WEIGHTED SIMILARITY MEASURE ON INTERVAL-VALUED FUZZY SETS AND ITS APPLICATION TO PATTERN RECOGNITION

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Abstract. A new approach to define the similarity measure between interval-valued fuzzy sets is presented. The proposed approach is based on a weighted measure in which the normalized similarities between lower functions and also between upper functions are combined by a weight parameter. The properties of this similarity measure are investigated. It is shown that, the proposed measure has some advantages in comparison with the commonly used similarity measures.

1. Introduction

In the real world, there are vaguely specified data values in many applications. Fuzzy set theory has been proposed to handle such vagueness by generalizing the notion of membership in a set. In a fuzzy set, each element is associated with a point-value selected from the unit interval [0,1], which is termed the grade of membership in the set [33].

The intuitionistic fuzzy sets (IFSs) [1, 2], the vague sets [14], and the interval-valued fuzzy sets [4, 15] are some generalizations of a fuzzy set. Since these generalizations can present the degrees of membership and non-membership with a degree of indeterminacy, the knowledge and semantic representation becomes more meaningful and applicable (see [1, 3, 2]). These generalizations have been widely studied and applied to a variety of areas such as decision making, pattern recognition, medical diagnostics, etc (see, e.g. [9, 11, 17, 20, 21, 24, 25, 26, 30, 31]).

An important issue related to interval-values fuzzy sets, which is very important in the applied area especially in pattern recognition, is the similarity measure between such sets. This topic has been studied by some researchers. For instance, Chen [7, 8] and Chen and Tan [10] studied several similarity measures for measuring the degree of similarity of vague sets. Li and Cheng [22] investigated the similarity measures on intuitionistic fuzzy sets and used these measures for the problem of pattern recognition. Li and Cheng's similarity measures may not be effective in some cases, and so to overcome the drawbacks of their methods, Liang and Shi [23] and Mitchell [24] proposed several new similarity measures. Numerical comparisons showed that these similarity measures are more reasonable than

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Li and Cheng's ones. Julian et al. [21] reviewed and revised the similarity measures introduced by Mitchell [24], and then, proposed a more scattered similarity measure for pattern recognition. Hung and Yang [18] presented a new method to calculate the degree of similarity between IFSs based on the Hausdorff distance (see also the approach introduced by Hung and Yang [19] based on L_p metric). Wang and Xin [28] provided the definitions of distance measures between IFSs, and then applied these measures to pattern recognition. Chachi and Taheri [6] introduced two general classes of similarity measures on IFSs, which include several commonly used similarity measures between IFSs. Dinagar and Anbalagan [12] provided an extended version of similarity measure on the type-2 fuzzy numbers. For studying some other researches about the similarity measure on interval-valued (intuitionistic) fuzzy sets, see Hwang et al. [20], Wei et al. [29], Ye [32], Zeng and Guo [34], Zeng and Li [35], and Zhang et al. [36].

In this paper, a new weighted similarity measure for interval-valued fuzzy sets is introduced. Since, the similarity measures introduced by Li and Cheng's [22], Liang and Shi's [23], Wang and Xin's [28], and Zeng and Guo's [34] can not be effective in some cases, we compare our proposed method with the similarity measures for pattern recognition. Numerical comparisons show that our proposed similarity measure is more reasoned.

The paper is organized as follows: In Section 2, we review some preliminary concepts about interval-valued fuzzy sets. In Section 3, we introduce a new weighted similarity measure between interval-valued fuzzy sets when the universal set is as discrete or continuous. In Section 4, we compare our method with some other methods. Application of the proposed method to pattern recognition is studied in Section 5, indicating the performance of the proposed method. Finally, a brief conclusion is provided in Section 6.

2. Preliminary Concepts

In the following, we review some notations and preliminary concepts of interval-valued fuzzy sets. For more details, the reader is referred to Atanassov [1, 3, 2] and Atanassov and Gargov [4].

Definition 2.1. An interval-valued fuzzy set (IVFS) $\widetilde{A}$ on the universal set X is defined as

$$\widetilde{A} = \{(x, [\mu_{\widetilde{A}}(x), 1 - \nu_{\widetilde{A}}(x)])|x \in X\},$$

where, $\mu_{\widetilde{A}}(.) : X \to [0, 1]$ is the "degree of membership", $\nu_{\widetilde{A}}(.) : X \to [0, 1]$ is the "degree of nonmembership", and $0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1$ for all $x \in X$. Also, the value $\tau_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x) - \nu_{\widetilde{A}}(x)$ is called the "degree of indeterminacy" of the element $x \in X$ to the IVFS $\widetilde{A}$.

In the above definition, $\mu_{\widetilde{A}}(x)$ and $1 - \nu_{\widetilde{A}}(x)$ are the lower and upper bounds for degree of membership of x into $\widetilde{A}$. Therefore, the degree of membership of x into the interval-valued fuzzy set $\widetilde{A}$ is characterized by the interval $[\mu_{\widetilde{A}}(x), 1 - \nu_{\widetilde{A}}(x)]$ (see [4, 16, 15, 27]).

