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SET-NORM EXHAUSTIVE SET MULTIFUNCTIONS

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Abstract. In this paper we present some properties of set-norm exhaustive set multifunctions and also of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity.

1. Introduction

The subject of the present paper concerns fuzzy set multifunctions. Non-additive set functions and fuzzy sets have been intensively studied by many authors (e.g., Asahina [1], Choquet [4], Daneshgar and Hashemi [7], Denneberg [9], Drewnowski [10], Dubois and Prade [11], Funiokova [12], Li [16], Merghadi and Aliouche [17], Pap [18], Precupanu [19], Shafer [20], Sugeno [21], Suzuki [22], Zadeh [23], Vaezpour and Karini [24], Wen, Shi and Li [25], Wu and Bo [26]), due to its applications in statistics, economy, theory of games, human decision making.

In our previous papers [5,6,13-15] we extended and studied different concepts (such as pseudo-atom, Darboux property, continuity, exhaustivity, regularity) to the set-valued case.

In [5] we introduced the notion of set-norm on the family of non-empty subsets of a real linear space and studied different notions of continuous set multifunctions with respect to a set-norm.

This paper contains three sections. In the second section, properties of set-norm exhaustive set multifunctions are presented. These set multifunctions (such as probability multimeasures) are used in control, robotics, decision theory (in Bayesian estimation) or in statistical inference (Dempster [8]). In the third section, we present different properties of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. Our results generalize to set-valued case important problems in measure theory, such as non-atomicity or pure atomicity (Aumann and Shapley [2]), that have applications in coincidence and rigidity phenomena (Chiţescu [3]).

2. Non-Additive Set Multifunctions

Let T be an abstract nonvoid set, $\mathcal{C}$ a ring of subsets of T and $\mathcal{P}(T)$ the family of all subsets of T.

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In the sequel, $(X, +, 0)$ will be a commutative semigroup with unity 0 and $\mathcal{P}_0(X)$ the family of non-empty subsets of X. On $\mathcal{P}_0(X)$ we consider an order relation denoted by " $\leq$ ". We write $E < F$ if $E \leq F$ and $E \neq F$, for every $E, F \in \mathcal{P}_0(X)$. The notation $F \geq E$ ($F > E$ respectively) will often be used in the place of $E \leq F$ ($E < F$ respectively). We shall write $(\mathcal{P}_0(X), \leq)$. For every $E, F \in \mathcal{P}_0(X)$, let $E + F = \{x + y | x \in E, y \in F\}$.

Example 2.1. I. The usual set inclusion "$\subseteq$" is an order relation on $\mathcal{P}_0(X)$.

If X is a normed space, then $\mathcal{P}_c(X)$ is the family of non-empty closed subsets of X and $\mathcal{P}_{cb}(X)$ is the family of non-empty closed bounded subsets of X.

The set of all real numbers is denoted by $\mathbb{R}$. We denote $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, where $\mathbb{N}$ is the set of all positive integers.

Definition 2.2. [5] A function $|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]$ is called *a set-norm* on $\mathcal{P}_0(X)$ if it satisfies the conditions:

(i) $|E| = 0 \Leftrightarrow E = \{0\}, \forall E \in \mathcal{P}_0(X)$.

(ii) $|E + F| \leq |E| + |F|, \forall E, F \in \mathcal{P}_0(X)$.

Definition 2.3. [5] A set-norm $|\cdot|$ on $(\mathcal{P}_0(X), \leq)$ is called *monotone* if for every sets $E, F \in \mathcal{P}_0(X)$, $E \leq F \Rightarrow |E| \leq |F|$. We denote $(\mathcal{P}_0(X), \leq, |\cdot|)$ when $(\mathcal{P}_0(X), \leq)$ is endowed with a monotone set-norm $|\cdot|$.

Example 2.4. Let $(X, \|\cdot\|)$ be a real normed space and $|E|_s = \sup_{x \in E} \|x\|$, for every $E \in \mathcal{P}_0(X)$. Then the function $|\cdot|_s$ is a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$ and we denote this by $(\mathcal{P}_0(X), \subseteq, |\cdot|_s)$.

Definition 2.5. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is called:

(i) a *multimeasure* if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

(ii) *null-additive* if for every $A, B \in \mathcal{C}$,

$$\mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(A).$$

(iii) *null-null-additive* if for every $A, B \in \mathcal{C}$,

$$\mu(A) = \mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(\emptyset).$$

Definition 2.6. Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ be a set multifunction. μ is said to be:

(i) *monotone* if $\mu(A) \leq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

(ii) *fuzzy* if μ is monotone and $\mu(\emptyset) = \{0\}$.

(iii) *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$.

(iv) *a multisubmeasure* if μ is fuzzy and subadditive.

