CONVERGENCE OF A SEMI-ANALYTICAL METHOD ON THE FUZZY LINEAR SYSTEMS

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Abstract. In this paper, we apply the homotopy analysis method (HAM) for solving fuzzy linear systems and present the necessary and sufficient conditions for the convergence of series solution obtained via the HAM. Also, we present a new criterion for choosing a proper value of convergence-control parameter $\hbar$ when the HAM is applied to linear system of equations. Comparisons are made between the results of the HAM and several well-known numerical algorithms such as Jacobi method (JM), Gauss-Seidel method (GSM), successive over relaxation method (SOR), Adomian decomposition method (ADM) and homotopy perturbation method (HPM).

1. Introduction

Systems of fuzzy linear equations occur in many fields, such as control problems, information, physics, statistics, engineering, economics, finance and even social sciences. In the 1990s, Buckley et al. [13, 14, 15] investigated them in series. Subsequently, Friedman et al. [18] considered a fuzzy linear system (FLS) as follows,

$$\begin{cases} a_{1\,1}x_1 + a_{1\,2}x_2 + \cdots + a_{1\,n}x_n = y_1, \\ a_{2\,1}x_1 + a_{2\,2}x_2 + \cdots + a_{2\,n}x_n = y_2, \\ \vdots \\ a_{n\,1}x_1 + a_{n\,2}x_2 + \cdots + a_{n\,n}x_n = y_n, \end{cases} \tag{1}$$

where the coefficient matrix $\mathbf{A} = (a_{ij})$ is a crisp matrix and $\mathbf{y} = (y_i)$ is a fuzzy vector, $1 \leqslant i, j \leqslant n$. They used the embedding method and replaced the original FLS by a crisp linear system with a nonnegative coefficient matrix $\mathbf{S}$, which may be singular even if $\mathbf{A}$ is nonsingular. They also presented conditions for the existence of unique fuzzy solution to the system. In [2, 3], Allahviranloo investigated several numerical methods, such as Jacobi method (JM), Gauss-Seidel method (GSM) and successive over relaxation method (SOR) for solving such fuzzy linear systems for the first time. Also, he proposed the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) for solving FLS in [4, 5]. Also, a simple and practical method to obtain fuzzy symmetric solutions of fuzzy linear systems is proposed in [10]. In 2012, Allahviranloo et al. [9, 11] obtained the fuzzy exact solutions and the nearest fuzzy symmetric solutions of fuzzy linear systems

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by proposing a new metric. Recently, Allahviranloo and Ghanbari [7, 8] proposed a new approach for solving fuzzy linear systems by introducing a new concept, namely "interval inclusion linear system". Nuraei et al. [30] obtained algebraic solution of a fuzzy linear system by limiting it's solution set. In 2009, Jafari et al. [23] considered the well-known homotopy analysis method (HAM) for solving fuzzy linear systems. Unfortunately, they have used the HAM in a commonplace way and did not discuss about the convergence of HAM when applied to fuzzy linear systems.

The HAM proposed by Liao [25, 26, 27, 28, 29] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-difference equations. This method is unique among other similar methods as it allows us to effectively control the region of convergence and rate of convergence of a series solution to a nonlinear problem, via control of an initial approximation, an auxiliary linear operator, an auxiliary function and a convergence-control parameter [29]. Recently, Van Gorder et al. [35] have discussed the selection of initial approximation, auxiliary linear operator, auxiliary function and convergence-control parameter in the application of HAM. They presented methods by which one may select the mentioned items when attempting to solve a nonlinear differential equation by using HAM. They Also, presented necessary and sufficient conditions for the convergence of series solution obtained via HAM [35]. In 2010, Odibat [31] discussed a reliable approach and presented the sufficient condition for convergence of HAM when is applied to nonlinear problems. since the HPM (proposed in [20, 21, 22]) is indeed a special case of the HAM it has been pointed out by many researchers, such as Abbasbandy [1], Liao et al. [28], Bataineh et al. [12], Van Gorder et al. [34], Hayat and Sajid [19, 32], and Song et al. [33].

The aim of this paper is to apply the HAM, in a different way from Jafari et al.'s [23], to obtain approximate solutions of the fuzzy linear systems. Also, unlike Jafari et al.'s, we will discuss the convergence of HAM and present the necessary and sufficient conditions for the convergence of series solution obtained via HAM when it is applied to linear system of equations. It is known that the series solution obtained by HAM, contains the convergence-control parameter $\hbar$, which provides us with a simple way to adjust and control the convergence of the series solution. Liao [27] suggested to choose a proper value of $\hbar$ by plotting the so-called $\hbar$-curve. However, in this paper we present a new way for choosing an appropriate value of convergence-control parameter $\hbar$ by the spectral radius of matrix, when the HAM is applied to linear system of equations.

We also compare solutions obtained by the HAM with those obtained by the various well-known methods, such as JM, GSM, SOR, ADM and HPM. Unfortunately, it is found that one may get the divergent results using the above mentioned methods for a FLS which its coefficient matrix is not strictly diagonally dominant, while the HAM provides a convenient way to ensure the convergence of results by introducing convergence-control parameter $\hbar$.

2. **Preliminaries**

Following [16], a fuzzy number is defined as an ordered pair of functions $(\underline{x}(r), \overline{x}(r))$, $0 \leqslant r \leqslant 1$, which satisfies the following requirements:

1. $\underline{x}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\overline{x}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{x}(r) \leqslant \overline{x}(r),\ 0 \leqslant r \leqslant 1$.

A crisp number α is simply represented by $\underline{x}(r) = \overline{x}(r) = \alpha,\ 0 \leqslant r \leqslant 1$. We recall that for $a \leqslant b \leqslant c \leqslant d,\ a, b, c, d \in \mathbb{R}$, the trapezoidal fuzzy number $x = (a, b, c, d)$ determined by a, b, c, d is denoted as $x = (\underline{x}(r), \overline{x}(r))$ such that $\underline{x}(r) = a + (b-a)r$ and $\overline{x}(r) = d - (d-c)r$.

To define a solution for system (1) we should recall the arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$ and real number k,

1. $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$, $\forall r \in [0, 1]$.
2. $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.
3. $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \geqslant 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.1. [18] A fuzzy number vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ given by

$$x_i = (\underline{x_i}(r), \overline{x_i}(r)), \qquad 1 \leqslant i \leqslant n, \qquad 0 \leqslant r \leqslant 1,$$

is called a *solution* of the fuzzy linear system (1) if

$$\begin{cases} \underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y_i}\ , \\ \\ \overline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{y_i}\ , \end{cases} \qquad i = 1, 2, \ldots, n. \tag{2}$$

Now suppose that $x_i = (\underline{x_i}(r), \overline{x_i}(r))$ is an exact fuzzy solution and $x'_i = (\underline{x'_i}(r), \overline{x'_i}(r))$ is the estimated fuzzy solution. Following [17], our aim is to maximize the possibility of closeness of exact solutions and our estimated solutions. So we define the criterion to this objective:

$$E_i = \int_0^1 \Big(|\underline{x_i}(r) - \underline{x'_i}(r)| + |\overline{x_i}(r) - \overline{x'_i}(r)| \Big)\, dr, \qquad i = 1, 2, \ldots n. \tag{3}$$

Obviously, the closer this criterion to zero, the better the possibility of closeness of these solutions becomes well.

