BI-MATRIX GAMES WITH INTUITIONISTIC FUZZY GOALS

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Abstract. In this paper, we present an application of intuitionistic fuzzy programming to a two person bi-matrix game (pair of payoffs matrices) for the solution with mixed strategies using linear membership and non-membership functions. We also introduce the intuitionistic fuzzy(IF) goal for a choice of a strategy in a payoff matrix in order to incorporate ambiguity of human judgements; a player wants to maximize his/her degree of attainment of the IF goal. It is shown that this solution is the optimal solution of a mathematical programming problem. Finally, we present a numerical example to illustrate the methodology.

1. **Introduction**

We study a two person non zero sum bi-matrix game with a single pair of payoff matrices which has interesting applications in the analysis of conflicting interests situations with players or decision makers (DM) who select their strategies from a set of available strategies. Fuzziness in bi-matrix game theory has been studied by various researchers [3, 17, 7] who viewed the goals as fuzzy sets, but they are very limited in scope and, in many cases, do not represent the real problem very well. In practice, due to insufficiency of the information available, it is not easy to describe the fuzzy constraint conditions by ordinary fuzzy sets and consequently, the evaluation of membership values up to DM's satisfaction is not always possible. Similarly, the evaluation of non- membership values is not always possible and consequently, there remains an indeterministic part of which survives. In such situations, the intuitionistic fuzzy sets (IFS), of Atanassov [1] serve our purpose better.

Intuitionistic fuzziness in bi-matrix games can appear in many ways, but two cases of fuzziness seem very natural: when the DMs have IF goals and when the elements of the payoff matrix are given by IF numbers. These two classes of fuzzy matrix games are referred to as matrix games with IF goals and matrix games with IF payoffs [11]. However, the solutions of bi- matrix games with IF goal have not yet been studied. We assume that each DM has an IF goal for the choice of the strategy and that the DMs want to maximize the degree of attainment of this goal.

This paper is organized as follows : In section 2, we give some basic definitions and notations on IFS. In sections 3 and 4, we discuss the solution procedure of the

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bi-matrix game on the basis of defining degree of attainment of the IF goal. Finally, in section 5, we give a numerical example.

2. Intuitionistic Fuzzy Sets

In this section, we first recall some basic preliminaries, notations and definitions of IFS [1]. An IFS in a given universal set U is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in U\}$$

where $\mu_A : U \to [0,1]$ is the lower bound on the degree of association of x with the IFSs A derived from the evidence of $u \in U$ and $\nu_A : U \to [0,1]$ is the lower bound on the negation of $u \in U$ derived from the evidence against $u \in U$. These functions satisfy the following conditions :

(i) μ_A and ν_A are restricted to $[0,1]$ and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in U.$$

(ii) $\exists$ a unique real number x, such that $\mu_A(x) = 1$ and $\nu_A(x) = 0$.

(iii) Both $\mu_A(x)$ and $\nu_A(x)$ are convex and continuous.

For all $x \in U$, the indeterministic part of $x, \prod_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of uncertainty of the IFS A; $0 \leq \prod_A(x) \leq 1$ for all $x \in U$. Thus the numbers $\mu_A(x)$ and $\nu_A(x)$ reflect the degrees of acceptance and rejection of the element x by the set A, respectively and the number $\prod_A(x)$ is the extent of the indeterminacy. The set of all IFS on U is denoted by $IFSs(U)$.

2.1. Operations on IFSs.

Let A and B be two IFSs of the set U, then their operations are defined by membership of their functions as follows:

(i) $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle | x \in U\}$

(ii) $A.B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle | x \in U\}$

(iii) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in U\}$

(iv) $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in U\}$

(v) $\overline{A} = \{< x, \nu_A(x), \mu_A(x) >: x \in U\}$.

The degree of acceptance $\mu_A(x)$ and of nonacceptance $\nu_A(x)$ can be arbitrary.

2.2. Inequality Relations on IFSs.

It is difficult to compare sets directly due to their intuitionistic nature. However, we can define a fuzzy preference or objective using the theory of IFS, introduced by Atanassov [1]. He suggested an order relation of IFS as follows:

For all $A, B \in IFSs(U)$, (i) $A \preceq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x); \forall x \in U$

(ii) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x); \forall x \in U$,

where '$\preceq$' is the IF version of '$\leq$' [1].

2.3. Intuitionistic Fuzzy Optimization Model.

Generally, an optimization problem includes objectives and constraints. Intuitionistic fuzzy optimization (IFO), a method of uncertainty optimization, is derived from the basis of IFS [1]. According to IFO theory [2], we must maximize the degree of acceptance and minimize the degree of rejection of the IF objectives and constraints as follows:

$$\begin{aligned}
&\max_x\{\mu_k(x)\}; && k=1,2,\ldots,n\\
&\min_x\{\nu_k(x)\}; && k=1,2,\ldots,n\\
&\mu_k(x),\nu_k(x)\geq 0; && k=1,2,\ldots,n\\
&\mu_k(x)\geq \nu_k(x); && k=1,2,\ldots,n\\
&0\leq \mu_k(x)+\nu_k(x)\leq 1; && k=1,2,\ldots,n
\end{aligned}$$

where $\mu_k(x)$ and $\nu_k(x)$ denote the degrees of acceptance and rejection of x from the k^{th} IFS, respectively. It is an extension of fuzzy optimization in which the degrees of rejection of objectives and constraints are considered together with the degrees of satisfaction. The above inequalities are equivalent to the following [14]:

$$\begin{aligned}
&\max\alpha,\quad \min\beta &&\\
&\alpha\leq \mu_k(x); && k=1,2,\ldots,n\\
&\beta\geq \nu_k(x); && k=1,2,\ldots,n\\
&\alpha\geq\beta; && \text{and } \alpha+\beta\leq 1;\alpha,\beta\geq 0
\end{aligned}$$

where α denotes the minimal acceptable degree of objectives and constraints and β denotes the maximal degree of rejection of objectives and constraints. The IFO model can be changed into the following certainty (non-fuzzy) optimization model [2, 14] as follows:

$$\begin{aligned}
&\max(\alpha-\beta) &&\\
&\alpha\leq \mu_k(x); && k=1,2,\ldots,n\\
&\beta\geq \nu_k(x); && k=1,2,\ldots,n\\
&\alpha\geq\beta; && \text{and } \alpha+\beta\leq 1;\alpha,\beta\geq 0
\end{aligned}$$

which can be solved easily by some simplex methods. Thus the IFO model gives a richer apparatus for formulation of optimization problems than the analogous fuzzy optimization problem; moreover, it makes the fuzzy process for objectives and constraints more meticulous.