Remark 2.7. I. If X is a normed space and μ is $\mathcal{P}_c(X)$-valued, then in the definition of a multi(sub)measure, it usually appears " $\overset{\bullet}{+}$ " instead of " $+$ ", because the sum of two closed sets is not always closed.

II. The following implications hold:

(i) If μ is a multisubmeasure, then μ is null-additive.

(ii) If μ is null-additive, then μ is null-null-additive.

III. The concepts in Definitions 2.5 and 2.6 do not reduce to the usual single-valued case. The difficulty arises here since we have to consider an order relation on $\mathcal{P}_0(X)$ and many classical measure theory proof methods fail. For instance, if $\mu : (\mathcal{P}_0(\mathbb{R}), \subseteq)$ is single-valued and monotone, then μ reduces in fact to a constant function $\mu(A) = \{\mu(\emptyset)\}$, $\forall A \in \mathcal{C}$.

Moreover, $\mathcal{P}_0(X)$ (and also $\mathcal{P}_c(X)$) is not a linear space since $\mathcal{P}_0(X)$ is not a group with respect to the addition "+" defined by $M + N = \{x + y | x \in M, y \in N\}$, for every $M, N \in \mathcal{P}_0(X)$.

Definition 2.8. A set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is said to be:

(i) *set-norm exhaustive* (shortly, *sn-exhaustive*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

(ii) *set-norm continuous* (shortly, *sn-continuous*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, such that $A_n \searrow \emptyset$ (i.e. $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}^* \wedge \bigcap\limits_{n=1}^{\infty} A_n = \emptyset$).

(iii) *strongly-set-norm continuous* (shortly, *strongly sn-continuous*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$ for every sequence $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ such that $A_{n+1} \subseteq A_n$, $\forall n \in \mathbb{N}^*$ and $\mu(\bigcap\limits_{n=1}^{\infty} A_n) = \{0\}$.

(iv) *null-continuous* if $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \{0\}$ for every sequence $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{0\}$, for every $n \in \mathbb{N}^*$.

We now establish some relationships among the set multifunctions introduced in Definition 2.8.

We recall that $\mathcal{C}$ is a σ-ring if the following conditions hold:

(i) $A \backslash B \in \mathcal{C}$, for every $A, B \in \mathcal{C}$,

(ii) $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$, for every $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

Theorem 2.9. *If $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy and sn-continuous, then μ is sn-exhaustive.*

Proof. Let $(A_n)_{n\in\mathbb{N}^*}$ be a sequence of mutually disjoint sets of $\mathcal{C}$ and let $B_n = \bigcup\limits_{k=n}^{\infty} A_k$, for all $n \in \mathbb{N}^*$. Then $B_n \in \mathcal{C}$, for every $n \in \mathbb{N}^*$ and $B_n \searrow \emptyset$. Since μ is sn-continuous, it results $|\mu(B_n)| \to 0$, which implies $|\mu(A_n)| \to 0$. So μ is sn-exhaustive. $\square$

Theorem 2.10. *Suppose $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy. If μ is null-null-additive and strongly-sn-continuous, then μ is null-continuous.*

Proof. Let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{0\}$, for every $n \in \mathbb{N}^*$. Denote $A = \bigcup\limits_{n=1}^{\infty} A_n$. We recurrently define a subsequence (A_{n_k}) of (A_n) as follows.

Let $n_1 = 1$. For every $k \in \mathbb{N}^*$, since $\mu(A_{n_k}) = \{0\}$ and $A_{n_k} \cup (A \backslash A_n) \searrow A_{n_k}$ when $n \to \infty$, by the fact that μ is strongly-sn-continuous, we can choose n_{k+1} so that $n_{k+1} > n_k$ and

$$|\mu(A_{n_k} \cup (A \backslash A_{n_{k+1}}))| < \frac{1}{k}.$$

Denote $B = \bigcup\limits_{k=1}^{\infty} (A_{n_{2k}} \backslash A_{n_{2k-1}})$ and $C = A \backslash B = A_{n_1} \cup \bigcup\limits_{k=1}^{\infty} (A_{n_{2k+1}} \backslash A_{n_{2k}})$. For all $k \in \mathbb{N}^*$, since $B \subseteq A_{n_{2k}} \cup (A \backslash A_{n_{2k+1}})$, it results:

$$|\mu(B)| \leq |\mu(A_{n_{2k}} \cup (A \backslash A_{n_{2k+1}}))| < \frac{1}{2k}.$$

It follows $\mu(B) = \{0\}$. Analogously, for every $k \in \mathbb{N}^*$, since $C \subseteq A_{n_{2k-1}} \cup (A \backslash A_{n_{2k}})$, it follows that:

$$|\mu(C)| \leq |\mu(A_{n_{2k-1}} \cup (A \backslash A_{n_{2k}}))| < \frac{1}{2k-1},$$

which implies that $\mu(C) = \{0\}$. Since μ is null-null-additive, we obtain $\mu(A) = \mu(B \cup C) = \{0\}$. So, μ is null-continuous. □

Example 2.11. I. If $\mathcal{C}$ is finite, then every set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$, with $\mu(\emptyset) = \{0\}$, is sn-exhaustive and sn-continuous.