Generally, the idea of interval-valued fuzzy sets was attributed to Gorzalczany [16] and Turksen [27]. Some well-known generalizations of a fuzzy set are, the so-called intuitionistic fuzzy set, introduced by Atanassov [1, 2], and the vague set, defined by Gau and Buehrer [14]. Essentially, these three approaches are not independent and there exist relationship among them. Sometimes, they are even mathematically equivalent, however they have arisen on different ground and they have different semantics (for more details, see [1, 2, 4, 5, 14]).

Note that, in a special case, if for each $x \in X$, $\nu_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x)$, then the interval-valued fuzzy set $\widetilde{A}$ is reduced to a fuzzy set with the membership function $\mu_{\widetilde{A}}(.)$.

Definition 2.2. [19] If $\widetilde{A}$ and $\widetilde{B}$ are two IVFSs on X, then

i) $\widetilde{A} \subseteq \widetilde{B}$ if and only if $\forall\, x \in X$, $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x)$ and $\nu_{\widetilde{A}}(x) \geq \nu_{\widetilde{B}}(x)$,

ii) $\widetilde{A} = \widetilde{B}$ if and only if $\forall\, x \in X$, $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)$ and $\nu_{\widetilde{A}}(x) = \nu_{\widetilde{B}}(x)$.

3. A New Weighted Similarity Measure

In this section, we introduce a new weighted similarity measure between the interval-valued fuzzy sets. We also state and prove the properties of the proposed definition.

Definition 3.1. Let $\widetilde{A}$ and $\widetilde{B}$ be two IVFSs of the universal set X. The weighted similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as follows:

i) Discrete case: Let $X = \{x_1, x_2, ..., x_n\}$. We first introduce the sum of lower bounds and the sum of upper bounds of subintervals $[\mu_{\widetilde{A}}(x_i), 1 - \nu_{\widetilde{A}}(x_i)]$ and $[\mu_{\widetilde{B}}(x_i), 1 - \nu_{\widetilde{B}}(x_i)]$, $i = 1, 2, ..., n$, as follows

$$ST_\mu = \sum_{i=1}^n \left[\mu_{\widetilde{A}}(x_i) + \mu_{\widetilde{B}}(x_i)\right], \; ST_\nu = \sum_{i=1}^n \left[(1 - \nu_{\widetilde{A}}(x_i)) + (1 - \nu_{\widetilde{B}}(x_i))\right].$$

Also, the proportions of the end points for each $i = 1, 2, ..., n$ are given by

$$\begin{aligned}
\varphi_\mu(x_i) &= \frac{|\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)|}{ST_\mu}, \\
\varphi_\nu(x_i) &= \frac{|(1 - \nu_{\widetilde{A}}(x_i)) - (1 - \nu_{\widetilde{B}}(x_i))|}{ST_\nu} = \frac{|\nu_{\widetilde{A}}(x_i) - \nu_{\widetilde{B}}(x_i)|}{ST_\nu}.
\end{aligned}$$

Then, the weighted similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as follows

$$\begin{aligned}
S_w(\widetilde{A}, \widetilde{B}) &= 1 - \left[w \cdot \sum_{i=1}^n \varphi_\mu(x_i) + (1 - w) \cdot \sum_{i=1}^n \varphi_\nu(x_i)\right] \\
&= 1 - w \cdot \frac{\sum_{i=1}^n |\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)|}{ST_\mu} - (1 - w) \cdot \frac{\sum_{i=1}^n |\nu_{\widetilde{A}}(x_i) - \nu_{\widetilde{B}}(x_i)|}{ST_\nu} \\
&= 1 - w \cdot \frac{\sum_{i=1}^n |\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)|}{\sum_{i=1}^n \mu_{\widetilde{A}}(x_i) + \sum_{i=1}^n \mu_{\widetilde{B}}(x_i)} - (1 - w) \cdot \frac{\sum_{i=1}^n |\nu_{\widetilde{A}}(x_i) - \nu_{\widetilde{B}}(x_i)|}{\sum_{i=1}^n (1 - \nu_{\widetilde{A}}(x_i)) + \sum_{i=1}^n (1 - \nu_{\widetilde{B}}(x_i))}.
\end{aligned}$$

ii) Continuous case: Let $X = R$. Then, in a similar way to the discrete case, the weighted similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as follows (subject to existence of integrals)

$$\begin{aligned}
S_w(\widetilde{A}, \widetilde{B}) &= 1 - \left[w \cdot \int_{-\infty}^{\infty} \varphi_\mu(x)dx + (1 - w) \cdot \int_{-\infty}^{\infty} \varphi_\nu(x)dx\right] \\
&= 1 - w \cdot \frac{\int_{-\infty}^{\infty} |\mu_{\widetilde{A}}(x) - \mu_{\widetilde{B}}(x)|dx}{ST_\mu} - (1 - w) \cdot \frac{\int_{-\infty}^{\infty} |\nu_{\widetilde{A}}(x) - \nu_{\widetilde{B}}(x)|dx}{ST_\nu} \\
&= 1 - w \cdot \frac{\int_{-\infty}^{\infty} |\mu_{\widetilde{A}}(x) - \mu_{\widetilde{B}}(x)|dx}{\int_{-\infty}^{\infty} \mu_{\widetilde{A}}(x)dx + \int_{-\infty}^{\infty} \mu_{\widetilde{B}}(x)dx} - (1 - w) \cdot \frac{\int_{-\infty}^{\infty} |\nu_{\widetilde{A}}(x) - \nu_{\widetilde{B}}(x)|dx}{\int_{-\infty}^{\infty} (1 - \nu_{\widetilde{A}}(x))dx + \int_{-\infty}^{\infty} (1 - \nu_{\widetilde{B}}(x))dx},
\end{aligned}$$

where, $0 \leq w \leq 1$.