3. **Bi-matrix Game**

A bi-matrix game, Nash [9], can be considered as a natural extension of the matrix game. Let I, II denote two players and let $M=\{1,2,...,m\}$ and $N=\{1,2,...,n\}$ be the sets of all pure strategies available for players I and II respectively. Let α_{ij} and γ_{ij} denote the payoffs that the player I and II receive when player I plays the pure strategy i and player II plays the pure strategy j. Then we have the following payoff matrix

$$A=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \cdots\alpha_{1n}\\ \alpha_{21} & \alpha_{22} & \cdots\alpha_{2n}\\ \cdots & \cdots & \cdots\\ \alpha_{m1} & \alpha_{m2} & \cdots\alpha_{mn}\end{pmatrix};B=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \cdots\gamma_{1n}\\ \gamma_{21} & \gamma_{22} & \cdots\gamma_{2n}\\ \cdots & \cdots & \\ \gamma_{m1} & \gamma_{m1} & \cdots\gamma_{mn}\end{pmatrix}$$

where we assume that the payoff for each of the player is a crisp, when each of them choose a pure strategy. We shall denote this game by $\Gamma=\langle\{I,II\},A,B\rangle$.

4. **Nash Equilibrium Solution**

Nash [9] defined the concept of Nash equilibrium solutions (NES) in bi-matrix games for single pair of payoff matrices and presented the methodology for obtaining these.

4.1. **Pure Strategy.** Let I, II denote two players and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all pure strategies available for players I, II respectively.

Definition 4.1. A pair of strategies $(row\ \ r, column\ \ s)$ is said to constitute a NES to a bi-matrix game Γ if the following inequalities are satisfied for all $i = 1, 2, ..., m$ and all $j = 1, 2, ..., n$:

$$\alpha_{is} \leq \alpha_{rs}; \quad \gamma_{rj} \leq \gamma_{rs}$$

Since the strategy sets are finite, these strategies may exist. In this case we say that the bi-matrix game admits a NES for pure strategies and the pair $(\alpha_{rs}, \gamma_{rs})$ is called a *Nash equilibrium outcome* of the bi-matrix game. A bi-matrix game can admit more than one NES, with the equilibrium outcomes being different in each case.

Example 4.2. Consider a 2×2 bi-matrix game whose payoff matrices are

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is clear that this game admits two Nash equilibriums $(row2, column1)$ and $(row1, column2)$. The corresponding equilibrium outcomes are $(2, 2)$ and $(0, 0)$ respectively.

4.2. **Mixed Strategy.** In the previous section, we have encountered only the cases in which a given bi- matrix game admits a one or more Nash equilibria. There are other cases, where no Nash equilibrium exists in pure strategies. The sets of all mixed strategies, called strategy spaces, available for players I, II are as follows:

$$S_I = \{(x_1, x_2, ..., x_m) \in \Re^m_+ : x_i \geq 0; i = 1, 2, ..., m \text{ and } \sum_{i=1}^{m} x_i = 1\}$$

$$S_{II} = \{(y_1, y_2, ..., y_n) \in \Re^n_+ : y_i \geq 0; i = 1, 2, ..., n \text{ and } \sum_{i=1}^{n} y_i = 1\}$$

where $\Re^m_+$ denotes the $m-$dimensional non negative Euclidean space. Since the player is uncertain about what strategy he/she will choose, he/she will choose a probability distribution over the set of alternatives available to him/her; i.e. a mixed strategy x in terms of game theory. We shall denote this bi-matrix game by $\Gamma_1 = \langle \{I, II\}, S_I \times S_{II}, A, B \rangle$. For $x \in S_I, y \in S_{II}$, $x^T Ay$ and $x^T By$ are the expected payoff to the player I and player II respectively.

Definition 4.3. If player I and player II use the mixed strategies x and y, respectively then their expected payoffs are respectively

$$x^T Ay = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \alpha_{ij} y_j \text{ and } x^T By = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \gamma_{ij} y_j.$$

Definition 4.4. A pair $(x^* \in S_I, y^* \in S_{II})$ is called a NES for the bi-matrix game Γ_1 if the following inequalities are satisfied:

$$x^T A y^* \leq x^{*T} A y^*; \ \ \forall x \in S_I$$

$$x^{*T} B y \leq x^{*T} B y^*; \ \ \forall y \in S_{II}.$$

x^* and y^* are also called the optimal strategies for the player I and player II respectively. The pair of numbers $V = \langle x^{*T} A y^*, x^{*T} B y^* \rangle$ is said to be the Nash equilibrium outcome of Γ_1 and the triplet (x^*, y^*, V) is called a solution of the bi-matrix game.

4.3. **Bi-matrix Game with IF Goal.** Here we discuss the computational procedure for NES with respect to the degree of attainment of the IF goal in single objective bi-matrix games. First, we define some terms useful in the solution procedure. Let the domain for players I and II be defined by $D_1 = \{x^T A y : (x, y) \in S_I \times S_{II} \subset \Re^m \times \Re^n\} \subseteq \Re$ and $D_2 = \{x^T B y : (x, y) \in S_I \times S_{II} \subset \Re^m \times \Re^n\} \subseteq \Re, respectively.$

Definition 4.5. (IF goal): An IF goal $\hat{G}_1$ for player I is defined as an IFS on D_1 characterized by the membership and nonmembership functions

$$\mu_{\hat{G}_1} : D_1 \to [0,1] \text{ and } \nu_{\hat{G}_1} : D_1 \to [0,1]$$
$$\text{or simply, } \mu_1 : D_1 \to [0,1] \text{ and } \nu_1 : D_1 \to [0,1]$$

such that $0 \leq \mu_1(x) + \nu_1(x) \leq 1$. Similarly, an IF goal for player II is an IFS on D_2 characterized by the membership function $\mu_{\hat{G}_2} : D_2 \to [0,1]$ and nonmembership function $\nu_{\hat{G}_2} : D_2 \to [0,1]$ such that $0 \leq \mu_{\hat{G}_2}(x) + \nu_{\hat{G}_2}(x) \leq 1$.