II. Let $T = \mathbb{R}$, $\mathcal{C} = \{A \subseteq T | A \text{ is finite}\}$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_s)$ be defined by $\mu(A) = [0, \nu(A)]$, where $\nu(A) = \begin{cases} 0, & A = \emptyset \\ 1 + \text{card}\, A, & A \neq \emptyset \end{cases}$, $\forall A \in \mathcal{C}$ and card$A$ is the number of elements in A. Then μ is sn-continuous, but not sn-exhaustive.

III. Let $T = \mathbb{N}$, $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_s)$ be defined by

$$\mu(A) = \begin{cases} \{0\}, & A \neq \mathbb{N} \\ \{0,1\} \cup [3,7], & A = \mathbb{N}. \end{cases}$$

μ is not null-null-additive, because there exist $A = \{0\}$ and $B = \mathbb{N}^*$ so that $\mu(A) = \mu(B) = \{0\}$, but $\mu(A \cup B) = \mu(\mathbb{N}) \neq \{0\}$.
μ is not null-continuous since there is $(A_n)_{n \in \mathbb{N}}$, $A_n = \{0, 1, 2, \ldots, n\}$, for every $n \in \mathbb{N}$, such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{0\}$, for all $n \in \mathbb{N}$, but $\mu(\bigcup_{n=0}^{\infty} A_n) = \mu(\mathbb{N}) \neq \{0\}$.

3. Atoms and Pseudo-Atoms

We now give some properties regarding atoms and pseudo-atoms of set multifunctions taking values in the family of nonvoid subsets of a commutative semigroup with unity.
In the sequel, $(X, +, 0)$ is a commutative semigroup with unity 0 and "$\leq$" is an order relation on $\mathcal{P}_0(X)$.

Definition 3.1. Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ be a set multifunction.

(i) A set $A \in \mathcal{C}$ is said to be *an atom* of μ if $\mu(A) > \mu(\emptyset)$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \mu(\emptyset)$ or $\mu(A \backslash B) = \mu(\emptyset)$.

(ii) A set $A \in \mathcal{C}$ is called *a pseudo-atom* of μ if $\mu(A) > \mu(\emptyset)$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \mu(\emptyset)$ or $\mu(B) = \mu(A)$.

(iii) μ is called *non-atomic* (*non-pseudo-atomic* respectively) if it has no atoms (pseudo-atoms respectively).

Remark 3.2. If μ is fuzzy, then μ is non-atomic (non-pseudo-atomic respectively) if and only if for every $A \in \mathcal{C}$ with $\mu(A) > \{0\}$, there exists $B \in \mathcal{C}$, so that $B \subseteq A$, $\mu(B) > \{0\}$ and $\mu(A \backslash B) > \{0\}$ ($\mu(B) < \mu(A)$ respectively).

Proposition 3.3. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ is fuzzy and null-additive (or a multimeasure). Then every atom of μ is a pseudo-atom of μ.*

Proof. (i) Suppose μ is fuzzy and null-additive and let $A \in \mathcal{C}$ be an atom of μ. Let $B \in \mathcal{C}$, $B \subseteq A$ so that $\mu(B) \neq \{0\}$. Since A is an atom of μ, it results $\mu(A \backslash B) = \{0\}$. Since μ is null-additive, it follows $\mu(A) = \mu(B \cup (A \backslash B)) = \mu(B)$ which proves that A is a pseudo-atom of μ.

(ii) Suppose μ is a multimeasure and let $A \in \mathcal{C}$ be an atom of μ. Let $B \in \mathcal{C}$, $B \subseteq A$ so that $\mu(B) \neq \{0\}$. Since A is an atom of μ, it results in $\mu(A \backslash B) = \{0\}$. Since μ is a multimeasure, we have:

$$\mu(A) = \mu(B \cup (A \backslash B)) = \mu(B) + \mu(A \backslash B) = \mu(B) + \{0\} = \mu(B).$$

So A is pseudo-atom of μ. □

Remark 3.4. The converse of Proposition 3.3 is not valid (see Example 3.7). As we shall see in the sequel, if X is a normed space, then this converse is true. To prove this we need the following lemma.