Using the embedding method given in [16] and equation (2), Friedman et al. [18] extended FLS (1) to a $2n \times 2n$ crisp linear system

$$\mathbf{Sx} = \mathbf{y}, \tag{4}$$

where $\mathbf{S} = (s_{kl})$, s_{kl} is determined as follows

$$\begin{array}{lll} a_{ij} \geq 0 \ \Rightarrow \ s_{ij} = a_{ij}, & s_{i+n,j+n} = a_{ij}, & \\ a_{ij} < 0 \ \Rightarrow \ s_{n+i,j} = -a_{ij}, & s_{i,j+n} = -a_{ij}, & \end{array} \qquad 1 \leq i, j \leq n,$$

and any s_{kl} which is not determined by the above items is zero, $1 \le k, l \le 2n$, and

$$\mathbf{x} = \begin{bmatrix} \underline{x_1}(r) \\ \vdots \\ \underline{x_n}(r) \\ -\overline{x_1}(r) \\ \vdots \\ -\overline{x_n}(r) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} \underline{y_1}(r) \\ \vdots \\ \underline{y_n}(r) \\ -\overline{y_1}(r) \\ \vdots \\ -\overline{y_n}(r) \end{bmatrix}.$$

In terms of [18], we know that $\mathbf{S}$ has the following structure

$$\begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} \end{bmatrix},$$

where $\mathbf{B}$ contains the positive entries of $\mathbf{A}$, $\mathbf{C}$ the absolute values of the negative entries of $\mathbf{A}$ and $\mathbf{A} = \mathbf{B} - \mathbf{C}$, $|\mathbf{A}| = \mathbf{B} + \mathbf{C}$.

The equation (4) is now a $2n \times 2n$ crisp linear system and can be uniquely solved for $\mathbf{x}$, if and only if the matrix $\mathbf{S}$ is nonsingular. The fact is however that $\mathbf{S}$ may be singular even if the original matrix $\mathbf{A}$ is not. The following theorem implies that when FLS (1) has unique solution.

Theorem 2.2. [18] *The matrix $\boldsymbol{S}$ is nonsingular if and only if the matrices $\boldsymbol{A} = \boldsymbol{B} - \boldsymbol{C}$ and $|\boldsymbol{A}| = \boldsymbol{B} + \boldsymbol{C}$ are both nonsingular.*

Assume that $\mathbf{S}$ is nonsingular, we obtain

$$\mathbf{x} = \mathbf{S}^{-1}\mathbf{y},$$

the solution vector is indeed unique, but the question we have to answer is: "Do the components of the $2n$-dimensional solution vector $\mathbf{x}$ represent an n-dimensional solution fuzzy vector to the fuzzy system given by equation (1)?"

The following result provides a sufficient condition for the unique solution to be a fuzzy vector.

Theorem 2.3. [2] *The unique solution $\boldsymbol{x}$ of equation (4) is a fuzzy vector for an arbitrary fuzzy vector $\boldsymbol{y}$ if $\boldsymbol{S}^{-1}$ is nonnegative.*

We can show that a nonnegative square matrix has a nonnegative inverse if and only if it is the product of a permutation matrix by a diagonal matrix. Another note is that a nonnegative square matrix has a nonnegative inverse if and only if its entries are all zero except for a single positive entry in each row and column.

Restricting to trapezoidal fuzzy numbers and having calculated vector solution $\mathbf{x}$ which solves (4), we can define the fuzzy solution to the original system given by (1) as follows:

Definition 2.4. [18] Let $X = \{(\underline{x_i}(r), -\overline{x_i}(r)),\ 1 \leqslant i \leqslant n\}$ denote the unique solution of the $2n \times 2n$ crisp linear system (4). The fuzzy number vector $U = \{(\underline{u_i}(r), \overline{u_i}(r)),\ 1 \leqslant i \leqslant n\}$ defined by

$$\underline{u_i}(r) = \min\{\underline{x_i}(r), \overline{x_i}(r), \underline{x_i}(1), \overline{x_i}(1)\},$$

$$\overline{u_i}(r) = \max\{\underline{x_i}(r), \overline{x_i}(r), \underline{x_i}(1), \overline{x_i}(1)\},$$

is called the fuzzy solution of $\mathbf{Sx} = \mathbf{y}$. If $\big(\underline{x_i}(r), \overline{x_i}(r)\big)$, $1 \leqslant i \leqslant n$, are all fuzzy numbers then $\underline{u_i}(r) = \underline{x_i}(r)$, $\overline{u_i}(r) = \overline{x_i}(r)$, $1 \leqslant i \leqslant n$ and U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

Remark 2.5. Recently, Allahviranloo et al. [6] showed that for some of special fuzzy numbers the weak fuzzy solution defined in Definition 2, may not be a fuzzy number vector. However, we can show that for trapezoidal fuzzy numbers, weak fuzzy solution is always a fuzzy number vector.

3. The Homotopy Analysis Method

Consider the system

$$\mathbf{Sx} = \mathbf{y}, \tag{5}$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} \end{bmatrix}_{2n\times 2n}, \qquad \mathbf{x} = \begin{bmatrix} \underline{x_1}(r) \\ \vdots \\ \underline{x_n}(r) \\ -\overline{x_1}(r) \\ \vdots \\ -\overline{x_n}(r) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} \underline{y_1}(r) \\ \vdots \\ \underline{y_n}(r) \\ -\overline{y_1}(r) \\ \vdots \\ -\overline{y_n}(r) \end{bmatrix}.$$

For application of HAM, we rewrite the above system (5) in the form:

$$\mathcal{N}(\mathbf{x}) = \mathbf{S}\,\mathbf{x} - \mathbf{y} = \mathbf{0}, \tag{6}$$

and construct the zero-order deformation equation

$$(1-p)\,\mathcal{L}[\mathbf{u}(r;p) - \mathbf{x}_0] = p\,\hbar\; H(r)\; \mathcal{N}[\mathbf{u}(r;p)], \tag{7}$$

where $p \in [0,1]$ is called homotopy-parameter [29], $\hbar$ is non-zero auxiliary parameter which is called convergence-control parameter [29], $H(r)$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $\mathbf{x}_0$ is an initial guess of $\mathbf{x}$ and $\mathbf{u}(r;p) = (u_1(r;p), u_2(r;p), \ldots, u_{2n}(r;p))^t$ is an unknown vector function.