A membership, non-membership function value for an IF goal can be interpreted as the degree of attainment [17] of the IF goal for a strategy of a payoff. According to Atanassov's property [1] of IFS (seen in section 2.1 in this paper), the intersection of IF objectivesand constraints is defined as

$$C = A \cap B = \{\langle x, \mu_C(x), \nu_C(x) \rangle : x \in X\}$$

where $\mu_C(x) = \min[\mu_A(x), \mu_B(x)]$; and$\nu_C(x) = \max[\nu_A(x), \nu_B(x)]$.
A denotes integrated IF objective and B denotes the integrated IF constraint set. Also, $\mu_C(x)$ is the degree of acceptance of IF decision set and $\nu_C(x)$ is the degree of rejection of IF decision set.

Definition 4.6. (Degree of attainment of IF goal): For any pair of mixed strategies,$(x, y) \in S_I \times S_{II}$, let $\hat{G}_1(x, y)$ denotes the IF goal for player I. Then the degree of attainment of the IF goal is defined as

$$\max_x \{\mu_{\hat{G}_1}(x, y)\} \text{ and } \min_x \{\nu_{\hat{G}_1}(x, y)\}.$$

The degree of attainment of the IF goal can be considered to be a degree of satisfaction of the fuzzy decision [1, 18], when the constraint is replaced by expected payoff. When the player I and II choose strategies x and y respectively, the maximum degree of attainment of the IF goal is determined by the above relations.

Let the membership functions for the aggregated IF goal for player I and and the player II, be respectively $\mu_1(x,y)$ and $\mu_2(x,y)$ and the corresponding non-membership functions be $\nu_1(x,y)$ and $\nu_2(x,y)$. Thus, when a player has two different strategies, he/she prefers the strategy possessing the higher membership function value and lower nonmembership function value in comparison with the other.

Definition 4.7. A pair of mixed strategies $x^* \in S_I$ and $y^* \in S_{II}$ is said to be the NES of a bi-matrix game with respect to the degree of attainment of the aggregate IF goal, if for any other mixed strategies x and y, we have

$$\mu_1(x^*,y^*) \geq \mu_1(x,y^*) \text{ and } \nu_1(x^*,y^*) \leq \nu_1(x,y^*)$$
$$\mu_2(x^*,y^*) \geq \mu_2(x^*,y) \text{ and } \nu_2(x^*,y^*) \leq \nu_2(x^*,y)$$

Since the membership and non-membership functions are convex and continuous [6, 1], the solution with respect to the degree of attainment of the aggregated IF goal for Γ_1 always exists.

Let A and B be the payoff matrices of a single objective bi-matrix game Γ. If player I chooses the mixed strategy x, the player II chooses the mixed strategy y, the membership and non membership functions for player I and player II are $\mu_1(x,y), \nu_1(x,y)$ and $\mu_2(x,y), \nu_2(x,y)$ respectively, then their expected payoffs are x^TAy and x^TBy respectively and we have

$$\mu_1(x,y) = \mu_1(x^TAy); \quad \nu_1(x,y) = \nu_1(x^TAy)$$
$$\mu_2(x,y) = \mu_2(x^TBy); \quad \nu_2(x,y) = \nu_2(x^TBy).$$

Definition 4.8. (Equilibrium solution of IF bi-matrix game): A pair $(x^*,y^*) \in S_I \times S_{II}$ is called a NES of the IF matrix game, with respect to the degree of attainment of the IF goal in a single objective bi-matrix game if for all other mixed strategies x and y we have

$$\mu_1(x^{*T}Ay^*) \geq \mu_1(x^TAy^*) \text{ and } \nu_1(x^{*T}Ay^*) \leq \nu_1(x^TAy^*); \forall x \in S_I$$
$$\mu_2(x^{*T}By^*) \geq \mu_2(x^TBy^*) \text{ and } \nu_2(x^{*T}By^*) \leq \nu_2(x^TBy^*); \forall y \in S_{II},$$

where '$\geq$' and '$\leq$' are defined in section 2.2. $(x^{*T}Ay^*, x^{*T}By^*)$ is called the Nash equilibrium outcome of the bi-matrix game in mixed strategies.

In order to obtain the NES of the given matrix game with respect to the degree of attainment of the IF goal, we now analyze the optimization problems for player I and player II By using the above definitions, we construct the following IF linear programming problem (LPP) for player I and II for NES.

4.4. **Optimization Problem for Player I.** The linear membership and nonmembership functions [1] of the IF goal $\mu_1(x^TAy)$ and $\nu_1(x^TAy)$, for the player I can mathematically be represented as :

$$\mu_1(x^TAy) = \begin{cases} 1; & x^TAy \geq \overline{a} \\ \frac{x^TAy-\underline{a}}{\overline{a}-\underline{a}}; & \underline{a} < x^TAy < \overline{a} \\ 0; & x^TAy \leq \underline{a} \end{cases}$$

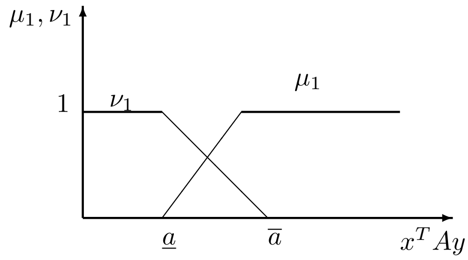

FIGURE 1

$$\nu_1(x^T Ay) = \begin{cases} 1; & x^T Ay \leq \underline{a} \\ \frac{x^T Ay - \overline{a}}{\underline{a} - \overline{a}}; & \underline{a} < x^T Ay < \overline{a} \\ 0; & x^T Ay \geq \overline{a} \end{cases}$$

where $\underline{a}$ and $\overline{a}$ are the tolerances of the expected payoff $x^T Ay$ and $\mu_1(x^T Ay)$ is in the objective allowable region $[\underline{a}, \overline{a}]$. For player I, $\underline{a}$ and $\overline{a}$ are the payoffs giving the worst and the best degree of satisfaction respectively. The parameters $\underline{a}$ and $\overline{a}$ may be any scalars with $\overline{a} > \underline{a}$. Nishizaki [17] has proposed the values

$$\overline{a} = \max_x \max_y x^T Ay = \max_i \max_j \alpha_{ij}$$

$$\underline{a} = \min_x \min_y x^T Ay = \min_i \min_j \alpha_{ij}.$$

The conditions $\underline{a} \leq \min\limits_{i,j} \alpha_{ij}$ and $\overline{a} \geq \max\limits_{i,j} \alpha_{ij}$ hold. Thus player I is not satisfied by a payoff less than $\underline{a}$ but is fully satisfied by a payoff greater than $\overline{a}$. Now,