Lemma 3.5. *Let $(X, \|\cdot\|)$ be a normed-space and $A, B \in \mathcal{P}_0(X)$ so that $A + B = B$ and B is bounded. Then $A = \{0\}$.*

Proof. Since B is bounded, there is $M > 0$ so that $\|y\| \leq M$, for every $y \in B$. Let $a \in A$, $b \in B$. It results in $a + b \in A + B = B$. Then $2a + b = a + (a + b) \in A + B = B$. By induction we obtain that $na + b \in B$, for every $n \in \mathbb{N}$. It follows $\|na + b\| \leq M$. Consequently, we have $\|a\| \leq \frac{2M}{n}$, for every $n \in \mathbb{N}^*$, which proves that $a = 0$. So $A = \{0\}$. □

Proposition 3.6. *If X is a normed space and $\mu : \mathcal{C} \to (\mathcal{P}_b(X), \leq)$ is a multimeasure, then $A \in \mathcal{C}$ is a pseudo-atom of μ if and only if A is an atom of μ.*

Proof. I. Suppose A is an atom of μ and let $B \in \mathcal{C}$, $B \subseteq A$ such that $\mu(B) \neq \{0\}$. Since A is an atom of μ, it results in $\mu(A \backslash B) = \{0\}$. Since μ is a multimeasure, we have:

$$\mu(A) = \mu(A \backslash B) \cup B = \mu(A \backslash B) + \mu(B) = \{0\} + \mu(B) = \mu(B),$$

which proves that A is a pseudo-atom of μ.

II. Suppose A is a pseudo-atom of μ and let $B \in \mathcal{C}$, $B \subseteq A$ such that $\mu(B) \neq \{0\}$. Since A is a pseudo-atom of μ, we have $\mu(B) = \mu(A)$. Since μ is a multimeasure, it results in:

$$\mu(A) = \mu((A \backslash B) \cup B) = \mu(A \backslash B) + \mu(B) = \mu(A \backslash B) + \mu(A).$$

According to Lemma 3.5, it follows $\mu(A\backslash B) = \{0\}$. Consequently, A is an atom of μ. □

Example 3.7. I. Let $T = \{a, b, c\}$, $\mathcal{C} = \mathcal{P}(T)$, $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in \mathcal{C}$ by $\mu(A) = \begin{cases} [0,1], & \text{if } A \neq \emptyset, \\ \{0\}, & \text{if } A = \emptyset. \end{cases}$
Then μ is null-additive, $A = \{a, b\}$ is a pseudo-atom of μ, but not an atom of μ.

II. Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$, $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in \mathcal{C}$ by
$\mu(A) = \begin{cases} [0,2]\,, & \text{if } A = T \\ [0,1]\,, & \text{if } A = \{b\} \\ \{0\}, & \text{if } A = \emptyset \text{ or } A = \{a\}. \end{cases}$
Then μ is not null-additive, T is an atom of μ, but not a pseudo-atom of μ.

III. Let $T = 2\mathbb{N} = \{0, 2, 4, \ldots\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \subseteq)$ be defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A = \emptyset \\ \frac{1}{2}A \cup \{0\}, & \text{if } A \neq \emptyset \end{cases}$$

where $\frac{1}{2}A = \{\frac{x}{2} | \, x \in A\}$. Then μ is a multisubmeasure.

If $A \in \mathcal{C}$ has card$A = 1$ and $A \neq \{0\}$ or $A \in \mathcal{C}, A = \{0, 2n\}$, $n \in \mathbb{N}^*$, then A is an atom of μ (and a pseudo-atom of μ too).

If $A \in \mathcal{C}$ has card$A \geq 2$ and there exist $a, b \in A$ such that $a \neq b$ and $ab \neq 0$, then A is not a pseudo-atom of μ (and not an atom of μ).

IV. Let $T = \mathbb{N}$, $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A \text{ is finite} \\ \{0\} \cup [n_A, +\infty), & \text{if } A \text{ is infinite and} \\ & n_A = \min A. \end{cases}$$

Then μ is monotone and non-pseudo-atomic.

From definitions we obtain the following properties of pseudo-atoms.

Proposition 3.8. *Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ be a fuzzy set multifunction.*

I. If $A \in \mathcal{C}$ is a pseudo-atom of μ and $B \in \mathcal{C}, B \subseteq A$ is so that $\mu(B) > \{0\}$, then B is a pseudo-atom of μ and $\mu(B) = \mu(A)$.

