fuzzy domain is modified

L-sober in the sense of [30]. And the transformations between fuzzy domains and strongly completely distributive L-ordered set is a duality up into category setting. We also show that on any strongly completely distributive L-ordered set, the L-spectrum coincides with the fuzzy Scott topology on the set of all L-fuzzy points. In Section 5, we give some reasons for choosing a frame as the truth value table and point out some mistakes in [39]. In Section 6, we give a short discussion to indicate that fuzzy lattices can not be studied buy cut sets. Conclusion remarks are given in the last section.

2. **Basic Materials**

2.1. **Related Category Theory.** For category theory, please refer to [3].

A category $\mathbf{A}$ is called dually equivalent to (dual to, for simple) a category $\mathbf{B}$ if $\mathbf{A}$ and $\mathbf{B}^{op}$ are equivalent to each other, that is, there are two functors $F : \mathbf{A} \longrightarrow \mathbf{B}^{op}$ and $G : \mathbf{B} \longrightarrow \mathbf{A}^{op}$ such that $F \circ G \cong \mathrm{id}_{\mathbf{A}}$ and $G \circ F \cong \mathrm{id}_{\mathbf{B}}$, the equations here are the natural isomorphisms between functors. Such an environment is also called a duality between $\mathbf{A}$ and $\mathbf{B}$.

2.2. **Lattices, Fuzzy Sets and Fuzzy Topologies.** The contents in this subsection can be found in [14].

A complete lattice L is called a frame, or a complete Heyting algebra if L satisfies the infinite distributive law of finite meets over arbitrary joins, that is, $a \wedge \bigvee B = \bigvee\limits_{b\in B} a \wedge b$ for any $a \in L, B \subseteq L$, or equivalently, there exists an implication $\rightarrow$: $L \times L \longrightarrow L$ satisfying that $a \wedge b \leq c \iff a \leq b \rightarrow c$ for any $a, b, c \in L$. In this paper L always denotes a compete Heyting algebra. Properties of a compete Heyting algebra can be found in [14].

An element $a \in L$ is called prime if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$ for all $b, c \in L$.

Let X be a set, every $A \in L^X$ is called an L-subset of X. For an element a in L and $S \subseteq X$, we use the symbols a_S to stand for the map sending x to a if $x \in S$ and 0 otherwise. For $a \in L, A \in L^X$, the notations $a \wedge A$ or aA denote the L-subset $a_X \wedge A$.

For each ordinary map $f : X \longrightarrow Y$, we have a map $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called L-forward powerset operator [31]) defined by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ $(\forall y \in Y, A \in L^X)$. The right adjoint to $f_L^{\rightarrow}$ is denoted by $f_L^{\leftarrow}$ (called L-forward powerset operator [31]) and given by $f_L^{\leftarrow}(B) = B \circ f$ $(\forall B \in L^Y)$. It is well known that $(f_L^{\rightarrow}, f_L^{\leftarrow})$ is a Galois connection on $(L^X, \leq)$ and $(L^Y, \leq)$. Then $f_L^{\rightarrow}$ (resp., $f_L^{\leftarrow}$) preserves arbitrary joins of $(L^X, \leq)$ (resp., meets of $(L^Y, \leq)$) and $f_L^{\rightarrow}(a_X) = a_Y$, $f_L^{\leftarrow}(a_Y) = a_X$ $(\forall a \in L)$.

Let X be a set. A subfamily $\delta \subseteq L^X$ is called a stratified L-topology on X if it satisfies (o1) $a_X \in \delta$ for all $a \in L$; (o2) $A, B \in \delta$ implies $A \wedge B \in \delta$; (o3) $\{A_i |\ i \in I\} \subseteq \delta$ implies $\bigvee_i A_i \in \delta$. The pair (X, δ) is called a stratified L-topological space.

2.3. ***L*-ordered Sets.** The L-order used in this paper is independently introduced by Fan and Zhang [7, 46] and Bělohlávek [4, 5], and then was shown to be equivalent to each other in [42].

An L-fuzzy binary relation e on X is an L-subset of $X \times X$. An L-fuzzy binary relation e on X is called an L-order or a fuzzy order [43] if
(Ref) $\forall x \in X,\ e(x,x) = 1$;
(Tran) $\forall x,y,z \in X,\ e(x,y) \wedge e(y,z) \leq e(x,z)$;
(Antysym) $\forall x,y \in X,\ e(x,y) = e(y,x) = 1$ implies $x = y$.
The pair (X,e) is called an L-ordered set or a fuzzy poset [43].

A map $f : (X_1,e_1) \longrightarrow (X_2,e_2)$ between two L-ordered sets is called monotone if $e_1(x,y) \leq e_2(f(x),f(y))$ for all $x,y \in X_1$. A bijection $f : (X_1,e_1) \longrightarrow (X_2,e_2)$ between two L-ordered sets is called an isomorphism if both f and f^{-1} are monotone, in this case (X_1,e_1) and (X_2,e_2) are also called isomorphic.

On any set X, define $d(x,y) = 1$ if $x = y$ and 0, otherwise. Then d is an L-order on X, called the discrete L-order. For an L-ordered set (X,e) and $Y \subseteq X$, we still use e to denote the map e restricted to $Y \times Y$ and then (Y,e) is also an L-ordered set, called a sub-poset of (X,e). For an L-ordered set (X,e), the set $\leq_e = \{(x,y) |\ e(x,y) = 1\}$ is a crisp order on X, which is exactly the 1-cut of e, the corresponding poset is often denoted by $|A|$. Suppose e is an L-order on a set X, then $e^{op}(x,y) = e(y,x)$ $(\forall x,y \in X)$ is also an L-order on X, (X,e^{op}) is called the opposite poset of (X,e).

Two classical examples of L-ordered sets are

(1) Define $e_L : L \times L \longrightarrow L$ by $e_L(x,y) = x \to y$, for all $x,y \in L$. Then e_L is an L-order on L.

(2) For any $A,B \in L^X$, the subsethood degree [11] of A in B is defined by $sub_X(A,B) = \bigwedge_{x \in X} A(x) \to B(x)$. Then $sub_X : L^X \times L^X \longrightarrow L$ is an L-order on L^X.

The following definitions and propositions can be found in [4, 5, 7, 43, 46], etc.

Definition 2.1. Let (X,e) be an L-ordered set, $x_0 \in X$ and $A \in L^X$. The element x_0 is called a join (resp., meet) of A, in symbols $x_0 = \bigsqcup A$ (resp., $x_0 = \bigsqcap A$), if

(1) $\forall x \in X,\ A(x) \leq e(x,x_0)$ (resp., $A(x) \leq e(x_0,x)$);

(2) $\forall y \in X,\ \bigwedge_{x \in X} A(x) \to e(x,y) \leq e(x_0,y)$ (resp., $\bigwedge_{x \in X} A(x) \to e(y,x) \leq e(y,x_0)$).

It is easy to verify by (Antysym), that if x_1,x_2 are two joins (resp., meets) of A, then $x_1 = x_2$. That is each $A \in L^X$ has at most one join (resp., one meet).

Proposition 2.2. (1) $x_0 = \bigsqcup A$ *iff for all* $y \in X,\ e(x_0,y) = \bigwedge_{x \in X} A(x) \to e(x,y)$.

(2) $x_0 = \bigsqcap A$ *iff for all* $y \in X,\ e(y,x_0) = \bigwedge_{x \in X} A(x) \to e(y,x)$.

An L-ordered set (X,e) is called complete if for all $A \in L^X$, $\bigsqcup A$ (or equivalently, $\bigsqcap A$) exists [43]. For example,

(1) (L,e_L) is a complete L-ordered set, where $\bigsqcup A = \bigvee_{a \in L} A(a) \wedge a$ and $\bigsqcap A = \bigwedge_{a \in L} A(a) \to a$ for every $A \in L^L$.

(2) Let δ be a stratified topology on X, then (δ, sub_X) is a complete L-ordered set, where $\bigsqcup \mathcal{A} = \bigvee_{A \in \delta} \mathcal{A}(A) \wedge A$ for every $\mathcal{A} \in L^\delta$.

Suppose that (X, e) is a complete L-ordered set. Then $|X|$ is a complete lattice, where for $S \subseteq X$, $\bigvee S = \bigsqcup \chi_S$, $\bigwedge S = \bigsqcap \chi_S$ and χ_S is the characteristic function of S.