$$\frac{x^T Ay - \underline{a}}{\overline{a} - \underline{a}} = (\frac{-\underline{a}}{\overline{a} - \underline{a}}) + x^T(\frac{A}{\overline{a} - \underline{a}})y = c_1^{(1)} + x^T \hat{A} y,$$

where $c_1^{(1)} = \frac{-\underline{a}}{\overline{a} - \underline{a}}$ and $\hat{A} = \frac{1}{\overline{a} - \underline{a}} A$. If we define $c_1^{(2)} = \frac{-\overline{a}}{\underline{a} - \overline{a}}$, then the membership and nonmembership functions can be written as

$$\mu_1(x^T Ay) = \begin{cases} 1 & x^T Ay; \geq \overline{a} \\ c_1^{(1)} + x^T \hat{A} y; & \underline{a} < x^T Ay < \overline{a} \\ 0; & x^T Ay \leq \overline{a} \end{cases}$$

$$\nu_1(x^T Ay) = \begin{cases} 1; & x^T Ay; \leq \overline{a} \\ c_1^{(2)} - x^T \hat{A} y; & \underline{a} < x^T Ay < \overline{a} \\ 0; & x^T Ay \geq \overline{a} \end{cases}$$

Since all the membership and non-membership functions of the IF goals are linear, the optimal solution is equal to the degree of attainment of the fuzzy goal for the matrix game. Since S_I and S_{II} are convex polytopes, for the choice of linear membership and non-membership functions, the existence of a Nash equilibrium of the game is guaranteed.

4.5. **Optimization problem for player II.** Here we consider the solition for player II with respect to the degree of attainment of his fuzzy goal. The membership and nonmembership functions of the IF goal $\mu_2(x^TBy)$ and $\nu_2(x^TBy)$ can be represented as :

$$\mu_2(x^TBy) = \begin{cases} 1; & x^TBy \le \underline{b} \\ \frac{x^TBy-\underline{b}}{\overline{b}-\underline{b}}; & \underline{b} < x^TBy < \overline{b} \\ 0; & x^TBy \ge \overline{b} \end{cases}$$

$$\nu_2(x^TBy) = \begin{cases} 1; & x^TBy \ge \overline{b} \\ \frac{x^TBy-\overline{b}}{\underline{b}-\overline{b}}; & \underline{b} < x^TBy < \overline{b} \\ 0; & x^TBy \le \overline{b} \end{cases}$$

where $\underline{b}$ and $\overline{b}$ are the tolerances of the expected payoff x^TBy, $\mu_2(x^TBy)$ is deter-

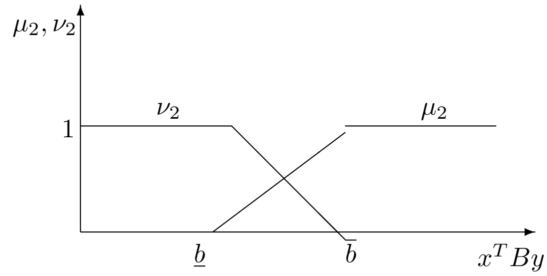

FIGURE 2

mined in the objective allowable region $[\underline{b}, \overline{b}]$ and $\underline{b} \le \min_{i,j} b_{ij}$, $\overline{b} \ge \max_{i,j} b_{ij}$. Also, the membership and nonmembership functions can be written as

$$\mu_2(x^TBy) = \begin{cases} 1; & x^TBy; \ge \overline{b} \\ c_2^{(1)} + x^T\hat{B}y; & \underline{b} < x^TBy < \overline{b} \\ 0; & x^TBy \le \overline{b} \end{cases}$$

$$\nu_2(x^TBy) = \begin{cases} 1; & x^TBy; \le \overline{b} \\ c_2^{(2)} - x^T\hat{B}y; & \underline{b} < x^TBy < \overline{b} \\ 0; & x^TBy \ge \overline{b} \end{cases}$$

where $\hat{B} = \frac{1}{\overline{b}-\underline{b}}B, c_2^{(1)} = \frac{-\underline{b}}{\overline{b}-\underline{b}}$ and $c_2^{(2)} = \frac{-\overline{b}}{\underline{b}-\overline{b}}$. Thus the NES is equal to the optimal solution of the following mathematical problem

$$P1 : \max_x \mu_1(x^T\hat{A}y^*) \text{ and } \min_x \nu_1(x^T\hat{A}y^*)$$
$$S.T. \sum_{i=1}^{m} x_i = 1; \quad x_i \ge 0; i = 1, 2, \ldots, m.$$
$$P2 : \max_y \mu_2(x^{*T}\hat{B}y) \text{ and } \min_y \nu_2(x^{*T}\hat{B}y)$$
$$S.T. \sum_{j=1}^{n} y_j = 1; \quad y_i \ge 0; i = 1, 2, \ldots, n.$$

where (x^*, y^*) are the optimal solution to $P1$ and $P2$ respectively. We see that the constraints are separable in the decision variables, x and y. Hence the two problems yield the following single objective mathematical programming problem [15]:

$$P3:\max_{x,y}\{\mu_1(x^T\hat{A}y^*)+\mu_2(x^{*T}\hat{B}y)\}+\min_{x,y}\{\nu_1(x^T\hat{A}y^*)+\nu_2(x^{*T}\hat{B}y)\}$$
$$S.T.\sum_{i=1}^{m}x_i=1 \text{ and } \sum_{j=1}^{n}y_j=1;\quad x_i\geq 0;\quad y_i\geq 0.$$