II. If $A, B \in \mathcal{C}$ are pseudo-atoms of μ and $\mu(A) \neq \mu(B)$, then $\mu(A \cap B) = \{0\}$.

III. Moreover, suppose μ is null-null-additive and let $A, B \in \mathcal{C}$ be pseudo-atoms of μ. Then the following statements hold:

- (i) *If $\mu(A \cap B) = \{0\}$, then $A\backslash B$ and $B\backslash A$ are pseudo-atoms of μ and $\mu(A\backslash B) = \mu(A)$, $\mu(B\backslash A) = \mu(B)$.*
- (ii) *If $\mu(A \cap B) > \{0\}$ and $\mu(A\backslash B) = \mu(B\backslash A) = \{0\}$, then $A \cap B$ is a pseudo-atom of μ and $\mu(A \triangle B) = \{0\}$ (where $A \triangle B = (A\backslash B) \cup (B\backslash A)$).*

Proof. I. Since $\mu(B) > \{0\}$ and A is a pseudo-atom of μ, it results in $\mu(B) = \mu(A)$. Let $C \in \mathcal{C}$, $C \subseteq B$ and suppose $\mu(C) > \{0\}$. Since $C \subseteq A$ and A is a pseudo-atom, it follows $\mu(C) = \mu(A)$. This shows that $\mu(C) = \mu(B)$. So B is a pseudo-atom of μ.

II. Let $a, B \in \mathcal{C}$ be pseudo-atoms of μ such that $\mu(A) \neq \mu(B)$. Suppose, by contrary, that $\mu(A \cap B) > \{0\}$. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and A, B are pseudo-atoms of μ, according to Proposition 3.8-I, we have $\mu(A \cap B) = \mu(A)$ and $\mu(A \cap B) = \mu(B)$. It follows $\mu(A) = \mu(B)$, false!

III. (i) Suppose $\mu(A \backslash B) = \{0\}$. Since $\mu(A \cap B) = \{0\}$ and μ is null-null-additive, it results $\mu((A \cup B) \cup (A \backslash B)) = \mu(A) = \{0\}$, that is false because A is a pseudo-atom. So $\mu(A \backslash B) > \{0\}$. By Proposition 3.8-I, it follows that $A \backslash B$ is a pseudo-atom and $\mu(A \backslash B) = \mu(A)$. In the same way it follows that $B \backslash A$ is a pseudo-atom and $\mu(B \backslash A) = \mu(B)$.

(ii) Since $A \cap B \subseteq A$ and $\mu(A \cap B) > \{0\}$, by Proposition 3.8-I, it results that $A \cap B$ is a pseudo-atom of μ. Since μ is null-null-additive, we have $\mu(A \triangle B) = \{0\}$. □

Proposition 3.9. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ is a null-additive fuzzy set multifunction and $A \in \mathcal{C}$ is an atom (pseudo-atom respectively). If $E \in \mathcal{C}$ is so that $\mu(E) = \{0\}$, then $B = A \cup E$ is also an atom (pseudo-atom respectively).*

Proof. Suppose A is a pseudo-atom of μ and let $C \in \mathcal{C}$, $C \subseteq B = A \cup E$. We can set $C = (C \cap A) \cup (C \cap E)$. By the monotonicity of μ, we have $\mu(C \cap E) = \{0\}$. Since μ is null-additive, the following relation holds:

$$\mu(C \cap A) = \mu(C). \tag{1}$$

But A is a pseudo-atom of μ and $C \cap A \subseteq A$. Then we have $\mu(C \cap A) = \{0\}$ or $\mu(C \cap A) = \mu(A)$.

(i) If $\mu(C \cap A) = \{0\}$, by (1) it results $\mu(C) = \{0\}$.

(ii) If $\mu(C \cap A) = \mu(A)$, by (1) it results $\mu(A) = \mu(C)$. Since $\mu(E) = \{0\}$ and μ is null-additive, we have $\mu(B) = \mu(A)$. So $\mu(C) = \mu(B)$.
This shows that B is a pseudo-atom of μ.
The case when A is an atom of μ analogously follows. □

Proposition 3.10. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a null-additive fuzzy set multifunction and let $A \in \mathcal{C}$ be an atom of μ. If $\{B_i\}_{i=1}^n \subset \mathcal{C}$ is a partition of A, then there exists a unique $i_0 \in \{1, \ldots, n\}$ such that $\mu(B_{i_0}) = \mu(A)$ and $\mu(B_i) = \{0\}$, for $i \in \{1, \ldots, n\}$, $i \neq i_0$.*

Proof. We have two cases:

I. $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\}$. From the null-additivity of μ, it results $\mu(A) = \{0\}$, which is false.

II. There exists $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$.
Suppose without loss in generality that $\mu(B_1) > \{0\}$ and $\mu(B_2) > \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_1) = \{0\}$. Since $B_2 \subseteq A \backslash B_1$ and μ is fuzzy, $\mu(B_2) = \{0\}$, which is false.