Remark 3.2. In Definition 3.1, we have two similarity measures on the lower bounds $\mu_{\widetilde{A}}(\cdot)$ and $\mu_{\widetilde{B}}(\cdot)$, and also on the upper bounds $1-\nu_{\widetilde{A}}(\cdot)$ and $1-\nu_{\widetilde{B}}(\cdot)$. We can control these similarity measures with the weight value w. It is obvious that the selection of w is more or less subjective, and it depends on the users opinion. The greater value of w means the more importance for the lower bounds.

Proposition 3.3. *The mapping S_w on $IVFS \times IVFS$ satisfies the following properties:*

P1: $0 \leq S_w(\widetilde{A}, \widetilde{B}) \leq 1$,
P2: $S_w(\widetilde{A}, \widetilde{B}) = 1$ *if and only if* $\widetilde{A} = \widetilde{B}$,
P3: $S_w(\widetilde{A}, \widetilde{B}) = S_w(\widetilde{B}, \widetilde{A})$,
P4: *If* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, *then* $S_w(\widetilde{A}, \widetilde{C}) \leq \min\{S_w(\widetilde{A}, \widetilde{B}), S_w(\widetilde{B}, \widetilde{C})\}$.

Proof. The properties $P1$, $P2$, and $P3$ are obviously held. We only prove the property $P4$ for the case $X = R$ (for discrete case, a similar proof can be used). Since $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, based on Definition 2.2, for each $x \in X$, we have $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x) \leq \mu_{\widetilde{C}}(x)$ and $\nu_{\widetilde{A}}(x) \geq \nu_{\widetilde{B}}(x) \geq \nu_{\widetilde{C}}(x)$, and, then

$$\int_{-\infty}^{\infty} \mu_{\widetilde{A}}(x)dx \leq \int_{-\infty}^{\infty} \mu_{\widetilde{B}}(x)dx \leq \int_{-\infty}^{\infty} \mu_{\widetilde{C}}(x)dx,$$

and

$$\int_{-\infty}^{\infty} \nu_{\widetilde{A}}(x)dx \geq \int_{-\infty}^{\infty} \nu_{\widetilde{B}}(x)dx \geq \int_{-\infty}^{\infty} \nu_{\widetilde{C}}(x)dx.$$

Hence, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |\mu_{\widetilde{A}}(x) - \mu_{\widetilde{C}}(x)|dx &= \int_{-\infty}^{\infty} \left(\mu_{\widetilde{C}}(x) - \mu_{\widetilde{A}}(x)\right) dx \\ &\geq \int_{-\infty}^{\infty} \left(\mu_{\widetilde{C}}(x) - \mu_{\widetilde{B}}(x)\right) dx \\ &= \int_{-\infty}^{\infty} |\mu_{\widetilde{C}}(x) - \mu_{\widetilde{B}}(x)|dx. \end{aligned}$$

But, $\int_{-\infty}^{\infty} \mu_{\widetilde{A}}(x)dx + \int_{-\infty}^{\infty} \mu_{\widetilde{C}}(x)dx \leq \int_{-\infty}^{\infty} \mu_{\widetilde{B}}(x)dx + \int_{-\infty}^{\infty} \mu_{\widetilde{C}}(x)dx$, and so

$$I_1 = \frac{\int_{-\infty}^{\infty} |\mu_{\widetilde{A}}(x) - \mu_{\widetilde{C}}(x)|dx}{\int_{-\infty}^{\infty} \mu_{\widetilde{A}}(x)dx + \int_{-\infty}^{\infty} \mu_{\widetilde{C}}(x)dx} \geq \frac{\int_{-\infty}^{\infty} |\mu_{\widetilde{C}}(x) - \mu_{\widetilde{B}}(x)|dx}{\int_{-\infty}^{\infty} \mu_{\widetilde{B}}(x)dx + \int_{-\infty}^{\infty} \mu_{\widetilde{C}}(x)dx} = I_2. \tag{1}$$