Using the above zero-order deformation equation, with $H(r) = 1$ and $\mathcal{L}[\mathbf{u}] = \mathbf{u}$, we have

$$(1-p)\,[\mathbf{u}(r;p) - \mathbf{x}_0] = p\,\hbar\,[\mathbf{S}\,\mathbf{u}(r;p) - \mathbf{y}]. \tag{8}$$

When $\hbar = -1$, it can be easily seen that the above equation (8) is exactly the same as equation (2) of Ref. [24]. Therefore, the HPM is indeed a special case of HAM, when $\mathcal{L}[\mathbf{u}] = \mathbf{u}$, $H(r) = 1$ and $\hbar = -1$.

Obviously, for $p = 0$ and $p = 1$, and we have

$$\mathbf{u}(r;0) = \mathbf{x}_0, \qquad \mathbf{u}(r;1) = \mathbf{x},$$

respectively. Thus, as p increases from 0 to 1, the solution $\mathbf{u}(r;p)$ varies from the initial guess $\mathbf{x}_0$ to the solution $\mathbf{x}$. Expanding $\mathbf{u}(r;p)$ in Maclaurin series with respect to p, we have

$$\mathbf{u}(r;p) = \mathbf{x}_0 + \sum_{m=1}^{\infty} \mathbf{x}_m p^m, \tag{9}$$

where

$$\mathbf{x}_m = \frac{1}{m!}\frac{\partial^m \mathbf{u}(r;p)}{\partial p^m}|_{p=0}. \tag{10}$$

Here, the series (9) is called homotopy-series and equation (10) is called the mth-order homotopy-derivative of $\mathbf{u}$ [29]. If the auxiliary linear operator, the auxiliary function, the initial guess, the convergence-control parameter $\hbar$ are so properly chosen, the homotopy-series (9) converges at $p = 1$, then using the relationship $\mathbf{u}(r;1) = \mathbf{x}$, one has the so-called homotopy-series solution [29]

$$\mathbf{x} = \mathbf{x}_0 + \sum_{m=1}^{\infty} \mathbf{x}_m.$$

Differentiating the zero-order deformation equation (8) m times with respect to the homotopy-parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have

$$\mathbf{x}_m - \alpha_m \mathbf{x}_{m-1} = \hbar\,(\mathbf{S}\,\mathbf{x}_{m-1} - \beta_m \mathbf{y}), \qquad (m \geq 1), \tag{11}$$

where

$$\alpha_m = \begin{cases} 0, & m = 1, \\ 1, & m \neq 1, \end{cases} \qquad\qquad \beta_m = \begin{cases} 1, & m = 1, \\ 0, & m \neq 1. \end{cases}$$

If we take $\mathbf{x}_0 = \mathbf{0}$, then we have

$$\mathbf{x}_m = -\hbar\,(\mathbf{I} + \hbar\,\mathbf{S})^{m-1}\mathbf{y}, \qquad (m \geq 1). \tag{12}$$

Hence, the solution is

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \cdots,$$

or

$$\mathbf{x} = -\hbar\,\left[\mathbf{I} + (\mathbf{I} + \hbar\,\mathbf{S}) + (\mathbf{I} + \hbar\,\mathbf{S})^2 + \cdots\right]\mathbf{y} = -\hbar\left[\sum_{i=0}^{\infty}(\mathbf{I} + \hbar\,\mathbf{S})^i\right]\mathbf{y}. \tag{13}$$

When $\hbar = -1$, the expression (13) is the same as the solution series obtained by HPM (see equation (4) of Ref. [24]). Therefore, the HAM logically contains the HPM.

For later numerical computations, we set

$$\mathbf{k}_n = -\hbar\left[\sum_{i=0}^{n}(\mathbf{I} + \hbar\,\mathbf{S})^i\right]\mathbf{y}, \qquad (n \geq 0), \tag{14}$$

to denote the nth-order approximation for the solution $\mathbf{x}$.

We know that if λ_i $(i = 1, 2, \ldots, 2n)$, is the eigenvalues of matrix $\mathbf{S}$, then

$$\rho(\mathbf{S}) := \max_{1\leq i\leq 2n} |\lambda_i|,$$

is called *spectral radius* of $\mathbf{S}$. It is easy to show that

$$\rho(\mathbf{S}) = \inf_{\|\cdot\|} ||\mathbf{S}\,\|,$$

where $\|\cdot\|$ is an arbitrary matrix norm.

Theorem 3.1. *If $\rho(\boldsymbol{I}+\hbar\,\boldsymbol{S}) < 1$, then the sequence $\{\boldsymbol{k}_n\}_{n\geq 0}$ is a Chauchy sequence.*

Proof. Since $\rho(\mathbf{I}+\hbar\,\mathbf{S}) < 1$, there exists vector norm $\|\cdot\|_{V}$ such that for it's corresponding matrix norm, denoted by $\|\cdot\|_{M}$, we have

$$\|\mathbf{I}+\hbar\,\mathbf{S}\|_{M} < 1.$$

We show that the sequence $\{\mathbf{k}_n\}_{n\geq 0}$ is a Chauchy sequence with respect to the vector norm $\|\cdot\|_{V}$. To this end, we must show that

$$\lim_{n\longrightarrow\infty} \|\mathbf{k}_{n+m}-\mathbf{k}_n\|_{V} = 0.$$

We can write

$$\mathbf{k}_{n+m}-\mathbf{k}_n = -\hbar\left[\sum_{i=n+1}^{n+m}(\mathbf{I}+\hbar\,\mathbf{S})^i\right]\mathbf{y}.$$

Therefore

$$\|\mathbf{k}_{n+m}-\mathbf{k}_n\|_{V} \leq |\hbar|\ \|\mathbf{y}\|_{V}\sum_{i=1}^{m}\|\mathbf{I}+\hbar\,\mathbf{S}\|_{M}^{i+n},$$

If $\alpha = \|\mathbf{I}+\hbar\,\mathbf{S}\|_{M}$, then

$$\|\mathbf{k}_{n+m}-\mathbf{k}_n\|_{V} \leq |\hbar|\ \|\mathbf{y}\|_{V}\ \alpha^n\sum_{i=1}^{m}\alpha^i = |\hbar|\ \|\mathbf{y}\|_{V}\ \alpha^n(\frac{\alpha^{m+1}-\alpha}{\alpha-1}).$$

Since $\alpha < 1$, we have

$$\lim_{n\longrightarrow\infty}\|\mathbf{k}_{n+m}-\mathbf{k}_n\|_{V} \leq |\hbar|\ \|\mathbf{y}\|_{V}\ (\frac{\alpha^{m+1}-\alpha}{\alpha-1})\cdot\lim_{n\longrightarrow\infty}\alpha^n = 0,$$

hence, we obtain

$$\lim_{n\longrightarrow\infty}\|\mathbf{k}_{n+m}-\mathbf{k}_n\|_{V} = 0,$$

which completes the proof. □

According to Theorem 3.1, if $\rho(\mathbf{I}+\hbar\,\mathbf{S}) < 1$, then the sequence $\{\mathbf{k}_n\}_{n\geq 0}$ converges with respect to the vector norm $\|\cdot\|_{V}$. On the other hand, it can be shown that the convergence of vectors in $\mathbb{R}^n$ is independent of the chosen norm. Then, we conclude that the sequence $\{\mathbf{k}_n\}_{n\geq 0}$ converges with respect to any vector norm and consequently the series

$$-\hbar\left[\sum_{i=0}^{\infty}(\mathbf{I}+\hbar\,\mathbf{S})^i\right]\mathbf{y},$$

is convergent. In the following theorem, we show that this series is converge to the unique solution $\mathbf{x} = \mathbf{S}^{-1}\mathbf{y}$.

