HÖLDER SUMMABILITY METHOD OF FUZZY NUMBERS AND A TAUBERIAN THEOREM

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Abstract. In this paper we establish a Tauberian condition under which convergence follows from Hölder summability of sequences of fuzzy numbers.

1. Introduction

The concept of the fuzzy set was introduced by Zadeh [22]. Matloka [7] introduced the concepts of bounded and convergent sequences of fuzzy numbers and showed that every convergent sequence is bounded. Nanda [8] studied the spaces of bounded and convergent sequences of fuzzy numbers and proved that every Cauchy sequence of fuzzy numbers is convergent. Furthermore, sequences of fuzzy numbers have been discussed in [5, 1, 3, 4] and many others.

Recently different classes of sequences of fuzzy real numbers have been investigated by Tripathy and Baruah [13, 14] , Tripathy and Sarma [20, 21], Tripathy and Borgohain [16, 17], Tripathy and Dutta [18, 19], Tripathy et. al [15] and many others. A number of authors including Subrahmanyam [10] and Talo and Cakan [12] have established Tauberian theorems for some summability methods for sequences of fuzzy numbers. Subrahmanyam [10] introduced Cesàro convergence of a sequence of fuzzy numbers and obtained a Tauberian theorem for Cesàro summability for the sequences of fuzzy numbers. Tripathy and Baruah [14] defined Nörlund and Riesz means of fuzzy real numbers and obtained an analogue of a Tauberian theorem for Riesz summability method for the sequences of fuzzy numbers. Talo and Cakan [12] obtained necessary and sufficient Tauberian conditions, under which convergence follows from Cesàro convergence of sequences of fuzzy numbers. Altın et. al [2] studied the concept of statistical summability $(C, 1)$ for sequences of fuzzy real numbers and obtained a Tauberian theorem.

In this paper, we introduce Hölder summability and obtain a Tauberian condition under which convergence follows from Hölder summability of sequences of fuzzy numbers.

2. Definitons and Notations

Let D denote the set of all closed and bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line $\mathbb{R}$.

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For $A, B \in D$, we define

$$d(A,B) = \max(|\underline{A}, \underline{B}|, |\overline{A}, \overline{B}|)$$

where $A = [\underline{A}, \overline{A}]$ and $B = [\underline{B}, \overline{B}]$. It is known that (D, d) is a complete metric space.

A fuzzy real number u is a fuzzy set on $\mathbb{R}$ and is a mapping $u : \mathbb{R} \to [0,1]$, associating each real number t with its grade of membership $u(t)$. If there exists $t_0 \in \mathbb{R}$, such that $u(t_0) = 1$, then the fuzzy real number u is called normal. A fuzzy real number u is called convex if $u(t) \geq u(s) \wedge u(r) = \min(u(s), u(r))$, where $s < t < r$. A fuzzy real number u is said to be upper semi-continuous if for each $\epsilon > 0$, $u^{-1}([0, a+\epsilon))$, for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$. Let E^1 denote the set of all fuzzy numbers which are upper semi-continuous and have a compact support.

For $0 < \alpha \leq 1$, the α-level set of a fuzzy number u denoted by u^α is defined by

$$u^\alpha = \{t \in \mathbb{R} : u(t) \geq \alpha\}.$$

u^0 is defined as the closure of the set $\{t \in \mathbb{R} : u(t) > 0\}$.

We define $D : E^1 \times E^1 \to \mathbb{R}_+ \cup \{0\}$ by

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d(u^\alpha, v^\alpha).$$

It is straightforward to see that the mapping D is a metric on E^1 and (E^1, D) is a complete metric space.

The following properties [6] are needed in the sequel.

(i) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^1$

(ii) $D(ku, kv) = |k| D(u,v)$ for all $u, v \in E^1, k \in \mathbb{R}$

(iii) $D(u+v, w+z) \leq D(u,w) + D(v,z)$ for all $u, v, w, z \in E^1$

A sequence $u = (u_n)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ of all positive integer into E^1.

A sequence (u_n) of fuzzy numbers is said to be convergent to ℓ, written as $\lim_{n\to\infty} u_n = \ell$, if for every $\epsilon > 0$ there exists a positive integer N_0 such that

$$D(u_n, \ell) < \epsilon$$

for $n > N_0$.

A sequence (u_n) is said to be slowly oscillating if

$$\lim_{\lambda \to 1^+} \limsup_{n\to\infty} \max_{n<k\leq\lambda_n} D(u_k, u_n) = 0,$$

where λ_n denotes the integer part of λn. The concept of slow oscillation was introduced for the sequences of real numbers by Stanojević [9].