Theorem 4.9. *If all membership and non membership functions of the IF goals are linear, the NES with respect to the degree of attainment of the IF goal aggregated by a minimum component is equal to the optimal solution of the following non-linear programming problem:*

$$P4:\max_{x,y,p_1,p_2,q_1,q_2,\lambda_1,\lambda_2,\beta_1,\beta_2}\{\lambda_1+\lambda_2+p_2+q_2-\beta_1-\beta_2-p_1-q_1\} \tag{1}$$

subject to

$$\hat{A}y+c_1^{(1)}e^m\leq p_1e^m;\quad -\hat{A}y+c_1^{(2)}e^m\geq p_2e^m \tag{2}$$
$$\hat{B}x+c_2^{(1)}e^n\leq q_1e^n;\quad -\hat{B}x+c_2^{(2)}e^n\geq q_2e^n \tag{3}$$
$$x^T\hat{A}y+c_1^{(1)}\geq\lambda_1;\quad -x^T\hat{A}y+c_1^{(2)}\leq\beta_1 \tag{4}$$
$$x^T\hat{B}y+c_2^{(1)}\geq\lambda_2;\quad -x^T\hat{B}y+c_2^{(2)}\leq\beta_2 \tag{5}$$
$$\lambda_1+\beta_1\leq 1;\quad \lambda_2+\beta_2\leq 1;\quad \lambda_1\geq\beta_1;\quad \lambda_2\geq\beta_2 \tag{6}$$
$$x_1+x_2+x_3=1;\quad y_1+y_2+y_3=1 \tag{7}$$
$$x_i,y_i,\lambda_i,\beta_i,p_i,q_i\geq 0,$$

where $e^m=(1,1,\ldots,1)^T; e^n=(1,1,\ldots,1)^T.$

Proof. The Kuhn-Tucker conditions corresponding to the problem are satisfied. Let S be the set of all feasible solutions of the non-linear programming problem $P4$, then $S\neq\phi$, the null set. From the constraint (2), we have,

$$\hat{A}y+c_1^{(1)}e^m\leq p_1e^m;\quad -\hat{A}y+c_1^{(2)}e^m\geq p_2e^m$$
$$\Rightarrow x^T\hat{A}y+c_1^{(1)}x^Te^m\leq p_1x^Te^m;\quad -x^T\hat{A}y+c_1^{(2)}x^Te^m\geq p_2x^Te^m$$
$$\Rightarrow \min[x^T\hat{A}y+c_1^{(1)}]\leq p_1;\quad \max[-x^T\hat{A}y+c_1^{(2)}]\geq p_2,\ \text{ as } x^Te^m=1.$$

Similarly, from constraint (3) we can write,

$$\min[x^T\hat{B}y+c_2^{(1)}]\leq q_1;\quad \max[-x^T\hat{B}y+c_2^{(2)}]\geq q_2.$$

and from constraints (4) and (5), it follows that

$$\min[x^T\hat{A}y+c_1^{(1)}]\geq\lambda_1;\quad \max[-x^T\hat{A}y+c_1^{(2)}]\leq\beta_1$$
$$\min[x^T\hat{B}y+c_2^{(1)}]\geq\lambda_2;\quad \max[-x^T\hat{B}y+c_2^{(2)}]\leq\beta_2.$$

Now, for arbitrary $\alpha=(x,y,p_1,p_2,q_1,q_2,\lambda_1,\lambda_2,\beta_1,\beta_2)\in S$, we have

$$\begin{aligned}
&\lambda_1+\lambda_2+p_2+q_2-\beta_1-\beta_2-p_1-q_1\\
&\quad\leq \min[x^T\hat{A}y+c_1^{(1)}]+\min[x^T\hat{B}y+c_2^{(1)}]+\max[-x^T\hat{A}y+c_1^{(2)}]\\
&\qquad\quad +\max[-x^T\hat{B}y+c_2^{(2)}]-\beta_1-\beta_2-p_1-q_1\\
&\quad\leq \min[x^T\hat{A}y+c_1^{(1)}]+\min[x^T\hat{B}y+c_2^{(1)}]+\max[-x^T\hat{A}y+c_1^{(2)}]\\
&\qquad +\max[-x^T\hat{B}y+c_2^{(2)}]-\max[-x^T\hat{A}y+c_1^{(2)}]\\
&\qquad -\max[-x^T\hat{B}y+c_2^{(2)}]-\min[x^T\hat{A}y+c_1^{(1)}]-\min[x^T\hat{B}y-c_2^{(1)}]\\
&\quad\leq 0
\end{aligned}$$

which shows that the optimal value of the objective function is non positive. Thus, the solution (x^*, y^*),

$$\begin{aligned} p_1^* &= x^{*T}\hat{A}y^* + c_1^{(1)}; \quad p_2^* = -x^{*T}\hat{A}y^* + c_1^{(2)};\\ q_1^* &= x^{*T}\hat{B}y^* + c_2^{(1)}; \quad q_2^* = -x^{*T}\hat{B}y^* + c_2^{(2)};\\ \lambda_1^* &= x^{*T}\hat{A}y^* + c_1^{(2)}; \quad \beta_1^* = -x^{*T}\hat{A}y^* + c_1^{(2)};\\ \lambda_2^* &= x^{*T}\hat{B}y^* + c_2^{(2)}; \quad \beta_2^* = -x^{*T}\hat{B}y^* + c_2^{(2)}; \end{aligned}$$

of the quadratic programming problem is feasible and optimal. Therefore

$$\alpha^* = (x^*, y^*, p_1^*, p_2^*, q_1^*, q_2^*, \lambda_1^*, \lambda_2^*, \beta_1^*, \beta_2^*) \in S.$$

The optimal value of the objective function of the non-linear programming problem $P4$ is 0. Conversely, let $\alpha^* \in S$ be the optimal solution to the non-linear programming problem $P4$. Since the optimal value of the problem $P4$ is zero, we get,

$$\lambda_1^* + \lambda_2^* + p_2^* + q_2^* - \beta_1^* - \beta_2^* - p_1^* - q_1^* = 0. \tag{8}$$

From (2) and (3), we have,

$$x^T\hat{A}y^* + c_1^{(1)} \le p_1^* x^T e^m = p_1^*; \quad -x^T\hat{A}y^* + c_1^{(2)} \ge p_2^* \tag{9}$$

$$x^T\hat{B}y^* + c_2^{(1)} \le q_1^*; \qquad -x^T\hat{B}y^* + c_2^{(2)} \ge q_2^* \tag{10}$$

and similarly, from (4) and (5), we have,

$$x^{*T}\hat{A}y^* + c_1^{(2)} \ge \lambda_1^*; \quad -x^{*T}\hat{A}y^* + c_1^{(2)} \le \beta_1^* \tag{11}$$

$$x^{*T}\hat{B}y^* + c_2^{(2)} \ge \lambda_2^*; \quad -x^{*T}\hat{B}y^* + c_2^{(2)} \le \beta_2^*. \tag{12}$$