Theorem 3.2. *Let $\boldsymbol{S}$ be nonsingular matrix and $\rho(\boldsymbol{I}+\hbar\,\boldsymbol{S}) < 1$. Then*

$$-\hbar\left[\sum_{i=0}^{\infty}(\boldsymbol{I}+\hbar\,\boldsymbol{S})^i\right]\boldsymbol{y} = \boldsymbol{S}^{-1}\boldsymbol{y}.$$

Proof. Since $\rho(\mathbf{I}+\hbar\,\mathbf{S}) < 1$, the matrix $\mathbf{I}+\hbar\,\mathbf{S}$ is a convergent matrix. Hence,

$$\sum_{i=0}^{\infty}(\mathbf{I}+\hbar\,\mathbf{S})^i = (\mathbf{I}-(\mathbf{I}+\hbar\,\mathbf{S}))^{-1} = -\frac{1}{\hbar}\mathbf{S}^{-1}.$$

and so,

$$-\hbar\left[\sum_{i=0}^{\infty}(\mathbf{I}+\hbar\,\mathbf{S})^i\right]\mathbf{y} = -\hbar\ \left[-\frac{1}{\hbar}\mathbf{S}^{-1}\right]\mathbf{y} = \mathbf{S}^{-1}\mathbf{y}.$$

□

Theorem 3.3. *Let $\boldsymbol{S}$ be a nonsingular matrix. Then:*
(a) the homotopy-series solution (13) obtained by HAM converges to the unique solution $\boldsymbol{x} = \boldsymbol{S}^{-1}\boldsymbol{y}$ if and only if $\rho(\boldsymbol{I} + \hbar\,\boldsymbol{S}) < 1$.
(b) $\| \boldsymbol{I} + \hbar\,\boldsymbol{S} \| < 1$ is a sufficient condition for the convergence of HAM, where $\| \,.\, \|$ is an arbitrary matrix norm.

Proof. (a): (1) Firstly, suppose that the homotopy-series solution (13) obtained by HAM converges to the unique solution $\mathbf{x} = \mathbf{S}^{-1}\mathbf{y}$. Then we have

$$-\hbar \left[\sum_{i=0}^{\infty} (\mathbf{I} + \hbar\,\mathbf{S})^i \right] \mathbf{y} = \mathbf{S}^{-1}\mathbf{y}.$$

It follows that

$$\sum_{i=0}^{\infty} (\mathbf{I} + \hbar\,\mathbf{S})^i = -\frac{1}{\hbar}\mathbf{S}^{-1} = (\mathbf{I} - (\mathbf{I} + \hbar\,\mathbf{S}))^{-1}.$$

Thus, the matrix $\mathbf{I} + \hbar\,\mathbf{S}$ is a convergent matrix and consequently $\rho(\mathbf{I} + \hbar\,\mathbf{S}) < 1$.
(2) If, conversely, $\rho(\mathbf{I} + \hbar\,\mathbf{S}) < 1$, it follows from Theorems 3.1 and 3.2 that the HAM converges to the unique solution $\mathbf{x} = \mathbf{S}^{-1}\mathbf{y}$.

(b): For an arbitrary norms one has $\rho(\mathbf{I} + \hbar\,\mathbf{S}) \leq \| \mathbf{I} + \hbar\,\mathbf{S} \|$. This proves the theorem. □

Theorem 3.4. $\rho(\boldsymbol{I} + \hbar\,\boldsymbol{S}) < 1$ *if and only if* $\rho(\boldsymbol{I} + \hbar(\boldsymbol{B} \pm \boldsymbol{C})) < 1$.

Proof.

$$\begin{aligned}
\det([\mathbf{I} + \hbar\,\mathbf{S}] - \lambda\mathbf{I}) &= \det(\hbar\,\mathbf{S} + (1-\lambda)\mathbf{I}) \\
&= \det\left(\begin{bmatrix} \hbar\,\mathbf{B} + (1-\lambda)\mathbf{I} & \hbar\,\mathbf{C} \\ \hbar\,\mathbf{C} & \hbar\,\mathbf{B} + (1-\lambda)\mathbf{I} \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \hbar(\mathbf{B} + \mathbf{C}) + (1-\lambda)\mathbf{I} & \hbar(\mathbf{B} + \mathbf{C}) + (1-\lambda)\mathbf{I} \\ \hbar\,\mathbf{C} & \hbar\,\mathbf{B} + (1-\lambda)\mathbf{I} \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \hbar(\mathbf{B} + \mathbf{C}) + (1-\lambda)\mathbf{I} & \mathbf{0} \\ \hbar\,\mathbf{C} & \hbar(\mathbf{B} - \mathbf{C}) + (1-\lambda)\mathbf{I} \end{bmatrix}\right) \\
&= \det([\mathbf{I} + \hbar(\mathbf{B} + \mathbf{C})] - \lambda\mathbf{I}) \cdot \det([\mathbf{I} + \hbar(\mathbf{B} - \mathbf{C})] - \lambda\mathbf{I}).
\end{aligned}$$

Thus eigenvalues of $(\mathbf{I} + \hbar\,\mathbf{S})$ are equal to the union of eigenvalues of matrices $(\mathbf{I} + \hbar(\mathbf{B} + \mathbf{C}))$ and $(\mathbf{I} + \hbar(\mathbf{B} + \mathbf{C}))$. This proves the theorem. □

In general, the condition $\rho(\mathbf{I} + \hbar\,\mathbf{S}) < 1$ and Theorem 3.4, provide us with a convenient way to adjust and control the convergence of the homotopy-series solution (13). To find the valid region of $\hbar$, we plot the curves of $\rho(\mathbf{I}+\hbar(\mathbf{B}+\mathbf{C})) \sim \hbar$ and $\rho(\mathbf{I} + \hbar(\mathbf{B} - \mathbf{C})) \sim \hbar$, and thus:

The valid region of $\hbar = \{\alpha \in \mathbb{R} \mid \quad \rho(\mathbf{I} + \alpha(\mathbf{B} + \mathbf{C})) < 1$ *and* $\rho(\mathbf{I} + \alpha(\mathbf{B} - \mathbf{C})) < 1\}$.

Therefore, when HAM is applied to a linear system of equations, the above criterion should be considered instead of the well-known h-curves.

Theorem 3.5. *If $\boldsymbol{S}$ is strictly diagonal dominant, $-1 \leq \hbar < 0$ and $\boldsymbol{D} = [d_{ij}]$, where*

$$d_{ij} = \begin{cases} 1, & i = j, \\ \frac{s_{ij}}{s_{ii}}, & i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, 2n,$$

then $\|\boldsymbol{I} + \hbar\,\boldsymbol{D}\|_\infty < 1$ *and consequently* $\rho(\boldsymbol{I} + \hbar\,\boldsymbol{D}) < 1$.