Similarly, since for each $x \in X$, $1 - \nu_{\widetilde{A}}(x) \leq 1 - \nu_{\widetilde{B}}(x) \leq 1 - \nu_{\widetilde{C}}(x)$, the similar relations are held for the nonmembership functions, as follows:

$$I_1' = \frac{\int_{-\infty}^{\infty} |\nu_{\widetilde{A}}(x) - \nu_{\widetilde{C}}(x)|dx}{\int_{-\infty}^{\infty} (1-\nu_{\widetilde{A}}(x))dx + \int_{-\infty}^{\infty} (1-\mu_{\widetilde{C}}(x))dx} \geq \frac{\int_{-\infty}^{\infty} |\nu_{\widetilde{C}}(x) - \nu_{\widetilde{B}}(x)|dx}{\int_{-\infty}^{\infty} (1-\nu_{\widetilde{B}}(x))dx + \int_{-\infty}^{\infty} (1-\nu_{\widetilde{C}}(x))dx} = I_2'. \tag{2}$$

Based on the relations (1) and (2), we can obtain the following relation between $S_w(\widetilde{A}, \widetilde{C})$ and $S_w(\widetilde{B}, \widetilde{C})$ as

$$S_w(\widetilde{A}, \widetilde{C}) = 1 - wI_1 - (1-w)I_1' \leq 1 - wI_2 - (1-w)I_2' = S_w(\widetilde{B}, \widetilde{C}).$$

Similarly, we can show that $S_w(\widetilde{A}, \widetilde{C}) \leq S_w(\widetilde{A}, \widetilde{B})$, and the proof is complete. □

Remark 3.4. It should be mentioned that, the most common similarity measures introduced by authors satisfy the properties **P1**, **P3**, **P4**, and a weak version of **P2** as "**P2′)** if $\widetilde{A} = \widetilde{B}$, then $S(\widetilde{A}, \widetilde{B}) = 1$". For instance, see [22, 23, 28, 34]. Note that, as we will explain in the next section, considering **P2′** instead of **P2** may lead to some mistakes in practical applications.

Remark 3.5. In Definition 3.1, if $\nu_{\widetilde{A}}(.) = 1 - \mu_{\widetilde{A}}(.)$ and $\nu_{\widetilde{B}}(.) = 1 - \mu_{\widetilde{B}}(.)$, then, two interval-valued fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ are reduced to two fuzzy sets. In such a case, the weighted similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is reduced as follows

$$S_w(\widetilde{A}, \widetilde{B}) = \begin{cases} 1 - \frac{\sum_{i=1}^{n} |\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)|}{\sum_{i=1}^{n} \mu_{\widetilde{A}}(x_i) + \sum_{i=1}^{n} \mu_{\widetilde{B}}(x_i)}, & if\ X = \{x_1, x_2, ..., x_n\} \\ \\ 1 - \frac{\int_{-\infty}^{\infty} |\mu_{\widetilde{A}}(x) - \mu_{\widetilde{B}}(x)| dx}{\int_{-\infty}^{\infty} \mu_{\widetilde{A}}(x) dx + \int_{-\infty}^{\infty} \mu_{\widetilde{B}}(x) dx}. & if\ X = R \end{cases}$$

Definition 3.6. [22] Assume that there exist n patterns in the universal set X, which are represented by IVFSs $\widetilde{A}_i$, $i = 1, 2, ..., n$, and there is a sample represented by IVFS $\widetilde{B}$ in X. Suppose that

$$S_w(\widetilde{A}_k, \widetilde{B}) = \max_{i=1,...,n} \{S_w(\widetilde{A}_i, \widetilde{B})\}.$$

Then, we decide that the sample $\widetilde{B}$ should belong to the pattern $\widetilde{A}_k$. This principle is called "principle of the maximum degree of similarity", and is commonly used in pattern recognition based on IVFSs.

4. Comparison Studies

In this section, we compare our method with some other works on the similarity measures of the interval-value (intuitionistic) fuzzy sets.

4.1. Comparison with Li and Cheng's Approach.

Let $\widetilde{A}$ and $\widetilde{B}$ be two IVFSs on the universal set X. Li and Cheng [22] based on the median values $\varphi_{\widetilde{A}}(i) = (\mu_{\widetilde{A}}(x_i) + 1 - \nu_{\widetilde{A}}(x_i))/2$ and $\varphi_{\widetilde{B}}(i) = (\mu_{\widetilde{B}}(x_i) + 1 - \nu_{\widetilde{B}}(x_i))/2$, $i = 1, 2, ..., n$, introduced a similarity measure between $\widetilde{A}$ and $\widetilde{B}$ as follows ($p \geq 1$)

$$S^p_{LC}(\widetilde{A}, \widetilde{B}) = \begin{cases} 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} \left(\varphi_{\widetilde{A}}(i) - \varphi_{\widetilde{B}}(i)\right)^p}, & if\ X = \{x_1, x_2, ..., x_n\} \\ \\ 1 - \frac{1}{\sqrt[p]{b-a}} \sqrt[p]{\int_a^b \left(\varphi_{\widetilde{A}}(x) - \varphi_{\widetilde{B}}(x)\right)^p dx}. & if\ X = [a, b] \end{cases}$$

Our proposed similarity measure has the following advantages over the Li and Cheng's one.