Lemma 2.3. *Let (X, e) be a complete L-ordered set. Then*

(1) for any $a \in X$, $\{b_i |\ i \in I\} \subseteq X$, $e(a, \bigwedge_i b_i) = \bigwedge_i e(a, b_i)$, where $\bigwedge_i b_i$ is the meet of $\{b_i\}$ taken in $|X|$;

(2) for any $a \in X$, $\{b_i |\ i \in I\} \subseteq X$, $e(\bigvee_i b_i, a) = \bigwedge_i e(b_i, a)$, where $\bigvee_i b_i$ is the join of $\{b_i\}$ taken in $|X|$;

(3) for any $a, b, c \in X$, $e(a, b) \leq e(a \wedge c, b \wedge c)$.

Proof. (1) and (2) can be found in [23]. For (3), by (1), $e(a \wedge c, b \wedge c) = e(a \wedge c, b) \wedge e(a \wedge c, c) = e(a \wedge c, b) \geq e(a, b)$. □

Let (X, e) be an L-ordered set. $A \in L^X$ is called an upper set if $A(x) \wedge e(x, y) \leq A(y)$ for any $x, y \in X$. For $x \in X$, $\uparrow x \in L^X$ defined by $\uparrow x(y) = e(x, y)$ $(\forall y \in X)$ is an upper set. The set of all upper sets of (X, e) is denoted by $\mathrm{Up}_L(X)$. Dually, $A \in L^X$ is called a lower set if $A(x) \wedge e(y, x) \leq A(y)$ for any $x, y \in X$. For $x \in X$, $\downarrow x \in L^X$ defined by $\downarrow x(y) = e(y, x)$ $(\forall y \in X)$ is a lower set. The set of all lower sets of (X, e) is denoted by $\mathrm{Low}_L(X)$. For any L-subset $S \in L^X$, define $\downarrow S \in L^X$ by $\downarrow S(x) = \bigvee\limits_{y \in X} S(y) \wedge e(x, y)$ $(\forall x \in X)$. Then $\downarrow S$ is the least lower set which is larger than or equal to S and if $\bigsqcup S$ exists, then $\bigsqcup \downarrow S = \bigsqcup S$ [42].

Lemma 2.4. *(Proposition 2.7 in [43]) Let (X, e) be an L-ordered set and $f : X \longrightarrow L$ be a map. Then for any $S \in L^X$, $\bigsqcup f_L^{\rightarrow}(S) = \bigvee\limits_{x \in X} f(x) \wedge S(x)$.*

Definition 2.5. Let $(X, e_X), (Y, e_Y)$ be two L-ordered sets and $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ two monotone maps. The pair (f, g) is called a fuzzy Galois connection between X and Y if

$$e_Y(f(x), y) = e_X(x, g(y))$$

for all $x \in X, y \in Y$, where f is called the left fuzzy adjoint of g and dually g the right fuzzy adjoint of f.

Remark 2.6. (1) $(f_L^{\rightarrow}, f_L^{\leftarrow})$ is a fuzzy Galois connection between (L^X, sub_X) and (L^Y, sub_Y).

(2) In [17, 22, 38], for two Ω-categories A and B, a pair of Ω-functors $f : A \longrightarrow B$ and $g : B \longrightarrow A$ is said to be an Ω-adjunction if

$$B(f(a), b) = A(a, g(b))$$

for all $a \in A, b \in B$ (cf. Definition 2.9 in [22]). A fuzzy Galois connection in this paper is an L-adjunction in the sense of [17, 22, 38].

Proposition 2.7. *(Theorem 4.5 in [42]) Let $f : (X, e_X) \longrightarrow (Y, e_Y)$ and $g : (Y, e_Y) \longrightarrow (X, e_X)$ be two maps between L-ordered sets. Then*

(1) *If X is complete, then f is monotone and has a right fuzzy adjoint if and only if $f(\bigsqcup A) = \bigsqcup f_L^{\rightarrow}(A)$ for all $A \in L^X$.*

(2) *If Y is complete, then g is monotone and has a left fuzzy adjoint if and only if $g(\bigsqcap B) = \bigsqcap g_L^{\rightarrow}(B)$ for all $B \in L^Y$.*

2.4. **Fuzzy dcpos, Their Continuity and the Fuzzy Scott Topology.** Fuzzy dcpos and their continuity are defined and studied in [20, 42] and the fuzzy Scott topology is defined and studied in [44].

Let (X, e) be an L-ordered set. An L-subset $D \in L^X$ is called directed (Definition 5.1 in [22, 42]) if

(FD1) $\bigvee_{x \in X} D(x) = 1$;

(FD2) $\forall x, y \in X,\ D(x) \wedge D(y) \leq \bigvee_{z \in X} D(z) \wedge e(x, z) \wedge e(y, z)$.

A directed L-subset is called a fuzzy ideal if it is a lower set additionally. We denote the set of all directed L-subsets and all fuzzy ideals on X by $\mathcal{D}_L(X)$ and $\mathcal{I}_L(X)$, respectively. An L-ordered set is called a fuzzy dcpo (a special case of an $\mathcal{I}$-cocomplete Ω-category in [22]) if every directed L-subset has a join, or equivalently, every fuzzy ideal has a join.

Let $f : X \longrightarrow Y$ be a monotone map between two L-ordered sets, then $f_L^{\rightarrow}(D) \in \mathcal{D}_L(Y)$ for any $D \in \mathcal{D}_L(X)$ (Proposition 5.3 in [42]). A map $f : X \longrightarrow Y$ between two fuzzy dcpos is called fuzzy Scott continuous if it is monotone and for any directed subset $D \in L^X$, $f(\bigsqcup D) = \bigsqcup f_L^{\rightarrow}(D)$ (which is a special case of the $\mathcal{I}$-cocontinuity in [22]). All fuzzy dcpos and fuzzy Scott continuous maps consist of a Cartesian-closed category **FDCPO** (Theorem 5.8 in [44]).

Let (X, e) be a fuzzy dcpo. For any $x \in X$, define $\Downarrow\! x \in L^X$ by

$$\forall y \in X,\ \Downarrow\! x(y) = \bigwedge_{I \in \mathcal{I}_L(X)} e(x, \bigsqcup I) \rightarrow I(y).$$

A fuzzy dcpo is called continuous or a fuzzy domain [42] if $\Downarrow\! x \in \mathcal{D}_L(X)$ (or equivalently, $\Downarrow\! x \in \mathcal{I}_L(X)$) and $x = \bigsqcup \Downarrow\! x$ for all $x \in X$. For $\Uparrow\! x \in L^X$, we mean the map $\Uparrow\! x(y) = \Downarrow\! y(x)$ ($\forall y \in X$).

Proposition 2.8. *In a complete L-ordered set (X, e) and $x \in X$, $\Downarrow\! x$ is always directed.*

Proof. Let $x \in X$.

(FD1) let 0 be the bottom element in $|X|$. Then $e(0, x) = 1$ for any $x \in X$. For any fuzzy ideal $I \in \mathcal{I}_L(X)$, we have $1 = \bigvee_{x \in X} I(x) = I(0)$ since I is a lower set. Then $\bigvee_{y \in X} \Downarrow\! x(y) \geq \Downarrow\! x(0) = 1$.

(FD2) Let $y_1, y_2 \in X$. For any $I \in \mathcal{I}_L(X)$, taking $\vee$ in $|X|$, we have

$$\begin{aligned} I(y_1) \wedge I(y_2) &\leq \bigvee_{y \in X} I(y) \wedge e(y_1, y) \wedge e(y_2, y) \\ &= \bigvee_{y \in X} I(y) \wedge e(y_1 \vee y_2, y) \\ &\leq I(y_1 \vee y_2). \end{aligned}$$

Then

$$\begin{aligned} \Downarrow\! x(y_1) \wedge \Downarrow\! x(y_2) &\leq \bigwedge_{I \in \mathcal{I}_L(X)} (e(x, \bigsqcup I) \rightarrow I(y_1)) \wedge (e(x, \bigsqcup I) \rightarrow I(y_2)) \\ &= \bigwedge_{I \in \mathcal{I}_L(X)} e(x, \bigsqcup I) \rightarrow (I(y_1) \wedge I(y_2)) \\ &\leq \bigwedge_{I \in \mathcal{I}_L(X)} e(x, \bigsqcup I) \rightarrow I(y_1 \vee y_2) \\ &= \Downarrow\! x(y_1 \vee y_2) \\ &\leq \bigvee_{y \in X} \Downarrow\! x(y) \wedge e(y_1, y) \wedge e(y_2, y). \end{aligned}$$

Notice that $e(y_1, y_1 \vee y_2) = e(y_2, y_1 \vee y_2) = 1$. □

If (X, e) is a fuzzy domain, then the map $\Downarrow$ has the property of interpolation, i.e., $\Downarrow y(x) = \bigvee_{z \in X} \Downarrow z(x) \wedge \Downarrow y(z)$ for all $x, y \in X$ (cf. Theorem 4.6 in [20], Theorem 5.9 in [42]). A fuzzy dcpo (X, e) is continuous iff $(\Downarrow, \bigsqcup)$ is a fuzzy Galois connection between (X, e) and $(\mathcal{I}_L(X), sub_X)$ (cf. Theorem 5.9 in [42]).