Proof. Let $\mathbf{F} = \mathbf{I} + \hbar\,\mathbf{D}$, then we can write

$$f_{ij} = \begin{cases} 1 + \hbar, & i = j, \\ \hbar\frac{s_{ij}}{s_{ii}}, & i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, 2n.$$

Since $\mathbf{S}$ is strictly diagonally dominant,

$$|s_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{2n} |s_{ij}|, \qquad i = 1, 2, \ldots, 2n,$$

or

$$1 - \sum_{\substack{j=1 \\ j\neq i}}^{2n} \frac{|s_{ij}|}{|s_{ii}|} > 0, \qquad i = 1, 2, \ldots, 2n.$$

Also, since $-1 \leq \hbar < 0$, $(1 + \hbar) \geq 0$ and $|\hbar| = -\hbar$.

Hence, for $i = 1, 2, \ldots, 2n$, we have

$$\sum_{j=1}^{2n} |f_{ij}| = |1 + \hbar| + |\hbar| \sum_{\substack{j=1 \\ j\neq i}}^{2n} \frac{|s_{ij}|}{|s_{ii}|} = 1 + \hbar(1 - \sum_{\substack{j=1 \\ j\neq i}}^{2n} \frac{|s_{ij}|}{|s_{ii}|}) < 1,$$

which implies

$$\|\mathbf{F}\|_\infty = \|\mathbf{I} + \hbar\,\mathbf{D}\|_\infty < 1.$$

Finally,

$$\rho(\mathbf{I} + \hbar\,\mathbf{D}) = \inf_{\|\cdot\|} ||\mathbf{I} + \hbar\,\mathbf{D}\,\|,$$

implies

$$\rho(\mathbf{I} + \hbar\,\mathbf{D}) < 1.$$

□

By Theorem 3.5, if the matrix $\mathbf{S}$ is strictly diagonal dominant and $-1 \leq \hbar < 0$, by changing the system $\mathbf{Sx} = \mathbf{y}$ to the system $\mathbf{Dx} = \mathbf{z}$, where

$$d_{ij} = \begin{cases} 1, & i = j, \\ \frac{s_{ij}}{s_{ii}}, & i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, 2n.$$

and $\mathbf{z} = \left(\underline{z_1}(r), \ldots, \underline{z_n}(r), \overline{z_1}(r), \ldots \overline{z_n}(r)\right)^t$ such that

$$\underline{z_i}(r) = \frac{\underline{y_i}(r)}{s_{ii}}, \qquad \overline{z_i} = -\frac{\overline{y_i}(r)}{s_{n+i,n+i}} \quad i = 1, 2, \ldots, n,$$

we can always get a convergent homotopy-series solution by HAM.

4. **Examples**

To illustrate the proposed method and show its ability, some examples are provided and presented here. The solutions obtained by the HAM are compared with results obtained by JM, GSM, SOR, ADM and HPM.

Example 4.1. Consider the 6×6 fuzzy system

$$\begin{cases} 9x_1 + 2x_2 - x_3 + x_4 + x_5 - 2x_6 = (1 + 35r, 65 - 29r), \\ -x_1 + 10x_2 + 2x_3 + x_4 - x_5 - x_6 = (3 + 47r, 85 - 35r), \\ x_1 + 3x_2 + 9x_3 - x_4 + x_5 + 2x_6 = (20 + 31r, 90 - 39r), \\ 2x_1 - x_2 + x_3 + 10x_4 - 2x_5 + 3x_6 = (23 + 22r, 95 - 50r), \\ x_1 + x_2 - x_3 + 2x_4 + 7x_5 - x_6 = (5 + 33r, 57 - 19r), \\ 3x_1 + 2x_2 + x_3 + x_4 - x_5 + 10x_6 = (25 + 27r, 81 - 29r), \end{cases} \tag{15}$$

with the strong fuzzy solution

$$\begin{aligned} x_1 &= (\underline{x_1}(r), \overline{x_1}(r)) = (1 + 2r, 5 - 2r), \\ x_2 &= (\underline{x_2}(r), \overline{x_2}(r)) = (1 + 4r, 7 - 2r), \\ x_3 &= (\underline{x_3}(r), \overline{x_3}(r)) = (2 + r, 6 - 3r), \\ x_4 &= (\underline{x_4}(r), \overline{x_4}(r)) = (3 + r, 7 - 3r), \\ x_5 &= (\underline{x_5}(r), \overline{x_5}(r)) = (1 + 3r, 5 - r), \\ x_6 &= (\underline{x_6}(r), \overline{x_6}(r)) = (2 + r, 4 - r). \end{aligned}$$

The extended 12×12 crisp linear system is

$$\mathbf{S}\mathbf{x} = \mathbf{y},$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} \underline{\mathbf{y}} \\ -\overline{\mathbf{y}} \end{bmatrix},$$

such that

$$\mathbf{B} = \begin{bmatrix} 9 & 2 & 0 & 1 & 1 & 0 \\ 0 & 10 & 2 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 2 \\ 2 & 0 & 1 & 10 & 0 & 3 \\ 1 & 1 & 0 & 2 & 7 & 0 \\ 3 & 2 & 1 & 1 & 0 & 10 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{x_1} \\ \underline{x_2} \\ \underline{x_3} \\ \underline{x_4} \\ \underline{x_5} \\ \underline{x_6} \end{bmatrix}, \qquad \overline{\mathbf{x}} = \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \\ \overline{x_4} \\ \overline{x_5} \\ \overline{x_6} \end{bmatrix}, \qquad \underline{\mathbf{y}} = \begin{bmatrix} 1 + 35r \\ 3 + 47r \\ 20 + 31r \\ 23 + 22r \\ 5 + 33r \\ 25 + 27r \end{bmatrix}, \qquad \overline{\mathbf{y}} = \begin{bmatrix} 65 - 29r \\ 85 - 35r \\ 90 - 39r \\ 95 - 50r \\ 57 - 19r \\ 81 - 29r \end{bmatrix}.$$