It is easy to see that every convergent sequence of fuzzy numbers is slowly oscillating.

Let (u_n) be a sequence of fuzzy numbers. The arithmetic means $\sigma_n^{(1)}$ of (u_n) are defined by $\sigma_n^{(1)} = \frac{1}{n+1}\sum_{j=0}^n u_j$ for all $n \in \mathbb{N} \cup \{0\}$. Repeating this averaging process, we define $\sigma_n^{(m)}$ for all $m \in \mathbb{N} \cup \{0\}$ by $\sigma_n^{(0)} = u_n$ and $\sigma_n^{(m+1)} = \frac{1}{n+1}\sum_{j=0}^n \sigma_j^{(m)}$.

A sequence (u_n) of fuzzy numbers is said to be summable by the Hölder's method (H,m) to ℓ, written as $\lim_{n\to\infty} u_n = \ell\,(H,m)$, if $D(\sigma_n^{(m)},\ell)\to 0$ as $n\to\infty$.

For any $m\in\mathbb{N}\cup\{0\}$, the identity

$$\sigma_n^{(m)} - \sigma_n^{(m+1)} = v_n^{(m)}, \tag{1}$$

where

$$v_n^{(m)} = \begin{cases} \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} v_j^{(m-1)} & ,m\geq 1 \\ \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} j(u_j-u_{j-1}) & ,m=0, \end{cases}$$

will be used in the sequel. The identity (1) for $m=0$ is known as the Kronecker identity.

By the expression of $\sigma_n^{(m+1)}$ in the form

$$\sigma_n^{(m+1)} = u_0 + \sum_{j=1}^{n} \frac{v_j^{(m)}}{j}, \tag{2}$$

we may rewrite (1) as

$$\sigma_n^{(m)} = v_n^{(m)} + \sum_{j=1}^{n} \frac{v_j^{(m)}}{j} + u_0.$$

The Hölder's method $(H,1)$ is the same as the Cesàro summability method. Since Cesàro summability method is regular ([10]), the Hölder's method (H,m) is also regular. Subrahmanyam [10] proved that every convergent sequence of fuzzy numbers is Cesàro summable and constructed a sequence of fuzzy numbers showing that the converse of this implication does not hold. The converse may be true under some additional conditions which are called Tauberian conditions. A theorem which states that convergence of a sequence of fuzzy numbers follows from a summability method and some Tauberian condition(s) is called a Tauberian theorem.

3. **Preliminary Results**

We need the following Lemmas for the proof of our result.

Lemma 3.1. *If (u_n) is slowly oscillating, then $D(v_n^{(0)},\overline{0}) = O(1)$ for all $n\in\mathbb{N}\cup\{0\}$, where $v_n^{(0)} = \frac{1}{n+1}\sum_{j=0}^{n} j(u_j-u_{j-1})$.*

Proof. Define $\rho(n,\lambda) = \max_{n<k\leq\lambda_n} D(u_k,u_n)$ for $\lambda>1$. Represent the finite sum $\sum_{j=1}^{n} j(u_j-u_{j-1})$ as the series $\sum_{k=0}^{\infty}\sum_{\frac{n}{2^{k+1}}\leq j\leq\frac{n}{2^k}} j(u_j-u_{j-1})$. Assume that $w(\lambda) := \limsup_{n\to\infty}\rho(n,\lambda)\to 0, \lambda\to 1^+$. If $w(\lambda)$ exists for $\lambda>1$, then $w(2)$ exists. By summation by parts, we have

$$\begin{aligned} D\left(\sum_{j=n_1}^{n_2} j(u_j-u_{j-1}),\overline{0}\right) &\leq \sum_{j=n_1}^{n_2-1} D(u_j,u_{n_2}) + n_2 D(u_{n_2},u_{n_1-1}) \\ &\leq (n_2-n_1)\rho(n_1,\frac{n_2}{n_1}) + n_1\rho(n_1,\frac{n_2}{n_1}) \\ &= n_2\rho(n_1,\frac{n_2}{n_1}). \end{aligned}$$

For $n_2 = [\frac{n}{2^k}]$ and $\frac{n_2}{n_1} \leq 2$, we obtain

$$D\left(\sum_{0}^{n} j(u_j - u_{j-1}), \overline{0}\right) \leq C \sum_{k=0}^{\infty} \frac{n}{2^k} = O(n), n \to \infty$$

for some positive constant C. By the last inequality, we have $D(v_n^{(0)}, \overline{0}) = O(1)$ for all $n \in \mathbb{N} \cup \{0\}$. □

The proof of the Lemma 3.1 was given by Szász [11] in his lecture notes for the sequences of real numbers. For the completeness of this paper we give the proof of the Lemma 3.1 for the sequences of fuzzy numbers by the minor changes in the proof of Szász.