An L-subset A of a fuzzy dcpo (X, e) is called fuzzy Scott open if it is an upper set and $A(\bigsqcup I) = \bigvee_{x \in X} A(x) \wedge I(x)$ for all $I \in \mathcal{I}_L(X)$. The family of all fuzzy Scott open sets of (X, e) forms a stratified L-topology on X, denoted by $\sigma_L(X)$, called the fuzzy Scott topology on X. It is shown by Theorem 3.12 in [44] that $A \in L^X$ is a fuzzy Scott open iff $A : (X, e) \longrightarrow (L, e_L)$ is a fuzzy Scott continuous map. Then by Lemma 2.4, for any $D \in \mathcal{D}_L(X)$, $A(\bigsqcup D) = \bigvee_{x \in X} A(x) \wedge D(x)$. If (X, e) is a fuzzy domain, then $\{a(\Uparrow x) |\ x \in X, a \in L\}$ is a basis for $\sigma_L(X)$ (Theorem 3.8 in [44]), in other words, any fuzzy Scott open set U can be represented as $U = \bigvee_{x \in X} U(x) \wedge \Uparrow x$, where $U(x)$ in the equality is a constant map with the value $U(x)$.

We end this subsection with a proposition, which can be easily proved and will be used later.

Proposition 2.9. *Suppose that U is an upper set of a fuzzy dcpo (X, e). Then for any $x \in X$, $U(x) = sub_X(\uparrow x, U) \leq sub_X(\Uparrow x, U)$.*

2.5. ***L*-frames.** A complete L-ordered set (C, e) is called an L-frame [43] if for any $c \in C$, $c \wedge \_ : C \longrightarrow C$ preserves the joins of any L-subsets, that is for any $S \in L^C$, $c \wedge \bigsqcup S = \bigsqcup (c \wedge \_)^{\rightarrow}_L(S)$, where $(c \wedge \_)^{\rightarrow}_L$ is the L-Zadeh function of $c \wedge \_ : C \longrightarrow C$ and $\wedge$ is taken in $|C|$, or equivalently, $c \wedge \_$ has a right fuzzy adjoint (notice that by Lemma 2.3(3), we know that $c \wedge \_$ always is monotone). A map $f : (A, e_A) \longrightarrow (B, e_B)$ between two complete L-ordered sets is called an L-frame homomorphism if f preserves finite meets (i.e., $f(1) = 1$ and $f(c_1 \wedge c_2) = f(c_1) \wedge f(c_2)$ for all $c_1, c_2 \in A$) and arbitrary joins (i.e., $f(\bigsqcup S) = \bigsqcup f^{\rightarrow}_L(S)$ for any $S \in L^{C_1}$), where the two 1s are the greatest elements of $|A|$ and $|B|$ respectively. For a complete L-ordered set (C, e), we denote $pt_L(C)$ the set of all L-frame homomorphism from (C, e) to (L, e_L), each member of $pt_L(C)$ will be called an L-fuzzy point of (C, e). Clearly, $(pt_L(C), sub_C)$ is an L-ordered set and it is easy to show that $pt_L(C) \subseteq \mathrm{Up}_L(C)$.

The two classical complete L-ordered sets mentioned in Subsection 2.2 are also examples of L-frames, that is, the frame L itself is an L-frame under the L-order e_L; for any stratified L-topological space (X, δ), the pair (δ, sub_X) is such a kind of an L-frame (See Example 3.4 in [43]).

Proposition 2.10. *For any complete L-ordered set (C, e), $(pt_L(C), sub_C)$ is a fuzzy dcpo.*

Proof. Suppose that $\mathcal{A}$ is a directed L-subset of $pt_L(C)$. Define $f : C \longrightarrow L$ by $f(c) = \bigvee_{g \in pt_L(C)} \mathcal{A}(g) \wedge g(c)$.

Claim 1. $f \in pt_L(C)$. In fact, (i)

$$f(1) = \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge g(1) = \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge 1 = \bigvee_{g\in pt_L(C)} \mathcal{A}(g) = 1;$$

(ii) for any $c_1, c_2 \in C$, we have $f(c_1\wedge c_2)\leq f(c_1)\wedge f(c_2)$ and

$$\begin{aligned}
& f(c_1)\wedge f(c_2)\\
= & \bigvee_{g_1,g_2\in pt_L(C)} \mathcal{A}(g_1)\wedge g_1(c_1)\wedge \mathcal{A}(g_2)\wedge g_2(c_2)\\
\leq & \bigvee_{g_1,g_2,g\in pt_L(C)} \mathcal{A}(g)\wedge g_1(c_1)\wedge g_2(c_2)\wedge sub_C(g_1,g)\wedge sub_C(g_2,g)\\
\leq & \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge g(c_1)\wedge g(c_2)\\
= & \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge g(c_1\wedge c_2)\\
= & f(c_1\wedge c_2);
\end{aligned}$$

(iii) for any $S\in L^C$, by Lemma 2.4,

$$\begin{aligned}
f(\bigsqcup S) &= \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge g(\bigsqcup S)\\
&= \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge \bigsqcup g_L^{\rightarrow}(S)\\
&= \bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge (\bigvee_{c\in C} S(c)\wedge g(c))\\
&= \bigvee_{c\in C} S(c)\wedge (\bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge \wedge g(c))\\
&= \bigvee_{c\in C} S(c)\wedge f(c)\\
&= \bigsqcup f_L^{\rightarrow}(S).
\end{aligned}$$

Claim 2. $f = \bigsqcup \mathcal{A}$**.** In fact, for any $h\in pt_L(C)$, we have

$$\begin{aligned}
& \bigwedge_{g\in pt_L(C)} \mathcal{A}(g)\rightarrow sub_C(g,h)\\
= & \bigwedge_{g\in pt_L(C)} \bigwedge_{c\in C} \mathcal{A}(g)\rightarrow (g(c)\rightarrow h(c))\\
= & \bigwedge_{c\in C} (\bigvee_{g\in pt_L(C)} \mathcal{A}(g)\wedge g(c))\rightarrow h(c)\\
= & sub_C(f,h).
\end{aligned}$$

□

Proposition 2.11. *(1) Suppose that (X,δ) is a stratified L-topological space. For any $x\in X$, the map $\Psi_x:\delta\longrightarrow L$ defined by $\Psi_x(A)=A(x)$ $(\forall A\in\delta)$ is an L-frame homomorphism.*

(2) Suppose that (X,δ) is a stratified L-topological space. Then $p:(\delta, sub_X)\longrightarrow (L, e_L)$ is an L-frame homomorphism iff $p:(\delta,\leq)\longrightarrow(L,\leq)$ is a frame homomorphism and $p(a_X)=a$ for any $a\in L$.

Proof. (1) is routine and (2) is precisely Proposition 5.2 in [43]. □

3. (Strongly) Completely Distributive L-ordered Sets

3.1. **Completely distributive L-ordered sets.** Completely distributivity of L-ordered sets are defined and studied using fuzzy Galois connections (cf. Proposition 3.3 below) in [21, 37, 45].

Let (C, e) be a complete L-ordered set. For any $c \in C$, define an L-subset $\overset{\shortmid}{\triangledown} c$ of C by

$$\overset{\shortmid}{\triangledown} c(d) = \bigwedge_{S \in \mathrm{Low}_L(C)} e(c, \bigsqcup S) \to S(d) \ (\forall d \in C).$$

Sometimes we use $\lhd(d, c)$ to denote $\overset{\shortmid}{\triangledown} c(d)$ for convenience.