Since the coefficient matrix $\mathbf{S}$ is strictly diagonally dominant, we write the new system as follows

$$\mathbf{D}\mathbf{x} = \mathbf{z},$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{N} & \mathbf{M} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} \underline{\mathbf{z}} \\ -\overline{\mathbf{z}} \end{bmatrix},$$

such that

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{10} & 0 & 0 \\ \frac{1}{9} & \frac{1}{3} & 1 & 0 & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{5} & 0 & \frac{1}{10} & 1 & 0 & \frac{3}{10} \\ \frac{1}{7} & \frac{1}{7} & 0 & \frac{2}{7} & 1 & 0 \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{10} & \frac{1}{10} & 0 & 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 & 0 & \frac{1}{9} & 0 & 0 & \frac{2}{9} \\ \frac{1}{10} & 0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} \\ 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & \frac{1}{7} \\ 0 & 0 & 0 & 0 & \frac{1}{10} & 0 \end{bmatrix},$$

and

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{x_1} \\ \underline{x_2} \\ \underline{x_3} \\ \underline{x_4} \\ \underline{x_5} \\ \underline{x_6} \end{bmatrix}, \ \overline{\mathbf{x}} = \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \\ \overline{x_4} \\ \overline{x_5} \\ \overline{x_6} \end{bmatrix}, \qquad \underline{\mathbf{z}} = \begin{bmatrix} \text{(1+35r)/9} \\ \text{(3+47r)/10} \\ \text{(20+31r)/9} \\ \text{(23+22r)/10} \\ \text{(5+33r)/7} \\ \text{(25+27r)/10} \end{bmatrix}, \ \overline{\mathbf{z}} = \begin{bmatrix} \text{(65-29r)/9} \\ \text{(85-35r)/10} \\ \text{(90-39r)/9} \\ \text{(95-50r)/10} \\ \text{(57-19r)/7} \\ \text{(81-29r)/10} \end{bmatrix}.$$

To find the valid region of $\hbar$, we plot the curves of $\rho(\mathbf{I} + \hbar(\mathbf{M} + \mathbf{N})) \sim \hbar$ and $\rho(\mathbf{I} + \hbar(\mathbf{M} - \mathbf{N})) \sim \hbar$. Figure 1 shows that if $\hbar$ is about in area $[-1.1, 0)$ the results are convergent. In Table 1, we present the solutions obtained by HAM for different values of $\hbar$ by 8th-order approximate solution. We know that when $\hbar = -1$, the solutions obtained by HAM is exactly the same as the HPM solutions. Also, the solutions obtained by the ADM, JM, GSM and SOR with 8 iterates are listed in Table 2. The number of iterations required for $\|E\|_\infty < 5 \times 10^{-10}$ are presented in Table 3, where $E = (E_1, E_2, \ldots, E_6)^t$ and E_i $(i = 1, 2, \ldots 6)$ is computed by equation (3). The results reveal that the HAM is very effective and simple.

	$\hbar = -1$ (HPM)	$\hbar = -0.8$	$\hbar = -0.7$	$\hbar = -0.6$
$\underline{x_1}(r)$	$0.77 + 2.23r$	$1.00 + 2.00r$	$1.00 + 2.00r$	$0.99 + 2.00r$
$\underline{x_2}(r)$	$0.81 + 4.19r$	$1.00 + 4.00r$	$1.00 + 4.00r$	$1.01 + 4.00r$
$\underline{x_3}(r)$	$1.74 + 1.26r$	$2.00 + 1.00r$	$2.00 + 1.00r$	$1.98 + 1.00r$
$\underline{x_4}(r)$	$2.73 + 1.28r$	$3.00 + 1.00r$	$3.00 + 1.00r$	$3.01 + 0.99r$
$\underline{x_5}(r)$	$0.73 + 3.27r$	$2.00 + 3.00r$	$1.00 + 3.00r$	$0.99 + 3.01r$
$\underline{x_6}(r)$	$1.76 + 1.24r$	$2.00 + 1.00r$	$2.00 + 1.00r$	$2.00 + 1.00r$
$\overline{x_1}(r)$	$5.23 - 2.23r$	$5.00 - 2.00r$	$5.00 - 2.00r$	$4.99 - 2.00r$
$\overline{x_2}(r)$	$7.19 - 2.19r$	$7.00 - 2.00r$	$7.00 - 2.00r$	$7.01 - 2.00r$
$\overline{x_3}(r)$	$6.26 - 3.26r$	$6.00 - 3.00r$	$6.00 - 3.00r$	$5.99 - 3.00r$
$\overline{x_4}(r)$	$7.28 - 3.28r$	$7.00 - 3.01r$	$7.00 - 3.00r$	$7.00 - 3.00r$
$\overline{x_5}(r)$	$5.27 - 1.27r$	$5.00 - 1.00r$	$5.00 - 1.00r$	$5.00 - 1.00r$
$\overline{x_6}(r)$	$4.24 - 1.24r$	$4.00 - 1.00r$	$4.00 - 1.00r$	$4.00 - 1.01r$

TABLE 1. The Solutions Obtained by HAM for Various $\hbar$ by 8th-Order Approximate Solution

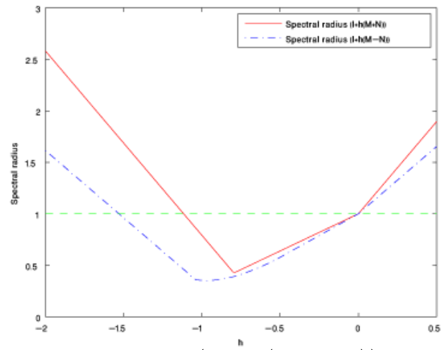

FIGURE 1. The Curves of $\rho(\mathbf{I}+\hbar(\mathbf{M}\pm\mathbf{N}))\sim\hbar$ for Example 1

	ADM	JM	GSM	SOR(ω= 0.85)
$\underline{x_1}(r)$	$0.77+2.23r$	$1.30+1.70r$	$1.00+2.00r$	$1.00+2.00r$
$\underline{x_2}(r)$	$0.81+4.19r$	$1.25+3.75r$	$1.00+4.00r$	$1.00+4.00r$
$\underline{x_3}(r)$	$1.74+1.26r$	$2.33+0.67r$	$2.00+1.00r$	$2.00+1.00r$
$\underline{x_4}(r)$	$2.73+1.28r$	$3.35+0.65r$	$3.00+1.00r$	$3.00+1.00r$
$\underline{x_5}(r)$	$0.73+3.27r$	$1.34+2.66r$	$1.00+3.00r$	$1.00+3.00r$
$\underline{x_6}(r)$	$1.76+1.24r$	$2.30+0.70r$	$2.00+1.00r$	$2.00+1.00r$
$\overline{x_1}(r)$	$5.23-2.23r$	$4.70-1.702r$	$5.00-2.00r$	$5.00-2.00r$
$\overline{x_2}(r)$	$7.19-2.19r$	$6.75-1.75r$	$7.00-2.00r$	$7.00-2.00r$
$\overline{x_3}(r)$	$6.26-3.26r$	$5.67-2.67r$	$6.00-3.00r$	$6.00-3.00r$
$\overline{x_4}(r)$	$7.28-3.28r$	$6.65-2.65r$	$7.00-3.00r$	$7.00-3.00r$
$\overline{x_5}(r)$	$5.27-1.27r$	$4.66-0.66r$	$5.00-1.00r$	$5.00-1.00r$
$\overline{x_6}(r)$	$4.24-1.24r$	$3.97-0.70r$	$4.00-1.00r$	$4.00-1.00r$