It results that there exists a unique $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_{i_0}) = \{0\}$. From the null-additivity of μ we obtain $\mu(A) = \mu(B_{i_0})$. Since $B_i \subseteq A \backslash B_{i_0}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$ and μ is fuzzy, it follows that $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$ and the proof is finished. □

Definition 3.11. For a set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(X), |\cdot|)$, the following set function (called the variation of μ) is introduced:

$\overline{\mu} : \mathcal{P}(T) \to [0, +\infty]$, $\overline{\mu}(E) = \sup\{\sum\limits_{i=1}^{n} |\mu(A_i)|; A_i \subseteq E, A_i \in \mathcal{C}, A_i \cap A_j = \emptyset, i \neq j, i, j \in \{1, \ldots, n\}, n \in \mathbb{N}^*\}$, for every $E \in \mathcal{P}(T)$.

Remark 3.12. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction. Then the following statements hold:

I. $|\mu(A)| \leq \overline{\mu}(A)$, for every $A \in \mathcal{C}$.

The inequality may be strict (see Example 3.13-II). It becomes an equality in supplementary hypothesis (see Proposition 3.14).

II. $\overline{\mu}(A) = 0 \Rightarrow |\mu(A)| = 0$, for every $A \in \mathcal{C}$.

III. Moreover, if $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy, then we also have:

$$|\mu(A)| = 0 \Rightarrow \overline{\mu}(A) = 0, \quad \forall A \in \mathcal{C}. \tag{2}$$

Indeed, let $\{B_i\}_{i=1}^n \subset \mathcal{C}$ be a partition of A. Since μ is fuzzy, $\sum\limits_{i=1}^{n} |\mu(B_i)| = 0$ and so, $\overline{\mu}(A) = 0$.

If μ is not fuzzy, then (2) may be false as we can see in Example 3.13-I.

Example 3.13. I. Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_h)$ be defined by:

$$\mu(A) = \begin{cases} \{0\}, & A = \emptyset \text{ or } A = T \\ [0,1] \cup \{2,3\}, & A = \{a\} \text{ or } A = \{b\}. \end{cases}$$

We have $|\mu(T)| = 0$ and $\overline{\mu}(T) = 6$.

II. Let $T = \{1, 2, 3\}$, $\mathcal{C} = \mathcal{P}(T)$, $X = \{f | f : [0, +\infty) \to [0, +\infty)\}$ and $|E| = \sup\limits_{f \in E} \|f\|_u$, for every $E \in \mathcal{P}_0(X)$, where $\|f\|_u = \sup\limits_{x \in E} |f(x)|$. Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), |\cdot|)$ be defined by

$$\mu(A) = \{\mathcal{X}_{[0,n]} | n \in A\}, \quad \forall A \in \mathcal{C},$$

where $\chi_{[0,n]}$ is the characteristic function of $[0, n]$.

In this setting we have $|\mu(T)| = 3 < 6 = \overline{\mu}(T)$.

Proposition 3.14. *If $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a fuzzy set multifunction and $A \in \mathcal{C}$ is an atom of μ, then $\overline{\mu}(A) = |\mu(A)|$.*

Proof. According to the Remark 3.12-I, we only have to prove:

$$\overline{\mu}(A) \leq |\mu(A)|. \tag{3}$$

Let $\{B_i\}_{i=1}^n \subset \mathcal{C}$ be an arbitrary partition of A, where $n \in \mathbb{N}^*$. We have two cases:

I. $\mu(B_i) = \{0\}$, for every $i \in \{1, \dots, n\}$. Then $\sum_{i=1}^{n} |\mu(B_i)| = 0 \leq |\mu(A)|$.

II. There exists $i_0 \in \{1, \dots, n\}$ so that $\mu(B_{i_0}) > \{0\}$.

Suppose without loss of generality that $\mu(B_1) > \{0\}$ and $\mu(B_2) > \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_1) = \{0\}$. Since $B_2 \subseteq A \backslash B_1$ and μ is fuzzy, it results in $\mu(B_2) = \{0\}$, which is false. It results there is a unique $i_0 \in \{1, \dots, n\}$ so that $\mu(B_{i_0}) > \{0\}$. Since $B_i \subseteq A \backslash B_{i_0}$, for every $i \in \{1, \dots, n\} \backslash \{i_0\}$ and μ is fuzzy, it follows that $\mu(B_i) = \{0\}$, for every $i \in \{1, \dots, n\} \backslash \{i_0\}$. So $\sum_{i=1}^{n} |\mu(B_i)| \leq |\mu(A)|$. Since $\{B_i\}_{i=1}^n$ is an arbitrary partition of A, it results (3). □

Proposition 3.15. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a sn-exhaustive fuzzy set multifunction. Then for every $E \in \mathcal{P}(T)$ and every $\varepsilon > 0$, there is $A \in \mathcal{C}$ so that $A \subseteq E$ and $|\mu(B \backslash A)| < \varepsilon$, for all $B \in \mathcal{C}$, $A \subseteq B \subseteq E$.*

Proof. Suppose on the contrary that there exist $E_0 \in \mathcal{P}(T)$ and $\varepsilon > 0$ so that for all $A \in \mathcal{C}$, with $A \subseteq E_0$, there is $B_0 \in \mathcal{C}$ so that $A \subseteq B_0 \subseteq E_0$ and $|\mu(B_0 \backslash A)| \geq \varepsilon$. We construct by recurrence a sequence of mutual disjoint sets $(L_n) \subset \mathcal{C}$ such that $L_n \subseteq E_0$ for every $n \in \mathbb{N}$ and $|\mu(L_n)| \geq \varepsilon$.