Lemma 3.2. *If (u_n) is slowly oscillating, then the sequences $(\sigma_n^{(1)})$ and $(v_n^{(0)})$ are slowly oscillating.*

Proof. By Lemma 3.1, we have $D(v_n^{(0)}, \overline{0}) \leq C$ for all $n \in \mathbb{N} \cup \{0\}$ and some positive constant C. Then for all $n < k$, we have

$$D(\sigma_k^{(1)}, \sigma_n^{(1)}) = D\left(\sum_{j=n+1}^{k} \frac{v_j}{j}, \overline{0}\right) \leq \sum_{j=n+1}^{k} \frac{D(v_j, \overline{0})}{j} \leq C \sum_{j=n+1}^{k} \frac{1}{j} \leq C \log\left(\frac{k+1}{n+1}\right).$$

Taking max of both sides of the inequality over k, we obtain

$$\max_{n<k\leq\lambda_n} D(\sigma_k^{(1)}, \sigma_n^{(1)}) \leq C \log\left(\frac{\lambda_n + 1}{n+1}\right).$$

Taking lim sup of both sides of the inequality above as $n \to \infty$, we have

$$\limsup_{n\to\infty} \max_{n<k\leq\lambda_n} D(\sigma_k^{(1)}, \sigma_n^{(1)}) \leq C \log \lambda.$$

Taking the limit of both sides as $\lambda \to 1^+$, we obtain

$$\lim_{n\to\infty} \limsup_{n\to\infty} \max_{n<k\leq\lambda_n} D(\sigma_k^{(1)}, \sigma_n^{(1)}) = 0,$$

which shows that $(\sigma_n^{(1)})$ is slowly oscillating.

It follows from the inequality

$$D(v_k^{(0)}, v_n^{(0)}) = D(u_k - \sigma_k^{(1)}, u_n - \sigma_n^{(1)}) \leq D(u_k, u_n) + D(\sigma_k^{(1)}, \sigma_n^{(1)})$$

that $(v_n^{(0)})$ is slowly oscillating. □

4. The Main Result

We prove that a sequence (u_n) of fuzzy numbers summable by the Hölder's method (H, m) is convergent if it is slowly oscillating.

Theorem 4.1. *If $\lim_{n\to\infty} u_n = \ell\,(H, m)$ for some positive integer m and (u_n) is slowly oscillating, then $\lim_{n\to\infty} u_n = \ell$.*

Proof. Let $\lim u_n = \ell\,(H, m)$. Then, we have $\lim u_n = \ell\,(H, m{+}1)$ and $\lim v_n^{(m)} = \overline{0}$. By definition, $v_n^{(m)} = \sigma_n^{(1)}(v^{(m-1)})$. Since (u_n) is slowly oscillating, $(v_n^{(m-1)})$ is slowly oscillating by Lemmas 3.1 and 3.2. We now prove that $v_n^{(m-1)} \to \overline{0}$ as $n \to \infty$. By the triangle inequality, we have

$$D(v_n^{(m-1)}, \overline{0}) \leq D(v_n^{(m-1)}, \tau_n^{(m-1)}) + D(\tau_n^{(m-1)}, v_n^{(m)}) + D(v_n^{(m)}, \overline{0}), \tag{3}$$

where $\tau_n^{(m-1)} = \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} v_k^{(m-1)}$.