Using (8), (11) and (12), we write,

$$\begin{aligned} p_1^* + q_1^* - p_2^* - q_2^* &= \lambda_1^* + \lambda_2^* - \beta_1^* - \beta_2^*\\ &\le [x^{*T}\hat{A}y^* + c_1^{(1)}] + [x^{*T}\hat{B}y^* + c_2^{(2)}] - [x^{*T}\hat{A}y^* + c_1^{(2)}] - [-x^{*T}\hat{B}y^* + c_2^{(2)}]\\ \Rightarrow [x^{*T}\hat{B}y^* + c_2^{(1)}] &- [-x^{*T}\hat{B}y^* + c_2^{(2)}]\\ &\ge -[x^{*T}\hat{A}y^* + c_1^{(1)}] + [-x^{*T}\hat{A}y^* + c_1^{(2)}] + p_1^* + q_1^* - p_2^* - q_2^*. \end{aligned}$$

If we put $x = x^*, y = y^*$ in equations (9) and (10), we get,

$$\begin{aligned} x^{*T}\hat{A}y^* + c_1^{(1)} \le p_1^*; &\quad -x^{*T}\hat{A}y^* + c_1^{(2)} \ge p_2^*\\ x^{*T}\hat{B}y^* + c_2^{(1)} \le q_1^*; &\qquad -x^{*T}\hat{B}y^* + c_2^{(2)} \ge q_2^*. \end{aligned}$$

Therefore,

$$\begin{aligned} q_1^* - q_2^* &\ge -[x^{*T}\hat{A}y^* + c_1^{(1)}] + [-x^{*T}\hat{A}y^* + c_1^{(2)}] + p_1^* + q_1^* - p_2^* - q_2^*\\ &\Rightarrow p_1^* - p_2^* \le [x^{*T}\hat{A}y^* + c_1^{(1)}] - [-x^{*T}\hat{A}y^* + c_1^{(2)}] \end{aligned}$$

$$\Rightarrow p_1^* \le [x^{*T}\hat{A}y^* + c_1^{(1)}]; \quad p_2^* \ge [-x^{*T}\hat{A}y^* + c_1^{(2)}]. \tag{13}$$

Similarly, $q_1^* \le [x^{*T}\hat{B}y^* + c_2^{(1)}]; \quad q_2^* \ge [-x^{*T}\hat{B}y^* + c_2^{(2)}].$ (14)

Hence from (9), (10), (13) and (14), we have

$$\begin{aligned} p_1^* &= x^{*T}\hat{A}y^* + c_1^{(1)}; \quad p_2^* = -x^{*T}\hat{A}y^* + c_1^{(2)};\\ q_1^* &= x^{*T}\hat{B}y^* + c_2^{(1)}; \quad q_2^* = -x^{*T}\hat{B}y^* + c_2^{(2)}; \end{aligned}$$

Now it follows from (9) and (10) that

$$\begin{aligned} x^{*T}\hat{A}y^* + c_1^{(1)} &\ge x^T\hat{A}y^* + c_1^{(1)};\\ x^{*T}\hat{B}y^* + c_2^{(1)} &\ge x^T\hat{B}y^* + c_2^{(1)}. \end{aligned}$$

Therefore, the pair (x^*, y^*) is a NES for the bi-matrix game $\Gamma = \langle\{I, II\}, A, B\rangle$. with respect to the degree of attainment of the IF goal aggregated by the minimum component. □

Theorem 4.10. *If all membership and non membership functions are linear and (x^*, y^*) is a NES for $\Gamma = \langle\{I, II\}, A, B\rangle$, then (x^*, y^*) is a NES for Γ with respect to the degree of attainment of the IF goal.*

Proof. Since (x^*, y^*) is a NES for the bi-matrix game Γ, so, by definition,

$$x^T Ay^* \leq x^{*T} Ay^*;\;\; \forall x \in S_I$$
$$x^{*T} By \leq x^{*T} By^*;\;\; \forall y \in S_{II}.$$

Now we consider the following five cases for $x^T Ay^* \leq x^{*T} Ay^*;\;\; \forall x \in S_I$.

(i) If $x^T Ay^* \leq x^{*T} Ay^* \leq \underline{a}$, then we have,

$$\mu_1(x^T Ay^*) = \mu_1(x^{*T} Ay^*) = 0;\;\; \nu_1(x^T Ay^*) = \nu_1(x^{*T} Ay^*) = 1.$$

(ii) If $x^T Ay^* \leq \underline{a} \leq x^{*T} Ay^*$, then we have

$$\mu_1(x^{*T} Ay^*) \geq \mu_1(x^T Ay^*) = 0;\;\; \nu_1(x^{*T} Ay^*) \leq \nu_1(x^T Ay^*) = 1.$$

(iii) If $\underline{a} \leq x^T Ay^* \leq x^{*T} Ay^* \leq \overline{a}$, then we have,

$$\mu_1(x^T Ay^*) = c_1^{(1)} + x^T \hat{A}y^*;\;\; \mu_1(x^{*T} Ay^*) = c_1^{(1)} + x^{*T} \hat{A}y^*.$$
$$\nu_1(x^T Ay^*) = c_1^{(2)} - x^T \hat{A}y^*;\;\; \nu_1(x^{*T} Ay^*) = c_1^{(2)} - x^{*T} \hat{A}y^*.$$

On the other hand, $\overline{a} - \underline{a} > 0$, $\overline{b} - \underline{b} > 0$ imply that (x^*, y^*) is a NES for $\Gamma = \langle\{I, II\}, A, B\rangle$. Therefore, we have, $x^T \hat{A}y^* \leq x^{*T} \hat{A}y^*;\;\; \forall x \in S_I$. So,

$$c_1^{(1)} + x^T \hat{A}y^* \leq c_1^{(1)} + x^{*T} \hat{A}y^*;\;\; c_1^{(2)} - x^T \hat{A}y^* \geq c_1^{(2)} - x^{*T} \hat{A}y^*.$$

Therefore,

$$\mu_1(x^T Ay^*) \geq \mu_1(x^{*T} Ay^*);\;\; \nu_1(x^T Ay^*) \leq \nu_1(x^{*T} Ay^*).$$

(iv) If $x^T Ay^* \leq \overline{a} \leq x^{*T} Ay^*$, then we have

$$\mu_1(x^{*T} Ay^*) = 1, \mu_1(x^T Ay^*) \leq 1;\;\; \nu_1(x^{*T} Ay^*) = 0, \nu_1(x^T Ay^*) \geq 0.$$