Definition 3.1. A complete L-ordered set (C, e) is called completely distributive or a completely distributive L-ordered set if $c = \bigsqcup \overset{\shortmid}{\triangledown} c$ for any $c \in C$. For $\underset{\shortmid}{\vartriangle} c \in L^C$, we mean the map $\underset{\shortmid}{\vartriangle} c(d) = \overset{\shortmid}{\triangledown} d(c)$ $(\forall d \in C)$.

Similar to Proposition 5.7 in [42], we can prove that

Proposition 3.2. *Let (C, e) be a complete L-ordered set and $c, d, u, v \in C$. Then*
(1) $\overset{\shortmid}{\triangledown} c \leq \Downarrow c \leq \downarrow c$;
(2) $e(u, c) \wedge \overset{\shortmid}{\triangledown} d(c) \wedge e(d, v) \leq \overset{\shortmid}{\triangledown} v(u)$.

Similar to Theorem 5.10 in [42], we have

Proposition 3.3. *A complete L-ordered set (C, e) is completely distributive iff $(\overset{\shortmid}{\triangledown}, \bigsqcup)$ is a fuzzy Galois connection between (C, e) and $(\mathrm{Low}_L(C), sub_C)$. This characterization is precisely the definition of completely distributivity in* [21, 37, 45].

Example 3.4. [21, 37] (1) For any L-ordered set (X, e), $(\mathrm{Low}_L(X), sub_X)$ is a completely distributive L-ordered set.
(2) (L, e_L) is a completely distributive L-ordered set.
(3) For any nonempty set X, (L^X, sub_X) is a completely distributive L-ordered set.

Similar to the interpolation of $\Downarrow$ in a fuzzy domain (Theorem 5.9 in [42]), we have

Proposition 3.5. *(Also in* [21, 37]*) If (X, e) is a completely distributive L-ordered set, then for any $x, y \in X$, $\overset{\shortmid}{\triangledown} y(x) = \bigvee_{z \in X} \overset{\shortmid}{\triangledown} z(x) \wedge \overset{\shortmid}{\triangledown} y(z)$.*

Lemma 3.6. *Let (C, e) be a complete L-ordered set. Then for any $c \in C, S \in L^C$, $\lhd(c, \bigsqcup S) \leq \downarrow S(c)$.*

Proof. $\lhd(c, \bigsqcup S) \leq e(\bigsqcup S, \bigsqcup \downarrow S) \to \downarrow S(c) = \downarrow S(c)$. □

In classical situation, we know that every complete distributive lattice is simultaneously a frame and continuous. In fuzzy setting, we have the similar result.

Proposition 3.7. *Every completely distributive L-ordered set is an L-frame.*

Proof. Let (C, e) be a completely distributive L-ordered set and $c \in C, S \in L^C$. We need to show that $c \wedge \bigsqcup S$ is the join of $(c \wedge \_)^{\rightarrow}_L(S)$, or for any $d \in C$,

$$\bigwedge_{d_1 \in C} (c \wedge \_)^{\rightarrow}_L(S)(d_1) \to e(d_1, d) = e(c \wedge \bigsqcup S, d).$$

In fact, $\bigwedge_{d_1\in C}(c\wedge \_)^{\rightarrow}_L(S)(d_1)\to e(d_1,d)=\bigwedge_{d_2\in C}S(d_2)\to e(d_2\wedge c,d)$
and $e(c\wedge\bigsqcup S,d)=e(\bigsqcup \downarrow_{\triangledown}(c\wedge\bigsqcup S),d)=\bigwedge_{d_3}\downarrow_{\triangledown}(c\wedge\bigsqcup S)(d_3)\to e(d_3,d)$.
On one hand, since

$$\begin{aligned}&\downarrow_{\triangledown}(c\wedge\bigsqcup S)(d_3)\le\downarrow_{\triangledown}c(d_3)\wedge\downarrow_{\triangledown}(\bigsqcup S)(d_3)\le e(d_3,c)\wedge(\downarrow S)(d_3)\\ &=\bigvee_{d_2\in C}e(d_3,c)\wedge S(d_2)\wedge e(d_3,d_2)=\bigvee_{d_2\in C}S(d_2)\wedge e(d_3,d_2\wedge c),\end{aligned}$$

we have

$$\begin{aligned}e(c\wedge\bigsqcup S,d)&\ge\bigwedge_{d_2,d_3\in C}(S(d_2)\wedge e(d_3,d_2\wedge c))\to e(d_3,d)\\ &=\bigwedge_{d_2,d_3\in C}S(d_2)\to(e(d_3,d_2\wedge c)\to e(d_3,d))\\ &\ge\bigwedge_{d_2\in C}S(d_2)\to e(d_2\wedge c,d)\\ &=\bigwedge_{d_1\in C}(c\wedge \_)^{\rightarrow}_L(S)(d_1)\to e(d_1,d).\end{aligned}$$

On the other hand, by Lemma 2.3(3), for any $d_2\in C$,

$$e(c\wedge\bigsqcup S,d)\to e(d_2\wedge c,d)\ge e(c\wedge d_2,c\wedge\bigsqcup S)\ge e(d_2,\bigsqcup S)\ge S(d_2).$$

Then $e(c\wedge\bigsqcup S,d)\le S(d_2)\to e(d_2\wedge c,d)$ and $e(c\wedge\bigsqcup S,d)\le\bigwedge_{d_2\in C}S(d_2)\to e(d_2\wedge c,d)$.
These complete the proof. □

Theorem 3.8. *(Corollary 2.9 in* [28]*) Every completely distributive L-ordered set is continuous.*

In crisp setting, we know that a complete lattice is completely distributive iff its dual poset is completely distributive. But in fuzzy setting, there is no similar conclusion.

Before giving a counterexample, we introduce the following concept. Let (C,e) be a complete L-ordered set. For any $c,d\in C$, define

$$\rhd(c,d)=\bigvee_{U\in Up_L(C)}e(\bigsqcap U,c)\to C(d).$$

Then for any $c\in C$, $\rhd(c,\cdot)$ is an upper set. We call (C,e) co-completely distributive if $c=\bigsqcap\rhd(c,\cdot)$ for any $c\in C$. It is routine to show that a complete L-ordered set (C,e) is co-completely distributive iff (C,e^{op}) is completely distributive and of course if $L=\{0,1\}$, then completely distributivity and co-completely distributivity coincide with each other.

Proposition 3.9. *(cf. Theorem 1.1 in* [21]*) (L,e_L) is co-completely distributive iff L is a Boolean algebra.*

3.2. Strongly Completely Distributive L-ordered Sets.

Definition 3.10. We call a completely distributive L-ordered set (C,e) strongly completely distributive if it additionally satisfies the following two conditions

(SCD1) for any $c,d\in C$, $\bigwedge_{g\in pt_L(C)}g(c)\to g(d)\le e(c,d)$.

(SCD2) $(pt_L(C),sub_C)$ is continuous as a fuzzy dcpo.

Remark 3.11. (1) The inequality in (SCD1) can be rewritten as an equality since any L-fuzzy point of (C, e) can be firstly considered as an upper set.

(2) For the case $L = \{0, 1\}$, conditions (SCD1) and (SCD2) automatically hold. In for detail for (SCD1), in a completely distributive lattice C, every point of C is exactly a complete prime filter of C, and has exactly the form $C\backslash{\downarrow}p$ for some prime elements $p \in C$ [15]. It is well known that the set of all prime elements are $\bigwedge$-generating in C. For $c, d \in C$, if $c \not\leq d$, then there exists a prime element $p \in C$ such that $c \not\leq p$ and $d \leq p$. Then $c \in C\backslash{\downarrow}p$, but $d \notin C\backslash{\downarrow}p$.

Proposition 3.12. *Suppose that the least element 0 of L is prime. Then for any nonempty set X, $pt_L(L^X) = \{\Psi_x |\ x \in X\}$ and then (L^X, sub_X) is strongly completely distributive.*

Proof. Equipped X with discrete L-order, that is, $e(x, y) = 1$ if $x = y$ and 0 otherwise. Firstly, if $D \in \mathcal{D}_L(X)$, then by condition (FD2) $D(x) \wedge D(y) \leq \bigvee_{z\in X} D(z) \wedge e(x, z) \wedge e(y, z) = 0$ for any two distinct elements $x, y \in X$. Since 0 is prime in L, there exists at most one element x such that $D(x) \neq 0$. By condition (FD1), we have $D(x) = 1$, for a unique $x \in X$. Then the directed L-subset of X is exactly fuzzy singleton $1_{\{x\}}$ of X and $\bigsqcup 1_{\{x\}} = x$. Then X is a fuzzy dcpo. Secondly, it is easy to show that ${\Downarrow}x = 1_{\{x\}}$ and $x = \bigsqcup {\Downarrow}x$. Therefore X is a fuzzy domain. Thirdly, since e is discrete, we have $\sigma_L(X) = L^X$. By Theorem 4.3 below, $pt_L(L^X) = \{\Psi_x |\ x \in X\}$. □

Clearly, for any nonempty set X, we have $pt_L(L^X) \supseteq \{\Psi_x |\ x \in X\}$. The following proposition shows that if 0 is not a prime element of L, then the equation $pt_L(L^X) = \{\Psi_x |\ x \in X\}$ needn't be hold.