For the first term on the right-hand side of the inequality (3), we have

$$\begin{aligned}
D(v_n^{(m-1)}, \tau_n^{(m-1)}) &\leq D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} v_n^{(m-1)}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} v_k^{(m-1)}\right) \\
&= \frac{1}{\lambda_n - n} D\left(\sum_{k=n+1}^{\lambda_n} v_n^{(m-1)}, \sum_{k=n+1}^{\lambda_n} v_k^{(m-1)}\right) \\
&\leq \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} D(v_n^{(m-1)}, v_k^{(m-1)}) \\
&\leq \max_{n<k\leq\lambda_n} D(v_n^{m-1}, v_k^{m-1}).
\end{aligned}$$

For the second term on the right-hand side of the inequality (3), we have

$$\begin{aligned}
D(\tau_n^{(m-1)}, v_n^{(m)}) &= D\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} v_k^{(m-1)}, \frac{1}{n+1}\sum_{k=0}^{n} v_k^{(m-1)}\right) \\
&= D\left(\frac{1}{\lambda_n - n}\left(\sum_{k=n+1}^{\lambda_n} v_k^{(m-1)} + \sum_{k=0}^{n} v_k^{(m-1)}\right), \left(\frac{1}{n+1} + \frac{1}{\lambda_n - n}\right)\sum_{k=0}^{n} v_k^{(m-1)}\right) \\
&= D\left(\frac{1}{\lambda_n - n}\sum_{k=0}^{\lambda_n} v_k^{(m-1)}, \frac{\lambda_n + 1}{(\lambda_n - n)(n+1)}\sum_{k=0}^{n} v_k^{(m-1)}\right) \\
&= D\left(\frac{\lambda_n + 1}{\lambda_n - n} v_{\lambda_n}^m, \frac{\lambda_n + 1}{\lambda_n - n} v_n^m\right) \\
&= \frac{\lambda_n + 1}{\lambda_n - n} D(v_{\lambda_n}^m, v_n^{(m)}).
\end{aligned}$$

Since

$$\frac{\lambda_n + 1}{\lambda_n - n} < \frac{2\lambda}{\lambda - 1} \tag{4}$$

for all $\lambda > 1$ and sufficiently large n,

$$D(\tau_n^{(m-1)}, v_n^{(m)}) \leq \frac{2\lambda}{\lambda - 1} D(v_{\lambda_n}^m, v_n^{(m)}). \tag{5}$$

From the calculations for the first and second terms of the right-hand side of the inequality (3), we have

$$D(v_n^{(m-1)}, \overline{0}) \leq \max_{n<k\leq\lambda_n} D(v_n^{m-1}, v_k^{m-1}) + \frac{2\lambda}{\lambda - 1} D(v_{\lambda_n}^m, v_n^{(m)}) + D(v_n^{(m)}, \overline{0}). \tag{6}$$

Note that since $\lim_{n\to\infty} v_n^{(m)} = \overline{0}$, the second and third terms on the right-hand side of the inequality of (6) vanishes as $n \to \infty$. Taking lim sup of both sides of the inequality (6) as $n \to \infty$, we conclude that

$$\limsup_{n\to\infty} D(v_n^{(m-1)}, \overline{0}) \leq \limsup_{n\to\infty} \max_{n<k\leq\lambda_n} D(v_n^{m-1}, v_k^{m-1}). \tag{7}$$

Taking the limit of both sides of the inequality (7) as $\lambda \to 1^+$, we obtain

$$\lim_{n\to\infty} D(v_n^{(m-1)}, \overline{0}) = 0. \tag{8}$$

It follows from the inequality

$$D(\sigma_n^{(m-1)}, \ell) = D(\sigma_n^{(m)} + v_n^{(m-1)}, \ell) \le D(\sigma_n^{(m)}, \ell) + D(v_n^{(m-1)}, \overline{0})$$

that $\lim_{n\to\infty} \sigma_n^{(m-1)} = \ell$. Continuing in this way, we obtain that $\lim_{n\to\infty} v_n^{(0)} = \overline{0}$ and $\lim_{n\to\infty} \sigma_n^{(1)} = \ell$. By the inequality

$$D(u_n, \ell) = D(\sigma_n^{(1)} + v_n^{(0)}, \ell) \le D(\sigma_n^{(1)}, \ell) + D(v_n^{(0)}, \overline{0})$$

we have $\lim_{n\to\infty} u_n = \ell$. □

Corollary 4.2. *If* $\lim u_n = \ell\,(H, m)$ *and* $D(u_n, u_{n-1}) = O(\frac{1}{n})$ *for all* $n \in \mathbb{N}$*, then* $\lim_{n\to\infty} u_n = \ell$.

Proof. Assume that $D(u_n, u_{n-1}) \le C$ for all $n \in \mathbb{N} \cup \{0\}$ and some positive constant C. Then for all $n < k$, we have

$$D(u_k, u_n) \le \sum_{j=n+1}^{k} D(u_j, u_{j-1}) \le C \sum_{j=n+1}^{k} \frac{1}{j} \le C \log\left(\frac{k+1}{n+1}\right). \tag{9}$$

It follows from the inequality (9) that

$$\lim_{n\to\infty} \limsup_{n\to\infty} \max_{n<k\le\lambda_n} D(u_k, u_n) = 0,$$

which shows that (u_n) is slowly oscillating. □

Since every convergent sequence of fuzzy numbers is slowly oscillating, we have the following immediate corollary.