The similarity measure introduced by Li and Cheng is based on the median values $\varphi_{\widetilde{A}}(i)$ and $\varphi_{\widetilde{B}}(i)$, $i = 1, ..., n$. Now, if median values for each subintervel $[\mu_{\widetilde{A}}(x_i), 1 - \nu_{\widetilde{A}}(x_i)]$ and $[\mu_{\widetilde{B}}(x_i), 1 - \nu_{\widetilde{B}}(x_i)]$ are equal, then the similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is equal to 1. Hence, in such cases, we can not distinguish different similarities. But, our method is based on the proportions of end points of subintervals in IVFSs (i.e. $\varphi_\mu(x_i) = \frac{|\mu_{\widetilde{A}}(x_i)-\mu_{\widetilde{B}}(x_i)|}{ST_\mu}$ and $\varphi_\nu(x_i) = \frac{|\nu_{\widetilde{A}}(x_i)-\nu_{\widetilde{B}}(x_i)|}{ST_\nu}$). Since ST_μ and ST_ν are depended on the degrees of membership and non-membership of IVFSs, we obtain different results, while the median values of IVFSs are the same (see Example 4.1).

Also, based on $S^p_{LC}(\widetilde{A}, \widetilde{B})$, we may have $\widetilde{A} \neq \widetilde{B}$, but $S^p_{LC}(\widetilde{A}, \widetilde{B}) = 1$. In fact, the definition proposed by Li and Cheng obeys the weak property **P2′** instead of **P2**.

Example 4.1. [23] Suppose that there are three patterns denoted by IFVSs on $X = \{x_1, x_2, x_3\}$ as follows:

$$\widetilde{A}_1 = \{(x_1, [0.3, 0.7]), (x_2, [0.2, 0.8]), (x_3, [0.1, 0.9])\};$$
$$\widetilde{A}_2 = \{(x_1, [0.2, 0.8]), (x_2, [0.2, 0.8]), (x_3, [0.2, 0.8])\};$$
$$\widetilde{A}_3 = \{(x_1, [0.4, 0.6]), (x_2, [0.4, 0.6]), (x_3, [0.4, 0.6])\}.$$

Assume that a sample $\widetilde{B} = \{(x_1, [0.3, 0.7]), (x_2, [0.2, 0.8]), (x_3, [0.1, 0.9])\}$ is given. According to Li and Cheng's definition, $S^p_{LC}(\widetilde{A}_1, \widetilde{B}) = 1$, $S^p_{LC}(\widetilde{A}_3, \widetilde{B}) = 1$, and $S^p_{LC}(\widetilde{A}_2, \widetilde{B}) = 1$. It is clear that the sample $\widetilde{B}$ is similar to the pattern $\widetilde{A}_1$, but the correct result is not obtained based on this definition. According to our proposed definition with $w = \frac{1}{2}$, $S_w(\widetilde{A}_1, \widetilde{B}) = 1$, $S_w(\widetilde{A}_3, \widetilde{B}) = 0.8958$, and $S_w(\widetilde{A}_2, \widetilde{B}) = 0.7619$. Hence, the sample $\widetilde{B}$ belongs to the pattern $\widetilde{A}_1$.

4.2. **Comparison with Liang and Shi's Approaches.** In the following, we compare our method with two methods introduced by Liang and Shi [23].

4.2.1. **Liang and Shi's First Approach.** The first similarity measure on $X = \{x_1, x_2, ..., x_n\}$ by Liang and Shi is introduced as

$$S^p_{LS1}(\widetilde{A}, \widetilde{B}) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} (\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i))^p}, \ p \geq 1$$

where, $\varphi_{\mu_{AB}}(i) = |\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)|/2$ and $\varphi_{\nu_{AB}}(i) = |(1-\nu_{\widetilde{A}}(x_i)) - (1-\nu_{\widetilde{B}}(x_i))|/2$.

Liang and Shi in this definition used the differences of two end points of the intervals in IVFSs. Hence, if the difference values of intervals between IVFSs are equal, since the denominators of $\varphi_{\mu_{AB}}(.)$ and $\varphi_{\nu_{AB}}(.)$ are fixed, then similarity measures between IVFSs are equal, and we can not obtain the correct result. But, in our approach, we use ST_μ and ST_ν in the denominators of $\varphi_\mu(x_i)$ and $\varphi_\nu(x_i)$, hence, the similarity measures between IVFSs are different (see Example 4.2).

Also, note that the definition proposed by Liang and Shi obeys the weak property **P2′**, but it does not satisfies the property **P2**.