Proposition 3.13. *Let $(L, \neg)$ be a nontrivial (the number of elements is larger then 2) Boolean algebra. Then for any set X with at least two elements, $pt_L(L^X) \not\subseteq \{\Psi_z |\ z \in X\}$.*

Proof. For $a \in L$ and $x, y \in X$, Define $p = (a \wedge \Psi_x) \vee (\neg a \vee \Psi_y)$, that is for any $A \in L^X$, $p(A) = (a \wedge A(x)) \vee (\neg a \wedge A(y))$.

Step 1. $p \in pt_L(L^X)$. Clearly, $p(b_X) = b$ for any $b \in L$. And the two maps $a \wedge \Psi_x, \neg a \wedge \Psi_y$ preserves nonempty arbitrary joins of $(L^X, \leq)$. We only need to show that $p(A \wedge B) = p(A) \wedge p(B)$ for any $A, B \in L^X$. In fact,

$$\begin{aligned} p(A) \wedge p(B) &= [(a \wedge A(x)) \vee (\neg a \wedge A(y))] \wedge [(a \wedge B(x)) \vee (\neg a \wedge B(y))] \\ &= [(a \wedge A(x)) \wedge (a \wedge B(x))] \vee [(a \wedge A(x)) \wedge (\neg a \wedge B(y))] \\ &\quad \vee [(\neg a \wedge A(y)) \wedge (a \wedge B(x))] \vee [(\neg a \wedge A(y)) \wedge (\neg a \wedge B(y))] \\ &= [a \wedge (A \wedge B)(x)] \vee [\neg a \wedge (A \wedge B)(y)] \\ &= p(A \wedge B). \end{aligned}$$

Step 2. $p = \Psi_z$ for some $z \in X$ iff $a \in \{0, 1\}$ or $x = y$. Sufficiency is routine. Necessity: If $a \notin \{0, 1\}$ and $x \neq y$, then for any $z \in X$, we have $z \neq x$ or $z \neq y$. Put $A = a_{\{x\}} \vee (\neg a)_{\{y\}}$. Then $p(A) = a \vee \neg a = 1$ while $\Psi_y(A) = A(z) \neq 1$.

Step 3. By Step 2, if $x \neq y, a \notin \{0, 1\}$, then $p \notin \{\Psi_z |\ z \in X\}$. □

The following proposition shows that for any frame L, $pt_L(L^{\{\bullet\}}) = \{\Psi_\bullet\}$ for any singleton $\{\bullet\}$. Clearly, $L \cong L^{\{\bullet\}}$ and $\Psi_\bullet$ is the identical map $id_L : L \longrightarrow L$.

Proposition 3.14. *The identical map $id_L : L \longrightarrow L$ is the unique L-frame homomorphism from (L, e_L) to (L, e_L).*

Proof. For any $a \in L$, $\bigsqcup a_L = \bigvee_{b \in L} b \wedge a = a \wedge 1 = a$ and then

$$p(a) = p(\bigsqcup a_L) = \bigsqcup p_L^{\rightarrow}(a_L) = \bigvee_{b \in L} p(b) \wedge a = a \wedge p(\bigvee_{b \in L} b) = a \wedge p(1) = a \wedge 1 = a.$$

□

Let (C, e) be a complete L-ordered set. For $c \in C$, define $\Phi_c : pt_L(C) \longrightarrow L$ by $\Phi_c(p) = p(c)$ $(\forall p \in pt_L(C))$. Then $\Phi(C) = \{\Phi_c | \ c \in C\}$ is a stratified L-topology on $pt_L(C)$ (Proposition 4.2 in [43]), the space $Pt_L(C) = (pt_L(C), \Phi(C))$ is called the L-spectrum on (C, e). (C, e) is called spatial if $\Phi_L : C \longrightarrow \Phi(C)$ is injective (equivalently, Φ is an L-frame homomorphism, or Φ is an isomorphism) (Lemma 5.5 in [43]).

Proposition 3.15. *Every complete L-ordered set with the condition (SCD1) is spatial.*

Proof. Suppose that (C, e) is a complete L-ordered set. For $c, d \in C$, if $\Phi_c = \Phi_d$, then for any $p \in pt_L(C)$, $p(c) = p(d)$. By condition (SCD1),

$$e(c, d) \wedge e(d, c) \geq \bigwedge_{p \in pt_L(C)} p(c) \rightarrow p(d) \wedge \bigwedge_{q \in pt_L(C)} q(d) \rightarrow q(c) = 1$$

and then $c = d$. Hence Φ is injective. □

4. A Duality Between the Category of Fuzzy Domains and the Category of Strongly Completely Distributive L-ordered Sets

4.1. Transformation Between Fuzzy Domains and Strongly Completely Distributive L-ordered Sets.

Proposition 4.1. *If (X, e) is a fuzzy domain, then $U(x) \leq \lhd(\Uparrow x, U)$ holds for all $x \in X$ and $U \in \sigma_L(X)$.*

Proof. Suppose that $x \in X$ and $U \in \sigma_L(X)$. By the definition of the relation $\overset{|}{\triangledown}$,

$$\lhd(\Uparrow x, U) = \bigwedge_{\mathcal{A} \in \mathrm{Low}(\sigma_L(X))} sub_X(U, \bigsqcup \mathcal{A}) \rightarrow \mathcal{A}(\Uparrow x).$$

Then we only need to show that for all $\mathcal{A} \in \mathrm{Low}(\sigma_L(X))$,

$$U(x) \leq sub_X(U, \bigsqcup \mathcal{A}) \rightarrow \mathcal{A}(\Uparrow x) \ \text{ or } \ U(x) \wedge sub_X(U, \bigsqcup \mathcal{A}) \leq \mathcal{A}(\Uparrow x).$$

In fact, by Proposition 2.8, we have

$$\begin{aligned} & U(x) \wedge sub_X(U, \bigsqcup \mathcal{A}) \\ = \ & sub_X(\uparrow x, U) \wedge sub_X(U, \bigsqcup \mathcal{A}) \\ \leq \ & sub_X(\uparrow x, \bigsqcup \mathcal{A}) \\ = \ & (\bigsqcup \mathcal{A})(x) \\ = \ & \bigvee_{A \in \sigma_L(X)} \mathcal{A}(A) \wedge A(x) \\ \leq \ & \bigvee_{A \in \sigma_L(X)} \mathcal{A}(A) \wedge sub_X(\Uparrow x, A) \\ \leq \ & \mathcal{A}(\Uparrow x). \end{aligned}$$

□

Proposition 4.2. *If (X, e) is a fuzzy domain, then $(\sigma_L(X), sub_X)$ is a completely distributive L-ordered set.*

Proof. Since $\sigma_L(X)$ is stratified, $(\sigma_L(X), sub_X)$ is a complete L-ordered set. For any $U \in \sigma_L(X)$, since $\Uparrow x \in \sigma_L(X)$ $(\forall x \in X)$, by Proposition 4.1,

$$\bigsqcup \downarrow_{\triangledown} U = \bigvee_{A \in \sigma_L(X)} \lhd(A, U) \wedge A \geq \bigvee_{x \in X} \lhd(\Uparrow x, U) \wedge \Uparrow x \geq \bigvee_{x \in X} U(x) \wedge \Uparrow x = U$$

and $\bigsqcup \downarrow_{\triangledown} U = \bigvee_{A \in \sigma_L(X)} \lhd(A, U) \wedge A \leq \bigvee_{A \in \sigma_L(X)} sub_X(A, U) \wedge A \leq U.$ □

Let (X, δ) be a stratified L-topological space. We call a frame homomorphism $p : \delta \longrightarrow L$ modified if $p(a_X) = a$ for any $a \in L$. The set of all modified frame homomorphisms from δ to L is denoted by $Lpt_{mod}(\delta)$ in [30]. By Proposition 2.10(2), $Lpt_{mod}(\delta) = pt_L(\delta)$ holds for any stratified L-topology δ. A stratified L-topological space (X, δ) is called modified L-sober [30] if $\Psi : X \longrightarrow pt_L(\delta)$ is bijective.