TABLE 2. The Solutions Obtained by the ADM, JM, GSM, and SOR with 8 Iterations

Example 4.2. Consider the 5×5 fuzzy system

$$\begin{cases} 8x_1+5x_2-x_3-2x_4+x_5=(23+17r,57-17r), \\ x_1+4x_2-x_3+4x_4+2x_5=(34+12r,58-12r), \\ -x_1-3x_2+6x_3+6x_4+2x_5=(25+18r,61-18r), \\ 7x_1-x_2+x_3+5x_4-x_5=(15+15r,45-15r), \\ 2x_1-3x_2+3x_3+x_4+5x_5=(6+14r,34-14r), \end{cases} \tag{16}$$

with the strong fuzzy solution

$$x_1=(\underline{x_1}(r),\overline{x_1}(r))=(1+r,3-r), \qquad x_2=(\underline{x_2}(r),\overline{x_2}(r))=(6+r,8-r),$$
$$x_3=(\underline{x_3}(r),\overline{x_3}(r))=(5+r,7-r), \qquad x_4=(\underline{x_4}(r),\overline{x_4}(r))=(3+r,5-r),$$
$$x_5=(\underline{x_5}(r),\overline{x_5}(r))=(2+r,4-r).$$

The extended 10×10 crisp linear system is

$$\mathbf{S}\,\mathbf{x}=\mathbf{y},$$

Method	Iterations required
ADM	91
JM	92
GSM	15
SOR ($\omega = 0.85$)	15
HAM ($\hbar = -1$) or HPM	91
HAM ($\hbar = -0.8$)	25
HAM ($\hbar = -0.7$)	28
HAM ($\hbar = -0.6$)	35

TABLE 3. Number of Iterations Required for $\|E\|_\infty < 5 \times 10^{-10}$

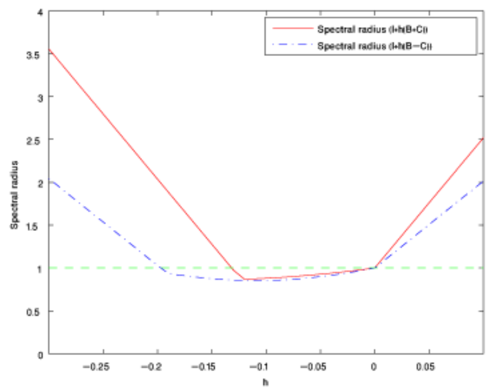

FIGURE 2. The Curves of $\rho(\mathbf{I} + \hbar(\mathbf{B} \pm \mathbf{C})) \sim \hbar$ for Example 2

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} \underline{\mathbf{y}} \\ -\overline{\mathbf{y}} \end{bmatrix},$$

such that

$$\mathbf{B} = \begin{bmatrix} 8 & 5 & 0 & 0 & 1 \\ 1 & 4 & 0 & 4 & 2 \\ 0 & 0 & 6 & 6 & 2 \\ 7 & 0 & 1 & 5 & 0 \\ 2 & 0 & 3 & 1 & 5 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{x_1} \\ \underline{x_2} \\ \underline{x_3} \\ \underline{x_4} \\ \underline{x_5} \end{bmatrix}, \qquad \overline{\mathbf{x}} = \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \\ \overline{x_4} \\ \overline{x_5} \end{bmatrix}, \qquad \underline{\mathbf{y}} = \begin{bmatrix} 23 + 17r \\ 34 + 12r \\ 25 + 18r \\ 15 + 15r \\ 6 + 14r \end{bmatrix}, \qquad \overline{\mathbf{y}} = \begin{bmatrix} 57 - 17r \\ 58 - 12r \\ 61 - 18r \\ 45 - 15r \\ 34 - 14r \end{bmatrix}.$$

In this example, the matrix $\mathbf{S}$ is not strictly diagonally dominant. To choose an appropriate range for $\hbar$ which ensures the convergence of the solution series, we plot the curves of $\rho(\mathbf{I} + \hbar(\mathbf{B} + \mathbf{C})) \sim \hbar$ and $\rho(\mathbf{I} + \hbar(\mathbf{B} - \mathbf{C})) \sim \hbar$. From the Figure 2, we could find that if $\hbar$ is about in area $[-0.13, 0)$ the result is convergent.

	$\hbar = -0.1$	$\hbar = -0.09$	$\hbar = -0.08$
$\underline{x_1}(r)$	$1.001 + 1.000r$	$0.999 + 1.000r$	$1.001 - 1.001r$
$\underline{x_2}(r)$	$5.998 + 1.000r$	$6.002 + 1.001r$	$5.998 + 0.999r$
$\underline{x_3}(r)$	$4.998 + 1.000r$	$5.001 + 1.001r$	$4.998 + 1.000r$
$\underline{x_4}(r)$	$3.000 + 1.000r$	$3.001 + 1.000r$	$2.999 + 0.999r$
$\underline{x_5}(r)$	$2.000 + 1.000r$	$2.001 + 1.000r$	$1.999 + 1.001r$
$\overline{x_1}(r)$	$3.001 - 1.000r$	$2.999 - 1.000r$	$3.002 - 1.001r$
$\overline{x_2}(r)$	$7.999 - 1.000r$	$8.003 - 1.001r$	$7.997 - 0.999r$
$\overline{x_3}(r)$	$6.998 - 1.000r$	$7.003 - 1.001r$	$6.998 - 1.000r$
$\overline{x_4}(r)$	$5.001 - 1.000r$	$5.000 - 1.000r$	$4.998 - 0.999r$
$\overline{x_5}(r)$	$4.000 - 1.000r$	$4.000 - 1.000r$	$4.001 - 1.001r$

Table 4. The Solutions Obtained by HAM for Various $\hbar$ by 50th-Order Approximate Solution

Method	25 iterations	50 iterations	75 iterations	100 iterations
HAM ($\hbar = -0.1$)	9.83e-02	2.02e-03	4.07e-05	1.11e-06
HAM ($\hbar = -0.09$)	8.63e-02	2.36e-03	4.52e-05	1.69e-06
HAM ($\hbar = -0.08$)	1.13e-01	2.80e-03	7.96e-05	3.57e-06
HPM	3.90e+28	2.46e+57	1.55e+86	9.79e+114
ADM	4.65e+06	4.22e+12	3.83e+18	3.47e+24
JM	2.69e+06	2.44e+12	2.21e+18	2.01e+24
GSM	8.65e+00	8.18e+00	8.25e+00	8.24e+00
SOR ($\omega = 0.85$)	8.59e+00	8.19e+00	8.25e+00	8.24e+00

Table 5. The Values of $\|E\|_\infty$ for Different Methods in the Various Iterations

In Table 4, we present the solutions obtained by HAM for different values of $\hbar$ by 50th-order approximate solution. Also, The values of $\|E\|_\infty$ for different methods in the various iterations are presented in Table 5, where $E = (E_1, E_2, \ldots, E_5)^t$ and E_i $(i = 1, 2, \ldots, 5)$ is computed by equation (3). In this example, the results reveal that only the HAM is convergent and other methods are divergent. We know that when $\hbar = -1$ the expression (13) is the same as the solution series obtained by HPM. In this case, it is clear that this value of $\hbar$ is out of range. So, $\hbar = -1$ is not a valid value to ensure the convergence of the solution series, which as shown in Table 5 explains why the HPM solution is divergent. Briefly speaking, HAM can provide us with a convenient way to adjust and control the convergence region and the rate of homotopy-series solutions by introducing convergence-control parameter $\hbar$.