4.2.2. **Liang and Shi's Second Approach.** Liang and Shi [23] also introduced another definition of similarity measure on $X = \{x_1, x_2, ..., x_n\}$. They first defined the median values of the IVF sets as $m_{\widetilde{A}}(x_i) = (\mu_{\widetilde{A}}(x_i) + 1 - \nu_{\widetilde{A}}(x_i))/2$ and $m_{\widetilde{B}}(x_i) = (\mu_{\widetilde{B}}(x_i) + 1 - \nu_{\widetilde{B}}(x_i))/2$, $i = 1, 2, ..., n$. Hence, the interval $[\mu_{\widetilde{A}}(x_i), 1 - \nu_{\widetilde{A}}(x_i)]$ is divided into two subintervals $[\mu_{\widetilde{A}}(x_i), m_{\widetilde{A}}(x_i)]$ and $[m_{\widetilde{A}}(x_i), 1 - \nu_{\widetilde{A}}(x_i)]$ (in a similar way, we obtain $[\mu_{\widetilde{B}}(x_i), m_{\widetilde{B}}(x_i)]$ and $[m_{\widetilde{B}}(x_i), 1-\nu_{\widetilde{B}}(x_i)]$). The median values of these subintervals are obtained as follows

$$m_{A1}(x_i)=\frac{\mu_{\widetilde{A}}(x_i)+m_{\widetilde{A}}(x_i)}{2},\quad m_{A2}(x_i)=\frac{m_{\widetilde{A}}(x_i)+1-\nu_{\widetilde{A}}(x_i)}{2},$$
$$m_{B1}(x_i)=\frac{\mu_{\widetilde{B}}(x_i)+m_{\widetilde{B}}(x_i)}{2},\quad m_{B2}(x_i)=\frac{m_{\widetilde{B}}(x_i)+1-\nu_{\widetilde{B}}(x_i)}{2}.$$

Then, the similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as follows

$$S^p_{LS2}(\widetilde{A},\widetilde{B})=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\varphi_{s1}(i)+\varphi_{s2}(i))^p},\ p\geq 1$$

where, for each $i=1,2,...,n$

$$\varphi_{s1}(i)=\frac{|m_{A1}(x_i)-m_{B1}(x_i)|}{2},\ \varphi_{s2}(i)=\frac{|m_{A2}(x_i)-m_{B2}(x_i)|}{2}.$$

In this definition, if the median values of subintervals of IVFSs are equal, since the denominators of $\varphi_{s1}(.)$ and $\varphi_{s2}(.)$ are fixed, then similarity measures between IVFSs are equal, and we can not obtain the correct result. But in such situations, since we use ST_μ and ST_ν in the denominators of $\varphi_\mu(.)$ and $\varphi_\nu(.)$, the results of similarity measures between IVFSs are different (see Example 4.2). Similar to the first definition of Liang and Shi, the second definition of Liang and Shi does not obey property **P2**.

Example 4.2. [23] Assume that there are two patterns denoted by IFVSs on $X=\{x_1,x_2,x_3\}$ as follows:

$$\widetilde{A}_1=\{(x_1,[0.2,0.8]),(x_2,[0.2,0.8]),(x_3,[0.2,0.8])\};$$
$$\widetilde{A}_2=\{(x_1,[0.4,0.6]),(x_2,[0.4,0.6]),(x_3,[0.4,0.6])\}.$$

Assume that a sample $\widetilde{B}=\{(x_1,[0.3,0.7]),(x_2,[0.3,0.7]),(x_3,[0.3,0.7])\}$ is given. According to the above definitions of Liang and Shi, $S^p_{LS1}(\widetilde{A}_1,\widetilde{B})=S^p_{LS1}(\widetilde{A}_2,\widetilde{B})$ and $S^p_{LS2}(\widetilde{A}_1,\widetilde{B})=S^p_{LS2}(\widetilde{A}_2,\widetilde{B})$. Hence, we can not make a decision about the sample $\widetilde{B}$ based on these definitions. But, based on our definition of similarity measure for the different values of w, we obtain the different values of $S_w(\widetilde{A}_1,\widetilde{B})$ and $S_w(\widetilde{A}_2,\widetilde{B})$. For example, if $w=\frac{1}{2}$ then, $S_w(\widetilde{A}_1,\widetilde{B})=0.8667$ and $S_w(\widetilde{A}_2,\widetilde{B})=0.8901$. Hence, the sample $\widetilde{B}$ belongs to the pattern $\widetilde{A}_2$.

4.3. **Comparison with Zeng and Guo's Approach.** Zeng and Guo [34] introduced a similarity measure between two IVFs on a discrete universal set $X=\{x_1,x_2,...,x_n\}$ based on some normalized distances. They defined the similarity measure between two IVFs $\widetilde{A}$ and $\widetilde{B}$ as follows

$$S_{ZG}(\widetilde{A},\widetilde{B})=\frac{f(d(\widetilde{A},\widetilde{B}))-f(1)}{f(0)-f(1)},$$

where, $f:[0,1]\to[0,1]$ is a strictly monotone decreasing function, and $d(.,.)$ is some normalized distances from Atanassov [2] as follows

i) Normalized Hamming distance:

$$d_1(\widetilde{A},\widetilde{B}) = \frac{1}{2n}\sum_{i=1}^{n}\left(|\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)| + |\nu_{\widetilde{A}}(x_i) - \nu_{\widetilde{B}}(x_i)|\right),$$

ii) Normalized Euclidean distance:

$$d_2(\widetilde{A},\widetilde{B}) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left((\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i))^2 + (\nu_{\widetilde{A}}(x_i) - \nu_{\widetilde{B}}(x_i))^2\right)}.$$

They also introduced some operations for $f(.)$ as follows:

$$\begin{aligned} f_1(x) &= 1-x, \\ f_2(x) &= 1-x^2, \\ f_3(x) &= 1/(1+x), \\ f_4(x) &= e^{-x}. \end{aligned}$$

This definition used the differences of the end points of subintervals in IVFSs for calculating the normalized distance $d(.,.)$. If the difference values between IVFSs are equal, then the normalized distance between IVFSs are equal, and hence, the similarity measures between IVFSs are similar. So, we can not obtain the correct result. But, in our approach, we use ST_μ and ST_ν in the denominators of $\varphi_\mu(x_i)$ and $\varphi_\nu(x_i)$, hence, the similarity measures between IVFSs are different (see Example 4.3).