Corollary 4.3. *If* $\lim u_n = \ell\,(H, 1)$ *and* $(v_n^{(0)})$ *is slowly oscillating, then* $\lim_{n\to\infty} u_n = \ell$.

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References

[1] Y. Altin, M. Et and M. Basarir, *On some generalized difference sequences of fuzzy numbers*, Kuwait J. Sci. Eng., **34(1A)** (2007), 1–14.

[2] Y. Altin, M. Mursaleen and H. Altinok, *Statistical summability* $(C, 1)$ *for sequences of fuzzy real numbers and a Tauberian theorem*, J. Intell. Fuzzy Syst., **21(6)** (2010), 379–384.

[3] H. Altinok, Y. Altin and M. Isik, *Statistical convergence and strong p-Cesàro summability of order* β *in sequences of fuzzy numbers*, Iranian Journal of Fuzzy Systems, **9(2)** 2012, 63–73.

[4] H. Altinok, R. Colak and Y. Altin, *On the class of* λ*-statistically convergent difference sequences of fuzzy numbers*, Soft Comput., **16(6)** (2012), 1029–1034.

[5] H. Altinok, R. Colak and M. Et, λ*-Difference sequence spaces of fuzzy numbers*, Fuzzy Sets and Syst., **160(21)** (2009), 3128–3139.

[6] D. Dubois and H. Prade, *Fuzzy numbers: an overview, analysis of fuzzy information*, Mathematical Logic, CRC Press, Boca, FL, **1** (1987), 3–39.

[7] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL, **28** (1986), 28–37.

[8] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets and Systems, **33(1)** (1989), 123–126.

[9] Č. V. Stanojević, *Analysis of divergence: control and management of divergent process*, Graduate Research Seminar Lecture Notes, edited by İ. Canak, University of Missouri - Rolla, Fall, 1998.

[10] P. V. Subrahmanyam, *Cesàro summability of fuzzy real numbers*, J. Anal., **7** (1999), 159–168.

[11] O. Szász, *Introduction to the theory of divergent series*, Revised ed. Department of Mathematics, Graduate School of Arts and Sciences, University of Cincinnati, Cincinnati, Ohio, 1952.

[12] O. Talo and C. Cakan, *On the Cesàro convergence of sequences of fuzzy numbers*, Appl. Math. Lett., **25** (2012), 676–681.

[13] B. C. Tripathy and A. Baruah, *New type of difference sequence spaces of fuzzy real numbers*, Math. Model. Anal., **14(3)** (2009), 391–397.

[14] B. C. Tripathy and A. Baruah, *Nörlund and Riesz mean of sequences of fuzzy real number*, Appl. Math. Lett., **23(5)** (2010), 651–655.

[15] B. C. Tripathy, A. Baruah, M. Et and M. Gungor, *On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers*, Iran. J. Sci. Technol., Trans. A, Sci., **36(2)** (2012), 147–155.

[16] B. C. Tripathy and S. Borgogain, *The sequence space* $m(M, \phi, \Delta_m^n, p)^F$, Math. Model. Anal., **13(4)** (2008), 577–586.

[17] B. C. Tripathy and S. Borgogain, *Some classes of difference sequence spaces of fuzzy real numbers defined by Orlicz function*, Adv. Fuzzy Syst., Article ID 216414, 6 pages, 2011.

[18] B. C. Tripathy and A. J. Dutta, *Bounded variation double sequence space of fuzzy real numbers*, Comput. Math. Appl., **59(2)** (2010), 1031–1037.

[19] B. C. Tripathy and A. J. Dutta, *On I-acceleration convergence of sequences of fuzzy real numbers*, Math. Model. Anal., **17(4)** (2012), 549–557.

[20] B. C. Tripathy and B. Sarma, *Sequence spaces of fuzzy real numbers defined by Orlicz functions*, Math. Slovaca, **58(5)** (2008), 621–628.

[21] B. C. Tripathy and B. Sarma, *On I-convergent double sequences of fuzzy real numbers*, Kyungpook Math. J., **52(2)** (2012), 189–200.

[22] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.

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