Theorem 4.3. [44] *If (X, e) is a fuzzy domain, then $(X, \sigma_L(X))$ is a modified L-sober space.*

Remark 4.4. The technique used in Theorem 4.3 is new even for the crisp case. Since in crisp setting, sobriety is defined using closed sets, in all of the monographs related to domain theory, Scott open sets and Scott closed sets are combined used to show the sobriety of the Scott topology on a domain (see Proposition 7.2.27 in [2], Proposition II-1.11, Corollary II-1.12 in [10] and the results in Pages 291-292 in [15]). In Theorem 4.3, only (fuzzy) Scott open sets are used.

Let (X, e) be a fuzzy dcpo. $B \in L^X$ is called a fuzzy Scott closed set if it is a lower set and $sub_X(D, B) \leq B(\bigsqcup D)$ for any $D \in \mathcal{D}_L(X)$ [44]. Clearly for any $x \in X$, $\downarrow x$ is a fuzzy Scott closed set. We claim that if we want to mutually use fuzzy Scott open sets and fuzzy closed set, then the lattice L is probably a Boolean algebra (see Proposition 4.5 below).

Proposition 4.5. *(1) Suppose (X, e) is a fuzzy dcpo and $(L, \neg)$ is a Boolean algebra. Then $U \in L^X$ is fuzzy Scott open iff $\neg U$ is fuzzy Scott closed.*

(2) For the special fuzzy dcpo $(L, \rightarrow)$, suppose that there is an order-reserving involution $\neg$ on L. If the result in (1) holds, then $(L, \neg)$ is a Boolean algebra.

Proof. (1) is routine since $a \rightarrow b = \neg a \vee b$ for any $a, b \in L$.

(2) For any $a \in L$, since $D = \downarrow a$ is fuzzy Scott closed, $\neg D$ is fuzzy Scott open and $\bigsqcup D = a$. Consider D as a fuzzy ideal, we have

$$0 = (\neg D)(\bigsqcup D) = \bigvee_{b \in L} \neg D(b) \wedge D(b) \geq \neg D(1) \wedge D(1) = \neg a \wedge a$$

and $a \vee \neg a = \neg(\neg a \wedge a) = \neg 0 = 1$. Hence $(L, \neg)$ is a Boolean algebra. □

By Theorem 4.3, we have

Proposition 4.6. *If X is a fuzzy domain, then $(\sigma_L(X), sub_X)$ is strongly completely distributive.*

Proof. By Definition 4.1 and Proposition 4.2, it is sufficient to show that $pt_L(\sigma_L(X)) = \{\Psi_x |\ x \in X\}$ is Equivalence to X as an L-ordered set. In fact, for any $x, y \in X$, on one hand,

$$sub_X(\Psi_x, \Psi_y) = \bigwedge_{A \in \sigma_L(X)} A(x) \to A(y)$$

$$\leq \bigwedge_{z \in X} \Uparrow z(x) \to \Uparrow z(y) = sub_X(\Downarrow x, \Downarrow y) = e(x, y).$$

On the other hand, we have $sub_X(\Psi_x, \Psi_y) = \bigwedge_{A \in \sigma_L(X)} A(x) \to A(y) \geq e(x, y)$, since every fuzzy Scott open set is an upper set, □

4.2. **The Isomorphism Between Fuzzy Domains and Strongly Completely Distributive Complete L-ordered Sets.** By Theorems 4.3 and 4.6, we have

Theorem 4.7. *Let (X, e) be a fuzzy domain. Then $(\sigma_L(X), sub_X)$ is a strongly completely distributive L-ordered set and (X, e) is isomorphic to $pt_L(\sigma_L(X))$ via the assignment $x \mapsto \Psi_x$ $(\forall x \in X)$.*

Suppose that (C, e) is a strongly completely distributive L-ordered set.

Proposition 4.8. *For $c \in C$, $\Phi_c \in \sigma_L(pt_L(C))$.*

Proof. For any $p, q \in pt_L(C)$,

$$\Phi_c(p) \wedge sub_C(p, q) \leq p(c) \wedge (p(c) \to q(c)) \leq q(x) = \Phi_c(q),$$

it follows that Φ_c is an upper set in $(pt_L(C), sub_C)$. Suppose that $D \in \mathcal{D}_L(pt_L(C))$. We have

$$\bigvee_{p \in pt_L(C)} \Phi_c(p) \wedge D(p) = \bigvee_{p \in pt_L(C)} p(c) \wedge D(p) = (\bigsqcup D)(c) = \Phi_c(\bigsqcup D).$$

Hence Φ_c is fuzzy Scott open. □

For $A \in \sigma_L(pt_L(C))$, define $D_A \in L^C$ by $D_A(c) = \bigwedge_{g \in pt_L(C)} g(c) \to A(g)$ and put $c_A = \bigsqcup D_A$.

Proposition 4.9. *For any $A \in \sigma_L(pt_L(C))$, we have $\Phi_{c_A} = A$.*

Proof. For any $g \in pt_L(C)$, we have

$$\Phi_{c_A}(g) = g(c_A) = g(\bigsqcup D_A) = \bigsqcup g_L^{\rightarrow}(D_A) = \bigvee_{d \in C} g(d) \wedge D_A(d).$$

On one hand,

$$\bigvee_{d \in C} g(y) \wedge D_A(d) = \bigvee_{d \in C} g(d) \wedge (\bigwedge_{h \in pt_L(C)} h(d) \to A(h))$$

$$\leq \bigvee_{d \in C} g(d) \wedge (g(d) \to A(g)) \leq A(g).$$

On the other hand,

$$\begin{array}{rcl} A(g) & = & A(\bigsqcup \Downarrow g) = \bigsqcup A_L^{\rightarrow}(\Downarrow g) \\ & = & \bigvee_{h\in pt_L(C)} A(h) \wedge \Downarrow g(h) \\ & = & \bigvee_{h\in pt_L(C)} A(h) \wedge (\bigvee_{d\in C} g(d) \wedge sub_C(h, \overset{\vartriangle}{\shortmid} d)) \\ & = & \bigvee_{d\in C} g(d) \wedge (\bigvee_{h\in pt_L(C)} A(h) \wedge sub_C(h, \overset{\vartriangle}{\shortmid} d)) \\ & = & \bigvee_{d\in C} g(d) \wedge A(\overset{\vartriangle}{\shortmid} d). \end{array}$$

We only need to show that $A(\overset{\vartriangle}{\shortmid} d) \le D_A(d)$. In fact, since A is an upper set,

$$\begin{array}{rcl} D_A(d) & = & \bigwedge_{g\in pt_L(C)} g(d) \to A(g) \\ & = & \bigwedge_{g\in pt_L(C)} sub_C(\uparrow d, g) \to A(g) \\ & \ge & \bigwedge_{g\in pt_L(C)} sub_C(\overset{\vartriangle}{\shortmid} d, g) \to A(g) \\ & \ge & A(\overset{\vartriangle}{\shortmid} d). \end{array}$$

□

By Propositions 4.8 and 4.9,

Theorem 4.10. *For a strongly completely distributive L-ordered set (C, e), its L-spectrum coincides with the fuzzy Scott topology on $pt_L(C)$.*

Proposition 4.11. $D_A = \downarrow c_A$.