Therefore,

$$1 = \mu_1(x^{*T} Ay^*) \geq \mu_1(x^T Ay^*);\;\; \nu_1(x^{*T} Ay^*) \leq \nu_1(x^T Ay^*).$$

(v) If $\overline{a} \leq x^T Ay^* \leq x^{*T} Ay^*$, then we have,

$$\mu_1(x^T Ay^*) = \mu_1(x^{*T} Ay^*) = 1;\;\; \nu_1(x^T Ay^*) = \nu_1(x^{*T} Ay^*) = 0.$$

The above cases imply that

$$\mu_1(x^{*T} Ay^*) \geq \mu_1(x^T Ay^*) \text{ and } \nu_1(x^{*T} Ay^*) \leq \nu_1(x^T Ay^*); \forall x \in S_I.$$

Similarly, for all $y \in S_{II}$, we have,

$$\mu_2(x^{*T} By^*) \geq \mu_2(x^{*T} By) \text{ and } \nu_2(x^{*T} By^*) \leq \nu_2(x^{*T} By).$$

The proof is complete. Thus an equilibrium solution for IF bi-matrix game is derived with respect to the degree of attainment of the fuzzy goals, in the sense of the definition 4.8. □

Corollary 4.11. *Suppose the conditions* $\underline{a} \leq \min_{i,j} \alpha_{ij}$, $\overline{a} \geq \max_{i,j} \alpha_{ij}$ *and* $\underline{b} \leq \min_{i,j} \gamma_{ij}$, $\overline{b} \geq \max_{i,j} \gamma_{ij}$ *hold. Then* (x^*, y^*) *is a NES for* $\Gamma = \langle \{I, II\}, A, B \rangle$ *if and only if* (x^*, y^*) *is a NES with respect to the degree of attainment of the IF goal.*

Proof. If the conditions of the statement (corollary 4.11) hold, then

$$\underline{a} \leq x^T Ay \leq \overline{a} \text{ and } \underline{b} \leq x^T By \leq \overline{b}, \quad \forall x \in S_I, \forall y \in S_{II},$$

which corresponds to the case (iii) for the proof of the theorem 4.10. Hence,

$$\mu_1(x^T Ay^*) = c_1^{(1)} + x^T \hat{A} y^*; \;\; \mu_1(x^{*T} Ay^*) = c_1^{(1)} + x^{*T} \hat{A} y^*.$$

$$\nu_1(x^T Ay^*) = c_1^{(2)} - x^T \hat{A} y^*; \;\; \nu_1(x^{*T} Ay^*) = c_1^{(2)} - x^{*T} \hat{A} y^*.$$

Since $\overline{a} - \underline{a} \geq 0$, $\overline{b} - \underline{b} \geq 0$ we have

$$\mu_1(x^T Ay^*) \geq \mu_1(x^{*T} Ay^*); \;\; \nu_1(x^T Ay^*) \leq \nu_1(x^{*T} Ay^*)$$

$$\mu_2(x^{*T} By) \geq \mu_2(x^{*T} By^*); \;\; \nu_2(x^{*T} By) \leq \nu_2(x^{*T} By^*)$$

i.e. (x^*, y^*) is a NES for $\Gamma = \langle \{I, II\}, A, B \rangle$. The converse follows by reversing the steps. $\square$

Thus, once an optimal solution $(x^*, y^*, p_1^*, p_2^*, q_1^*, q_2^*, \lambda_1^*, \lambda_2^*, \beta_1^*, \beta_2^*)$ of the non-linear programming problem (as in Theorem 4.9) has been obtained, (x^*, y^*) gives an equilibrium solution of the bi-matrix game. The degree of attainment of IF goals G_1 and G_2 can be determined by evaluating $x^{*T} Ay^*$ and $x^{*T} By^*$, and then employing the membership and non- membership function μ_1, ν_1 and μ_2, ν_2.

Solution Procedure: The process of solving an IFO model can be divided into two steps: aggregation of IF objectivesand constraints by Atanassov's property [2] of IFS and defuzzification of the aggregated function which transforms the IFO model to a crisp one. Let us consider the solution of the game with respect to the degree of attainment of the IF goal aggregated by minimum component [1]. Such a rule is often used in decision making [18]. Basically, in IF decision making the aggregation corresponds to the union of all the IFS intersections, and the solution is determined by maximizing the membership function and minimizing the non-membership function of aggregated function of the intersection.

5. **Numerical Example**

Suppose player I and player II have three strategies defined in the following pair of matrices:

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 5 & 8 & 9 \\ 6 & 2 & 7 \end{pmatrix}; \;\; B = \begin{pmatrix} 7 & 9 & 0 \\ 5 & 8 & 9 \\ 3 & 2 & 1 \end{pmatrix}.$$

The tolerances are arbitrarily fixed for each matrix and each membership and non-membership function of the players. Let the IF goals of the player I be represented by linear membership and non-membership functions as follows:

$$\mu_1(x^TAy) = \begin{cases} 1; & x^TAy \geq 9 \\ \frac{x^TAy-2}{7}; & 2 < x^TAy < 9 \\ 0; & x^TAy \leq 2 \end{cases}$$

$$\nu_1(x^TAy) = \begin{cases} 1; & x^TAy \leq 2 \\ \frac{x^TAy-9}{-7}; & 2 < x^TAy < 9 \\ 0; & x^TAy \geq 9 \end{cases}$$

Similarly, let the IF goals of the player II be represented by the following linear membership and non-membership functions:

$$\mu_2(x^TBy) = \begin{cases} 1; & x^TBy \geq 9 \\ \frac{x^TBy-0}{9}; & 0 < x^TBy < 9 \\ 0; & x^TBy \leq 0 \end{cases}$$

$$\nu_2(x^TBy) = \begin{cases} 1; & x^TBy \leq 0 \\ \frac{x^TBy-9}{-9}; & 0 < x^TBy < 9 \\ 0; & x^TBy \geq 9 \end{cases}$$