Note that, based on $S_{ZG}(\widetilde{A},\widetilde{B})$, we may have $\widetilde{A} \neq \widetilde{B}$, but $S^p_{LC}(\widetilde{A},\widetilde{B}) = 1$. Hence, the definition proposed by Zeng and Guo does not satisfy the property **P2**.

Example 4.3. Consider two patterns $\widetilde{A}_1$ and $\widetilde{A}_2$, and the sample $\widetilde{B}$ in Example 4.2. According to the above definitions of Zeng and Guo, since $d_1(\widetilde{A}_1,\widetilde{B}) = d_1(\widetilde{A}_2,\widetilde{B}) = 0.1$ and $d_2(\widetilde{A}_1,\widetilde{B}) = d_2(\widetilde{A}_2,\widetilde{B}) = 0.1$, the degrees of similarity measure $S_{ZG}(\widetilde{A}_1,\widetilde{B})$ and $S_{ZG}(\widetilde{A}_2,\widetilde{B})$ are equal for each $d_i(.,.)$, $i = 1,2$, and each $f_j(.)$, $j = 1,..,4$. Hence, we can not make a decision about the sample $\widetilde{B}$ based on this definition. But, based on our definition of similarity measure, we obtain the different values of $S_w(\widetilde{A}_1,\widetilde{B})$ and $S_w(\widetilde{A}_2,\widetilde{B})$. For example, if $w = \frac{1}{2}$ then, $S_w(\widetilde{A}_1,\widetilde{B}) = 0.8667$ and $S_w(\widetilde{A}_2,\widetilde{B}) = 0.8901$. Hence, the sample $\widetilde{B}$ belongs to the pattern $\widetilde{A}_2$.

4.4. **Comparison with Farhadinia's Approach.** Farhadinia [13] introduced a similarity measure between two IVFs using a special distance on the discrete universal set $X = \{x_1, x_2, ..., x_n\}$. This distance is introduced as follows

$$d_{IVF}(\widetilde{A},\widetilde{B}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{m+1}\sum_{j=0}^{m}[\chi_j(\widetilde{A}(x_i)) - \chi_j(\widetilde{B}(x_i))]^2\right)}$$

where $\chi_j(\widetilde{A}(x_i))$ (and $\chi_j(\widetilde{B}(x_i))$) is the convex combination of the lower and upper bound values of interval $[\mu_{\widetilde{A}}(x_i), 1-\nu_{\widetilde{A}}(x_i)]$. Finally, by inception of the Zeng and

Guo's approach [34], the similarity measure between two IVFs $\widetilde{A}$ and $\widetilde{B}$ is given by $S_F^d(\widetilde{A},\widetilde{B}) = 1 - d_{IVF}(\widetilde{A},\widetilde{B})$.

The similarity measure introduced by Farhadinia is based on the convex combination of lower and upper bound values $\chi_j(.)$, $j = 0, 1, ..., m$. Now, if the difference values of lower and upper bounds between IVFs are equal (i.e. $\mu_{\widetilde{A}}(x_i) = c_i + \mu_{\widetilde{B}}(x_i)$ and $1-\nu_{\widetilde{A}}(x_i) = c_i + 1 - \nu_{\widetilde{B}}(x_i)$, for some constant values c_i, $i = 1, 2, ..., n$), then, the similarity measure between these IVFs would be the same. Hence, in such cases, we can not distinguish different similarities. But, since in our method ST_μ and ST_ν are depended on the degrees of membership and non-membership of IVFSs, we obtain different results (see Example 4.4).

Note that, while our approach is introduced both for discrete and continuous cases, the Farhadinia's approach is introduced just for discrete case.

Example 4.4. Suppose that there are two patterns denoted with IFVSs on $X = \{x_1, x_2, x_3\}$ as follows:

$$\widetilde{A}_1 = \{(x_1, [0.1, 0.2]), (x_2, [0.2, 0.5]), (x_3, [0.2, 0.4])\};$$
$$\widetilde{A}_2 = \{(x_1, [0.5, 0.6]), (x_2, [0.6, 0.9]), (x_3, [0.8, 1.0])\}.$$

Assume that a sample $\widetilde{B} = \{(x_1, [0.3, 0.4]), (x_2, [0.4, 0.7]), (x_3, [0.5, 0.7])\}$ is given. According to Farhadinia's definition, $S_F^d(\widetilde{A}_1,\widetilde{B}) = S_F^d(\widetilde{A}_2,\widetilde{B}) = 0.7620$. Hence, we can not take a correct decision about the sample $\widetilde{C}$ based on this definition. But, based on our definition of similarity measure, we obtain $S_w(\widetilde{A}_1,\widetilde{B}) < S_w(\widetilde{A}_2,\widetilde{B})$ for each $w \in [0, 1]$. Hence, the sample $\widetilde{B}$ belongs to the pattern $\widetilde{A}_2$.