Proof. For all $c_1, c_2 \in C$,

$$\begin{array}{rl} D_A(c_1) \to D_A(c_2) \ge & \bigwedge_{g\in pt_L(C)} (g(c_1) \to A(g)) \to (g(c_2) \to A(g)) \\ \ge & \bigwedge_{g\in pt_L(C)} g(c_2) \to g(c_1) \ge e(c_2, c_1). \end{array}$$

Hence D_A is a lower set. Since $c_A = \bigsqcup D_A$, we have $D_A \le \downarrow c_A$. To show that $D_A = \downarrow c_A$, we only need to show that $D_A(c_A) = 1$. In fact,

$$\begin{array}{rcl} D_A(c_A) & = & \bigwedge_{g\in pt_L(C)} g(\bigsqcup D_A) \to A(g) \\ & = & \bigwedge_{g\in pt_L(C)} \bigsqcup g_L^{\rightarrow}(D_A) \to A(g) \\ & = & \bigwedge_{g\in pt_L(C)} \bigwedge_{c\in C} (g(c) \wedge D_A(c)) \to A(g) \\ & = & \bigwedge_{g\in pt_L(C)} \bigwedge_{c\in C} D_A(c) \to (g(c) \to A(g)) \\ & = & 1, \end{array}$$

as desired. □

Proposition 4.12. *For any $c \in C$, we have $c_{\Phi_c} = c$.*

Proof. We only need to show that $D_{\Phi_c} = \downarrow c$. In fact, since (C, e) is strongly completely distributive,

$$D_{\Phi_c}(d) = \bigwedge_{g \in pt_L(C)} g(d) \to \Phi_c(g) = \bigwedge_{g \in pt_L(C)} g(d) \to g(c) = e(d, c).$$

□

By Propositions 4.11 and 4.12,

Theorem 4.13. *Let (C, e) be a strongly completely distributive L-ordered set. Then $(pt_L(C), sub_C)$ is a fuzzy domain and (C, e) is isomorphic to $(\sigma_L(pt_L(C)), sub_{pt_L(C)})$ via the assignment $c \mapsto \Phi_c$ given by $\Phi_c(g) = g(c)$ $(\forall g \in pt_L(C))$.*

In any strongly completely distributive L-ordered set C, we have

Proposition 4.14. *For any $p, q \in pt_L(C)$,*

$$\Downarrow p(q) = p(\bigsqcap q) = \bigvee_{c \in C} p(c) \wedge sub_C(p, \uparrow c) = \bigvee_{c \in C} p(c) \wedge sub_C(p, \mathop{\Uparrow}\limits_{\mid} c).$$

4.3. **Up to category theory.** Let **FDom** denote the category of fuzzy domains with fuzzy Scott continuous maps. Let **SFCDL** denote the category of strongly completely distributive L-ordered sets with L-frame homomorphisms.

Define $\Sigma : \mathbf{FDom} \longrightarrow \mathbf{SFCDL}^{op}$ by $\Sigma(X, e) = (\sigma_L(X), sub_X)$ and $\Sigma(f) = f_L^{\leftarrow} : \sigma_L(X_2) \longrightarrow \sigma_L(X_1)$ $(\forall\ (X, e) \in |\mathbf{FDom}|$ and $\forall f : (X_1, e_1) \longrightarrow (X_2, e_2) \in \mathrm{Mor}(\mathbf{FDom}))$.

Define $\mathrm{Pt} : \mathbf{SFCDL} \longrightarrow \mathbf{FDom}^{op}$ by $\mathrm{Pt}(C, e) = (pt_L(C), sub_C)$ and $\mathrm{Pt}(g) = g_L^{\leftarrow} : pt_L(C_2) \longrightarrow pt_L(C_1)$ $(\forall\ (C, e) \in |\mathbf{SFCDL}|$ and $\forall g : (C_1, e_1) \longrightarrow (C_2, e_2) \in \mathrm{Mor}(\mathbf{SFCDL}))$.

Theorem 4.15. *Both Σ and* Pt *are functors and they define a dual equivalence between* **FDom** *and* **SFCDL**.

Proof. **(1)** Σ is a functor. Suppose that $f : (X_1, e_1) \longrightarrow (X_2, e_2) \in \mathrm{Mor}(\mathbf{FDom})$. We need to show that $\Sigma(f) = f_L^{\leftarrow} : \sigma_L(X_2) \longrightarrow \sigma_L(X_1) \in \mathrm{Mor}(\mathbf{SFCDL})$, which will be followed by Steps (a) and (b).

(a) $\Sigma(f)$ is a map. Suppose that $A \in \sigma_L(X_2)$. For any $x_1, x_2 \in X_1$,

$$\begin{aligned} e_1(x_1, x_2) \wedge f_L^{\leftarrow}(A)(x_1) &\le e_2(f(x_1), f(x_2)) \wedge A(f(x_1)) \\ &\le A(f(x_2)) = f_L^{\leftarrow}(A)(x_2). \end{aligned}$$

Then $f_L^{\leftarrow}(A)$ is an upper set in (X_1, e_1). For any $D \in \mathcal{D}_L(X_1)$,

$$\begin{aligned} &\Sigma(f)(A)(\bigsqcup D) = f_L^{\leftarrow}(A)(\bigsqcup D) = A(f(\bigsqcup D)) = A(\bigsqcup f_L^{\rightarrow}(D)) \\ &= \bigsqcup A_L^{\rightarrow}(f_L^{\rightarrow}(D)) = \bigsqcup (Af)_L^{\rightarrow}(D) = \bigvee_{x \in X_1} D(x) \wedge A(f(x)) \end{aligned}$$

and

$$\begin{aligned} &\bigsqcup (\Sigma(f)(A))_L^{\rightarrow}(D) = \bigsqcup (f_L^{\leftarrow}(A))_L^{\rightarrow}(D) \\ &= \bigvee_{x \in X_1} D(x) \wedge f_L^{\leftarrow}(A)(x) = \bigvee_{x \in X_1} D(x) \wedge A(f(x)). \end{aligned}$$

It follows that $\Sigma(f)(A)(\bigsqcup D) = \bigsqcup(\Sigma(f)(A))_L^{\rightarrow}(D)$. Hence $\Sigma(f)(A) \in \sigma_L(X_1)$.

(b) By Remark 2.6, we know that $f_L^{\leftarrow} : L^{X_2} \longrightarrow L^{X_1}$ is a left fuzzy adjoint and then it preserves arbitrary joins from the complete L-ordered set L^{X_2} to L^{X_1}. The fuzzy Scott topologies $\sigma_L(X_1)$ and $\sigma_L(X_2)$ are closed under arbitrary joins induced from L^{X_1} and L^{X_2}, respectively. Hence $\Sigma(f) = f_L^{\leftarrow} : \sigma_L(X_2) \longrightarrow \sigma_L(X_1)$ preserves arbitrary joins. The fact that $\Sigma(f)$ preserves finite meets is routine.

(2) Pt is a functor. Suppose that $g : (C_1, e_1) \longrightarrow (C_2, e_2) \in \mathrm{Mor}(\mathbf{SFCDL})$, we need to show that $\mathrm{Pt}(g) = g_L^{\leftarrow} : pt_L(C_2) \longrightarrow pt_L(C_1)$ is fuzzy Scott continuous. Similar to Step (b) above, we can see that $\mathrm{Pt}(g) = g_L^{\leftarrow}$ preserves arbitrary directed joins $pt_L(C_1), pt_L(C_2)$ are fuzzy dcpos, and hence $\mathrm{Pt}(g)$ is fuzzy Scott continuous.

(3) $\mathrm{Pt} \circ \Sigma$ is natural isomorphic of $\mathrm{id}_{\mathbf{FDom}}$. Suppose that $f : (A, e_A) \longrightarrow (B, e_B) \in \mathrm{Mor}(\mathbf{Fdom})$. Then both $\Psi_A : (A, e_A) \longrightarrow (\mathrm{Pt}(\sigma_L(A)), sub_A)$ and $\Psi_B : (B, e_B) \longrightarrow (\mathrm{Pt}(\sigma_L(B)), sub_B)$ are isomorphisms. Clearly, $\mathrm{Pt} \circ \Sigma(f) = (f_L^{\leftarrow})_L^{\leftarrow}$. For any $x \in A$ and for any $p \in pt(\sigma_L(B))$,

$$(\mathrm{Pt} \circ \Sigma(f))((\Psi_A)_x(p)) = (\Psi_A)_x(f_L^{\leftarrow}(p)) = p(f(x)) = (\Psi_B)_{f(x)}(p).$$

Then $(\mathrm{Pt} \circ \Sigma(f)) \circ (\Psi_A) = (\Psi_B) \circ f$.