By $P4$, we get the following mathematical programming problem

$$\max_{x,y,p_1,p_2,q_1,q_2,\lambda_1,\lambda_2,\beta_1,\beta_2} \{\lambda_1 + \lambda_2 + p_2 + q_2 - \beta_1 - \beta_2 - p_1 - q_1\}$$

$$\begin{aligned}
2y_1 + 4y_2 + 5y_3 - 7p_1 &\leq 2 \\
5y_1 + 8y_2 + 9y_3 - 7p_1 &\leq 2 \\
6y_1 + 2y_2 + 7y_3 - 7p_1 &\leq 2 \\
2y_1 + 4y_2 + 5y_3 + 7p_2 &\leq 9 \\
5y_1 + 8y_2 + 9y_3 + 7p_2 &\leq 9 \\
6y_1 + 2y_2 + 7y_3 + 7p_2 &\leq 9 \\
7x_1 + 9x_2 - 9q_1 &\leq 0
\end{aligned}$$

$$\begin{aligned}
5x_1 + 8x_2 + 9x_3 - 9q_1 &\leq 0 \\
3x_1 + 2x_2 + x_3 - 9q_1 &\leq 0 \\
7x_1 + 9x_2 + 9q_2 &\leq 1 \\
5x_1 + 8x_2 + 9x_3 + 9q_2 &\leq 1 \\
3x_1 + 2x_2 + x_3 + 9q_2 &\leq 1
\end{aligned}$$

$$\begin{aligned}
2x_1y_1 + 5x_2y_1 + 6x_3y_1 + 4x_1y_2 + 8x_2y_2 + 2x_3y_2 + 5x_1y_3 + 9x_2y_3 + 7x_3y_3 - 7\lambda_1 &\geq 2 \\
2x_1y_1 + 5x_2y_1 + 6x_3y_1 + 4x_1y_2 + 8x_2y_2 + 2x_3y_2 + 5x_1y_3 + 9x_2y_3 + 7x_3y_3 - 7\beta_1 &\leq -9 \\
7x_1y_1 + 5x_2y_1 + 3x_3y_1 + 9x_1y_2 + 8x_2y_2 + 2x_3y_2 + 9x_2y_3 + x_3y_3 - 9\lambda_2 &\geq 0 \\
7x_1y_1 + 5x_2y_1 + 3x_3y_1 + 9x_1y_2 + 8x_2y_2 + 2x_3y_2 + 9x_2y_3 + x_3y_3 + 9\beta_2 &\geq 9 \\
x_1 + x_2 + x_3 = 1 = y_1 + y_2 + y_3.&
\end{aligned}$$

The solutions obtained using LINGO software are shown in the following table:

x_1	x_2	x_3	y_1	y_2	y_3
0.000	0.8182	0.1818	0.8237	0.1763	0.000

Table1 : Intuitionistic fuzzy goals

The optimal strategies for the player I and II are respectively $(0.0000, 0.8182, 0.1818)$ and $(0.8237, 0.1763, 0.000)$. The Nash equilibrium outcome in mixed strategies is $(5.4863, 5.0371)$ and the degrees of satisfaction are $\mu_1 = 0.498$ and $\mu_2 = 0.560$. Here the solution of the IFO problem depends on the formulation of the respective functions of acceptance and rejection.

6. Conclusion

In this paper, we have presented a model for studying bi-matrix games with IF goals. In this approach, the degree of acceptance and the degree of rejection of the objective and the constraints are introduced together, These cannot be simply considered as a complement of each other and the sum of their value is less than or equal to 1. Since the strategy spaces of player I and player II could be polyhedral sets, we may also conceptualize the constrained IF bi-matrix game on the lines of crisp constrained bi-matrix games. On this basis, we define the solution in terms of degree of attainment of IF goal in IFS environment, and obtain it by solving a mathematical programming problem. A numerical example illustrates the proposed methods. This theory can be applied in decision making theory such as economics, operation research, management, war science, etc.

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References

[1] K. Atanassov, *Intuitionistic fuzzy sets: theory and applications,* Physica-Verlag, 1999.

[2] P. P. Angelov,*Optimization in an intuitinistic fuzzy enviornment*, Fuzzy Sets and Systems, **86** (1997), 299-306.

[3] C. R. Bector, S. Chandra and V. Vijay, *Bi-matrix games with fuzzy payoffs and fuzzy goals*, Fuzzy Optimization and Decision Making, **3** (2004), 327-344.

[4] T. Basar and G. J. Olsder, *Dynamic non-cooperative game theory*, Academic Press, New York, 1995.

[5] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets and Systems, **32** (1989), 275-289.

[6] D. Dubois and H. Prade, *Fuzzy sets and systems*, Academic Press, New York, 1980.

[7] T. Maeda, *Characterization of the equilibrium strategy of bi-matrix games with fuzzy payoffs*, Journal of Mathematical Analysis and Applications, **251** (2000), 885-896.

[8] J. V. Neumann and O. Morgenstern, *Theory of games and economic behaviour*, Princeton University Press, Princeton, New Jersey, 1944 .

[9] J. F. Nash, *Non cooperative games*, Annals of Mathematics, **54** (1951), 286-295.

[10] P. K. Nayak and M. Pal, *Solution of interval games using graphical method*, Tamsui Oxford Journal of Mathematical Sciences, **22(1)** (2006), 95-115.

[11] P. K. Nayak and M. Pal, *Bi-matrix games with intuitionistic fuzzy payoffs*, Notes on Intuitionistic Fuzzy Sets, **13(3)** (2007), 1-10.

[12] P. K. Nayak and M. Pal, *Bi-matrix games with interval payoffs and its Nash equilibrium strategy*, Asia specific Journal of Operational Research, **26(2)** (2009), 285-305.

[13] P. K. Nayak and M. Pal, *Bi-matrix games with interval payoffs and its Nash equyilibrium strategy*, Journal of Fuzzy Mathematics, **17(2)** (2009).

[14] Z. Peng, L. Dagang and W. Guangyuan, *Idea and principle of intuitionistic fuzzy optimization*, htpp://www.paper.edu.cn.

[15] S. K. Roy, M. P. Biswal and R. N. Tiwari, *An approach to multi-objective bimatrix games of Nash equilibrium solutions*, Ricerca Operativa , **30(93)** (2001), 47-64.

[16] M. Sakawa and I. Nishizaki, *Max-min solution for fuzzy multiobjective matrix games*, Fuzzy Sets and Systems, **67** (1994), 53-69.

[17] M. Sakawa and I. Nishizaki, *Equilibrium solution in bi-matrix games with fuzzy payoffs*, Japanse Fuzzy Theory and Systems, **9(3)** (1997), 307-324.

[18] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-352.

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