Suppose we obtained $L_1, L_2, \dots, L_n$ and let $K = \bigcup_{i=1}^{n} L_i$. Obviously, $K \in \mathcal{C}$ and $K \subseteq E_0$. Then there is $B \in \mathcal{C}$ so that $K \subseteq B \subseteq E_0$ and $|\mu(B \backslash K)| \geq \varepsilon$. If we set $L_{n+1} = B \backslash K$, then we have $L_{n+1} \in \mathcal{C}$, $L_{n+1} \subseteq E_0$, $|\mu(L_{n+1})| \geq \varepsilon$ and $L_{n+1} \cap L_i = \emptyset$, for every $i \in \{1, \dots, n\}$. Since μ is sn-exhaustive, we have $\lim_{n\to\infty} |\mu(L_n)| = 0$, that is a contradiction. □

Definition 3.16. Suppose $|\cdot|$ is a monotone set-norm on $(\mathcal{P}_0(X), \leq)$ and let $\mathcal{R}$ be a non-empty subset of $\mathcal{P}_0(X)$.

(i) A net $(Y_i) \subset \mathcal{P}_0(X)$ is called *sn-Cauchy* if the net $(|Y_i|)$ is Cauchy (i.e., for every $\varepsilon > 0$, there is i_ε such that $||Y_i| - |Y_j|| < \varepsilon$, for every $i, j \geq i_\varepsilon$).

(ii) A net $(Y_i) \subset \mathcal{R}$ is called *sn-convergent* in $\mathcal{R}$ if there is a unique $Y_0 \in \mathcal{R}$ so that $\lim_i |Y_i| = |Y_0|$. We denote this by $\lim_i Y_i = Y_0$.

(iii) $\mathcal{R}$ is called *sn-complete* if every sn-Cauchy net of $\mathcal{R}$ is convergent in $\mathcal{R}$.

Example 3.17. Let $\mathcal{R} = \{[0, x] | x \in [0, +\infty)\}$. Then $\mathcal{R}$ is a sn-complete subspace of $(\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_s)$.

Theorem 3.18. *Suppose $|\cdot|$ is a monotone set-norm on $(\mathcal{P}_0(X), \leq)$, let $\mathcal{R}$ be a sn-complete subset of $(\mathcal{P}_0(X), \leq, |\cdot|)$ and let $\mu : \mathcal{C} \to \mathcal{R}$ be an sn-exhaustive subadditive fuzzy set multifunction. Then the following statements hold:*

(i) For every $E \in \mathcal{P}(T)$, there exists $\lim_{\substack{A \in \mathcal{C} \\ A \subseteq E}} \mu(A) = \mu^*(E) \in \mathcal{R}$, where the net $(\mu(A))_{\substack{A \in \mathcal{C} \\ A \subseteq E}}$ is directed by the usual inclusion "$\subseteq$" of sets and the limit is in the sense of Definition 3.16-(ii). We obtain a set multifunction $\mu^* : \mathcal{P}(T) \to \mathcal{R}$.

(ii) $|\mu^*(A)| = |\mu(A)|, \forall A \in \mathcal{C}$.

(iii) $\forall E_1, E_2 \in \mathcal{P}(T)$, $E_1 \subseteq E_2 \Rightarrow |\mu^*(E_1)| \leq |\mu^*(E_2)|$.

(iv) μ^* *is sn-exhaustive.*

(v) If μ *is non-atomic, then* μ^* *is non-atomic.*

Proof. (i) According to Proposition 3.15, for every $E \in \mathcal{P}(T)$ and $\varepsilon > 0$, there exists $A_0 \in \mathcal{C}$ so that $A_0 \subseteq E$ and for every $A \in \mathcal{C}$ with $A_0 \subseteq A \subseteq E$, we have

$$0 \leq |\mu(A)| - |\mu(A_0)| \leq |\mu(A \backslash A_0)| < \varepsilon.$$

So $(\mu(A))_{\substack{A\in\mathcal{C}\\ A\subseteq E}}$ is sn-Cauchy in $\mathcal{R}$ and since $\mathcal{R}$ is sn-complete, the net $(\mu(A))_{\substack{A\in\mathcal{C}\\ A\subseteq E}}$ is sn-convergent in $\mathcal{R}$.