(4) $\Sigma \circ \mathrm{Pt}$ is natural isomorphic of $\mathrm{id}_{\mathbf{SFCDL}}$. Suppose that $g : (C_1, e_1) \longrightarrow (C_2, e_2) \in \mathrm{Mor}(\mathbf{SFCDL})$. Then both $\Phi_{C_1} : (C_1, e_1) \longrightarrow (\sigma_L(pt_L(C_1)), sub_{pt_L(C_1)})$ and $\Phi_{C_2} : (C_2, e_2) \longrightarrow (\sigma_L(pt_L(C_2)), sub_{pt_L(C_2)})$ are isomorphism. For any $c \in C_1$ and any $p \in pt_L(C_2)$,

$$\begin{aligned} & \Sigma(\mathrm{Pt}(g))(\Phi_{C_1}(c)(p)) = (g_L^{\leftarrow})_L^{\leftarrow}(\Phi_{C_1}(c))(p) \\ = \; & \Phi_{C_1}(c)(g_L^{\leftarrow}(p)) = p(g(c)) = \Phi_{C_2}(g(c))(p). \end{aligned}$$

Hence $\Sigma(\mathrm{Pt}(g)) \circ \Phi_{C_1} = \Phi_{C_2} \circ g$. □

5. The Reason for Choosing L a Frame

In [44], we have studied fuzzy Scott topology on fuzzy dcpos and shown the Cartesian-closeness of the category of fuzzy dcpos. Many readers are doubtful that, why the results are not established on some more general lattices than a frame, for example a commutative unital quantale? In this section, we would like to answer this question.

Firstly, we would like to point out a mistake in [39]. Let Ω be a commutative unital quantale.

Remark 5.1. In [39], Wagner also defines Scott open sets in quantitative setting, an Ω-functor $\phi : A \longrightarrow \Omega$ is Scott open if for all convergent sequences α in A, $\phi(\lim \inf \alpha) \leq \lim \inf(\phi \circ \alpha)$ (Definition 4.1 in [39]). And in Theorem 4.10, it is claimed that the family of all Scott open sets SA is a commutative unital quantale. Lemma 4.6 says that whenever ϕ and ψ are Scott open, so is $\phi \otimes \psi$. In the proof of Lemma 4.6, the first sentence says that the up-closedness of $\phi \otimes \psi$ is obviously or in our words, $\phi \otimes \psi$ is obvious an upper set. Unfortunately, it is not true.

Proposition 5.2. *The following are equivalent.*

(1) $\phi \otimes \psi$ is an upper set (hence Scott open) for any Scott open sets ϕ, ψ;

(2) $a \leq a \otimes a$ for every $a \in \Omega$.

Proof. (1)⇒(2). If (1) holds, then following the Wagner's method, we have that $S\Omega$ is a quantale. Let $\phi = \mathrm{id}_\Omega$. Then $\phi \in S\Omega$ and $\phi \otimes \phi \in S\Omega$. For any $a \in \Omega$, $(\phi \otimes \phi)(I) \otimes (I \to a) \leq (\phi \otimes \phi)(a)$ and hence $a \leq a \otimes a$.

(2)⇒(1). Suppose that ϕ, ψ are two upper sets, then for any $a, b \in \Omega$, $a \to b \leq \phi(a) \to \phi(b)$ and $a \to b \leq \psi(a) \to \psi(b)$ and then

$$\begin{aligned} a \to b &\leq (a \to b) \otimes (a \to b) \\ &\leq (\phi(a) \to \phi(b)) \otimes (\psi(a) \to \psi(b)) \\ &\leq (\phi \otimes \psi)(a) \to (\phi \otimes \psi)(b). \end{aligned}$$

Hence $\phi \circ \psi$ is an upper set. □

Remark 5.3. There is also another potential mistake in Lemma 4.14 and consequently in Proposition 4.15 [39]. Lemma 4.14 says that if $f : A \longrightarrow B$ is Scott continuous (inverse images of Scott open sets are Scott open sets, cf. Definition 4.11) and $\psi : B^{op} \longrightarrow \Omega$ is Scott closed (cf. Definition 4.4), then $\psi \circ f : A^{op} \longrightarrow \Omega$ is Scott closed. In the proof of Lemma 4.14, the last sentence says that if a sequence α converges to a, then $(f(\alpha_n))_{n\in\mathbb{N}}$ converges to $f(a)$. It is not true in our opinion and also has never mentioned or proved before Lemma 4.14.

Consequently, Proposition 4.15 says that a function is Scott continuous iff it is liminf continuous. It is also not true since Proposition 4.15 is based on Lemma 4.14. In fact, we can only prove that a function is liminf continuous if the inverse images of Scott closed sets are Scott closed. Notice that the classical counterparts of Proposition 4.15 are proved by Scott closed sets (cf. Proposition 2.3.4 in [2] and Proposition II-2.1 in [10]). If we want to correct Proposition 4.15 in [39], then Ω should be a Boolean algebra (cf. Proposition 4.5 in this paper).

Remark 5.4. (1) A commutative quantale satisfying condition (2) of Proposition 5.2 is called pre-idempotent in [12]. It is easy to show that a quantale is pre-idempotent iff $a \wedge b \leq a \otimes b$ for all elements a, b.

(2) The canonical lattice of truth values $([0,\infty]^{op}, +, 0)$ in generalized metric spaces theory [25] is not pre-idempotent. Thus the results related to Scott topology in [39] can not be applied to generalized metric spaces.

(3) Any pre-idempotent complete residuated lattice is precisely a frame.

(4) If we choose a commutative, unital, pre-idempotent quantale $(\Omega, \otimes, I)$ as a lattice of truth values, then the first condition of a direct L-subset D of an Ω-ordered set (X, e) should be

(FD′) $\bigvee_{x\in X} D(x) \geq I$ (cf. Definition 5.1 in [22]).

Then the proof of Proposition 4.3 should be revised, where (1) of Step 2 should be rewritten as

$\bigvee_{x\in X} P(x) = \bigvee_{x\in X} p(\Uparrow x) = p(\bigvee_{x\in X} \Uparrow x) \geq p(I_X) \geq I$ since for any $y \in X$, we have $\bigvee_{x\in X} \Uparrow x(y) = \bigvee_{x\in X} \Downarrow y(x) \geq I$ (notice that $\Downarrow y$ is directed).

But here $p(I_X) \geq I$ is not guaranteed by an L-frame homomorphism $p : \sigma_L(X) \longrightarrow L$ and such an inequality is difficult to be defined between L-frames or completely distributive L-ordered sets since there is probably no unit with them.

6. **Conclusions and Remarks**

In this paper, by the means of fuzzy Scott topology on fuzzy dcpos and fuzzy points of complete L-ordered sets, we establish a duality between the category of fuzzy domains and the category of strongly completely distributive L-ordered sets. More over, we prove that (i) Every completely distributive L-ordered set is simultaneously a fuzzy domain and an L-frame; (ii) The fuzzy Scott topology on a fuzzy domain is modified L-sober; (iii) The L-spectrum on an strongly completely distributive L-ordered set coincides with the fuzzy Scott topology on the set of its L-fuzzy points. These results establish close relations among (continuous) fuzzy dcpos, fuzzy Scott topology, L-frames, completely distributive L-ordered sets and modified L-sobriety and L-topological spaces.

The sobriety is a link which connects topology theory and lattice theory as well as domain theory. By using a new approach, Theorem 4.3 shows that the fuzzy Scott topology on a fuzzy domain is a modified L-sober space. In [43], we also establish a duality between the category of modified L-sober (stratified) L-topological spaces and the category of spatial L-frame. The above two main results indicate that the modified L-sobriety is a proper sobriety in fuzzy setting. We will continue the study of properties of modified L-sobriety and its relation to other kinds of fuzzy sobriety in future (cf. [30] for a discussion of many different kinds of fuzzy sobriety).

In this paper, there are two kinds of fuzzy versions of completely distributivity: the completely distributivity and the strongly fuzzy completely distributivity—the completely distributivity with the conditions (SCD1) and (SCD2). We show that every (resp., strongly) completely distributive L-ordered set is an (resp., spatial) L-frame. In crisp setting, (SCD1) and (SCD2) automatically hold for any ordinary completely distributive lattice, related methods of proof and intermediate results are

(1) to constructing a way-below chain between two elements;

(2) the method of reduction to absurdity;

(3) the results related to Scott closed sets;

(4) the dual poset of a completely distributive lattice is completely distributive; and so on (see in [2] for detail). All of these become difficulties in fuzzy setting since they are either difficult to be translated into fuzzy language, or will induce a naive result for the background lattice, or do not hold anymore in fuzzy setting (cf. Theorem 3.9 and Remark 4.4(2)) and so on. A future work should pay attention to that whether or not (SCD1) and (SCD2) already hold for a completely distributivity L-ordered set. Maybe we will try to find some alternative proof methods.

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