5. Conclusion

In this paper, the HAM was successfully applied to obtain the approximate solution of the fuzzy linear system. The necessary and sufficient conditions for the convergence of series solution obtained via the HAM were presented. Also, a new criterion for choosing a proper value of convergence-control parameter $\hbar$ was suggested. We illustrated the advantages of the HAM with respect to the JM, GSM, SOR, ADM and HPM. It is explicitly showed that one may get divergent results using the mentioned methods for some examples, while, the HAM provides a convenient way to control the convergence of approximation series. This is the fundamental qualitative difference in analysis between HAM and other methods. The results reveals the validity and the great potential of HAM in solving a fuzzy linear system.

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References

[1] S. Abbasbandy, *The application of homotopy analysis method to nonlinear equations arising in heat transfer*, Phys. Lett. A, **360** (2006), 109–113.

[2] T. Allahviranloo, *Numerical methods for fuzzy system of linear equations*, Applied Mathematics and Computation, **155** (2004), 493–502.

[3] T. Allahviranloo, *Successive over relaxation iterative method for fuzzy system of linear equations*, Applied Mathematics and Computation, **162** (2005), 189–196.

[4] T. Allahviranloo, *The Adomian decomposition method for fuzzy system of linear equations*, Applied Mathematics and Computation, **163** (2005), 553–563.

[5] T. Allahviranloo and M. Ghanbari, *Solving fuzzy linear systems by homotopy perturbation method*, International Journal of Computational Cognition, **8(2)** (2010), 24–30.

[6] T. Allahviranloo, M. Ghanbari, E. Haghi, A. Hosseinzadeh and R. Nouraei, *A note on "Fuzzy linear systems"*, Fuzzy Sets and Systems, **177(1)** (2011), 87–92.

[7] T. Allahviranloo and M. Ghanbari, *On the algebraic solution of fuzzy linear systems based on interval theory*, Applied Mathematical Modelling, **36** (2012), 5360–5379.

[8] T. Allahviranloo and M. Ghanbari, *A new approach to obtain algebraic solution of interval linear systems*, Soft Computing, **16** (2012), 121–133.

[9] T. Allahviranloo, E. Haghi and M. Ghanbari, *The nearest symmetric fuzzy solution for a symmetric fuzzy linear system*, An. St. Univ. Ovidius Constanta, **20(1)** (2012), 151–172.

[10] T. Allahviranloo and S. Salahshour, *Fuzzy symmetric solution of fuzzy linear systems*, Journal of Computational and Applied Mathematics, **235(16)** (2011), 4545–4553.

[11] T. Allahviranloo, R. Nuraei, M. Ghanbari, E. Haghi and A. A. Hosseinzadeh, *A new metric for L-R fuzzy numbers and its application in fuzzy linear systems*, Soft Computing, **16** (2012), 1743-1754.

[12] A. S. Bataineh, M. S. M. Noorani and I. Hashim, *Solutions of time-dependent Emden-Fowler type equations by homotopy analysis method*, Phys. Lett. A, **371** (2007) 72–82.

[13] J. J. Buckley, *Solving fuzzy equations in economics and finance*, Fuzzy Sets and Systems, **48** (1992), 289–296.

[14] J. J. Buckley, *Solving fuzzy equations*, Fuzzy Sets and Systems, **50** (1992), 1–14.

[15] J. J. Buckley and Y. Qu, *Solving systems of linear fuzzy equations*, Fuzzy Sets and Systems,**43** (1991) 33–43.

[16] W. Cong-Xin and M. Ming, *Embedding problem of fuzzy number space: Part I*, Fuzzy Sets and Systems, **44** (1991), 33–38.

[17] M. Dehghan and B. Hashemi, *Iterative solution of fuzzy linear systems*, Applied Mathematics and Computation, **175** (2006), 645–674.
[18] M. Friedman, M. Ming and A. Kandel, *Fuzzy linear systems*, Fuzzy Sets and Systems, **96** (1998), 201–209.
[19] T. Hayat and M. Sajid, *On analytic solution for thin film flow of a fourth grade fluid down a vertical cylinder*, Phys. Lett. A, **361** (2007), 316–322.
[20] J. H. He, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Eng., **178** (1999), 257–262.
[21] J. H. He, *Homotopy perturbation method: a new non-linear analytical technique*, Applied Mathematics and Computation, **135(1)** (2003), 73–79.
[22] J. H. He, *Application of homotopy perturbation method to non-linear wave equations*, Chaos, Solitons and Fractals, **26(3)** (2005), 695–700.
[23] H. Jafari, M. Saeidy and J. Vahidi, *The Homotopy analysis method for solving fuzzy system of linear equations*, International Journal of Fuzzy Systems, **11(4)** (2009), 208–313.
[24] B. Keramati, *An approach to the solution of linear system of equations by He's homotopy perturbation method*, Chaos, Solitons and Fractals, **41** (2009), 152–156.
[25] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PhD thesis, Shanghai Jiao Tong University, 1992.
[26] S. J. Liao, *An explicit totally analytic approximation of Blasius viscous flow problems*, Int. J. Non-Linear Mech., **34(4)** (1999), 759–78.
[27] S. J. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, Boca Raton: Chapman and Hall, CRC Press, 2003.
[28] S. J. Liao, *A general approach to obtain series solutions of nonlinear differential equations*, Stud. Appl. Math., **119** (2007), 297–355.
[29] S. J. Liao, *Notes on the homotopy analysis method: Some definitions and theorems*, Commun. Nonlinear Sci. Numer. Simulat., **14** (2009), 983–997.
[30] R. Nuraei, T. Allahviranloo and M. Ghanbari, *Finding an inner estimation of the solution set of a fuzzy linear system*, Applied Mathematical Modelling, **37** (2013), 5148–5161.
[31] Z. Odibat, *A study on the convergence of homotopy analysis method*, Applied Mathematics and Computation, **217(2)** (2010), 782–789.
[32] M. Sajid and T. Hayat, *Comparison of HAM and HPM methods in nonlinear heat conduction and convection equations*, Nonlinear Anal. (B), **9** (2008), 2290–2295.
[33] L. Song and H. Zhang, *Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation*, Phys. Lett. A, **367** (2007), 88–94.
[34] R. A Van Gorder and K. Vajravelu, *Analytic and numerical solutions to the Lane-Emden equation*, Phys. Lett. A, **372** (2008), 6060–6065.
[35] R. A. Van Gorder and K. Vajravelu, *On the selection of auxiliary functions, operators, and convergence control parameters in the application of the Homotopy Analysis Method to nonlinear differential equations: a general approach*, Commun. Nonlinear Sci. Numer. Simulat., **14** (2009), 4078–4089.

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