(ii) Let $A \in \mathcal{C}$. For every $\varepsilon > 0$ there is $B_0 \in \mathcal{C}$ so that $B_0 \subseteq A$ and for every $B \in \mathcal{C}$ with $B_0 \subseteq B \subseteq A$ we have $||\mu(B)| - |\mu^*(A)|| < \varepsilon$. Particularly, $||\mu(A)| - |\mu^*(A)|| < \varepsilon$ for every $\varepsilon > 0$. Hence $|\mu^*(A)| = |\mu(A)|$.

For $A = \emptyset$ it results in $|\mu^*(\emptyset)| = |\mu(\emptyset)| = 0$ which implies that $\mu^*(\emptyset) = \{0\}$.

(iii) Let $E_1, E_2 \in \mathcal{P}(T)$ be so that $E_1 \subseteq E_2$. Consider an arbitrary $\varepsilon > 0$. Then there exists $A_1 \in \mathcal{C}$ so that $A_1 \subseteq E_1$ and for every $A \in \mathcal{C}$, with $A_1 \subseteq A \subseteq E_1$, we have $||\mu(A)| - |\mu^*(E_1)|| < \frac{\varepsilon}{2}$. Particularly we have

$$||\mu^*(E_1)| - |\mu(A_1)|| < \frac{\varepsilon}{2}. \tag{4}$$

Analogously there exists $A_2 \in \mathcal{C}$ so that $A_2 \subseteq E_2$ and for every $A \in \mathcal{C}$, with $A_2 \subseteq A \subseteq E_2$ we have

$$||\mu^*(E_2)| - |\mu(A)|| < \frac{\varepsilon}{2}.$$

Let $A_0 = A_1 \cup A_2 \in \mathcal{C}$. Then $A_2 \subseteq A_0 \subseteq E_2$ and we have

$$||\mu^*(E_2)| - |\mu(A_0)|| < \frac{\varepsilon}{2}. \tag{5}$$

Now, from (4) and (5) we obtain:

$$|\mu^*(E_1)| < |\mu(A_1)| + \frac{\varepsilon}{2} \leq |\mu(A_0)| + \frac{\varepsilon}{2} < |\mu^*(E_2)| + \varepsilon$$

for every $\varepsilon > 0$. Hence $|\mu^*(E_1)| \leq |\mu^*(E_2)|$.

(iv) Let $(E_n) \subset \mathcal{P}(T)$ be so that $E_n \cap E_m = \emptyset$, for every $m \neq n$. For every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $A_n \in \mathcal{C}$ such that $A_n \subseteq E_n$ and $||\mu(A_n)| - |\mu^*(E_n)|| < \frac{\varepsilon}{2}$. Since $A_n \cap A_m = \emptyset$, for every $n \neq m$ and μ is sn-exhaustive, it results that there is $n_0 \in \mathbb{N}$ so that $|\mu(A_n)| < \frac{\varepsilon}{2}$, for every $n \geq n_0$. So we have:

$$|\mu^*(E_n)| \leq ||\mu^*(E_n)| - |\mu(A_n)|| + |\mu(A_n)| < \varepsilon, \ \ \forall n \geq n_0,$$

which proves that μ^* is sn-exhaustive.

(v) On the contrary suppose there is $E \in \mathcal{P}(T)$ an atom of μ^*. Then $\mu^*(E) > \{0\}$ and so $|\mu^*(E)| > 0$. It results that there is $A \in \mathcal{C}$ so that $A \subseteq E$ and $|\mu(A)| > 0$. Since $\mu(A) > \{0\}$ and μ is non-atomic, there exists $B \in \mathcal{C}$ such that $B \subseteq A$, $\mu(B) > \{0\}$ and $\mu(A \backslash B) > \{0\}$. Since $B \subseteq E$ and E is an atom of μ^*, it follows that $\mu^*(B) = \{0\}$ or $\mu^*(E \backslash B) = \{0\}$.

I. If $\mu^*(B) = \{0\}$, then by (ii) we have $|\mu(B)| = |\mu^*(B)| = 0$ and so $\mu(B) = \{0\}$ which is false.

II. If $\mu^*(E \backslash B) = \{0\}$, then by (ii) and (iii) we have:

$$0 < |\mu(A \backslash B)| = |\mu^*(A \backslash B)| \leq |\mu^*(E \backslash B)| = 0$$

that is a contradiction.
Consequently, μ^* is non-atomic. □

4. Conclusion

In this paper we presented properties of set-norm exhaustive set multifunctions and also some properties of atoms and pseudo-atoms of set-multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. It would be interesting to see what non-atomicity becomes in the absence of fuzzyness of μ.

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