4.5. **Comparison with Wang and Xin's Approach.** Wang and Xin [28] introduced a similarity measure between two IVFs on the discrete universal set $X = \{x_1, x_2, ..., x_n\}$ based on a certain distance measure. They first introduced a distance measure between $\widetilde{A}$ and $\widetilde{B}$ as follows

$$\begin{aligned} d_{WX}(\widetilde{A},\widetilde{B}) &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\Big[\tfrac{|\mu_{\widetilde{A}}(x_i)-\mu_{\widetilde{B}}(x_i)|+|\nu_{\widetilde{A}}(x_i)-\nu_{\widetilde{B}}(x_i)|}{4} \\ &\qquad + \tfrac{\max(|\mu_{\widetilde{A}}(x_i)-\mu_{\widetilde{B}}(x_i)|,|\nu_{\widetilde{A}}(x_i)-\nu_{\widetilde{B}}(x_i)|)}{2}\Big]. \end{aligned}$$

Then, the similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as $S_{WX}(\widetilde{A},\widetilde{B}) = 1 - d_{WX}(\widetilde{A},\widetilde{B})$.

This definition is similar to the definition of similarity measure introduced by Zeng and Guo [34] in Subsection 4.3 for the function $f_1(x) = 1 - x$. But, Wang and Xin in this definition have added the following expression to the normalized Hamming distance $d_1(\widetilde{A},\widetilde{B})$:

$$\frac{\max(|\mu_{\widetilde{A}}(x_i) - \mu_{\widetilde{B}}(x_i)|, |\nu_{\widetilde{A}}(x_i) - \nu_{\widetilde{B}}(x_i)|)}{2}.$$

Hence, if the differences of end points of subintervals in IVFSs and the values max(., .) between this differences are similar, then we can not obtain the correct result. Note that in this definition, the denominators are fixed, but, in our approach, we use ST_μ and ST_ν in the denominators of $\varphi_\mu(x_i)$ and $\varphi_\nu(x_i)$, and hence, the similarity measures between IVFSs are different (see Example 4.5).

Note that, in Wang and Xin's approach, the weak property **P2′**, rather than **P2**, is held.

Example 4.5. Consider two patterns $\widetilde{A}_1$ and $\widetilde{A}_2$ in Example 4.2. Now, assume that a sample $\widetilde{C} = \{(x_1, [0.3, 0.7]), (x_2, [0.3, 0.7]), (x_3, [0.5, 0.9])\}$ is given. According to Wang and Xin's definition, $S_{WX}(\widetilde{A}_1, \widetilde{C}) = S_{WX}(\widetilde{A}_2, \widetilde{C}) = 0.8500$. Hence, we can not make a correct decision about the sample $\widetilde{C}$ based on Wang and Xin's definition. But, based on our definition of similarity measure for each $w < 0.2620$, $S_w(\widetilde{A}_1, \widetilde{C}) > S_w(\widetilde{A}_2, \widetilde{C})$ and for each $w > 0.2620$, $S_w(\widetilde{A}_1, \widetilde{C}) < S_w(\widetilde{A}_2, \widetilde{C})$. Hence, for $w < 0.2620$, the sample $\widetilde{C}$ belongs to the pattern $\widetilde{A}_1$, and for $w > 0.2620$, the sample $\widetilde{C}$ belongs to the pattern $\widetilde{A}_2$.

5. Numerical Examples in Pattern Recognition

In this section, we apply the proposed similarity measure to several numerical examples given by Li and Cheng [22], Liang and Shi [23] and Wang and Xin [28], and explain advantages of the proposed measure.

Example 5.1. [23] Assume that there are three patterns denoted with IFVSs on $X = \{x_1, x_2, x_3\}$ as follows:

$$\widetilde{A}_1 = \{(x_1, [0.1, 0.9]), (x_2, [0.5, 0.9]), (x_3, [0.1, 0.1])\};$$
$$\widetilde{A}_2 = \{(x_1, [0.5, 0.5]), (x_2, [0.7, 0.7]), (x_3, [0.0, 0.2])\};$$
$$\widetilde{A}_3 = \{(x_1, [0.7, 0.8]), (x_2, [0.1, 0.2]), (x_3, [0.4, 0.6])\}.$$

Assume that a sample $\widetilde{B} = \{(x_1, [0.4, 0.6]), (x_2, [0.6, 0.8]), (x_3, [0.0, 0.2])\}$ is given. Some degrees of similarity measure are obtained in Table 1. Since, $S_w(\widetilde{A}_3, \widetilde{B}) < S_w(\widetilde{A}_1, \widetilde{B}) < S_w(\widetilde{A}_2, \widetilde{B})$, the sample $\widetilde{B}$ belongs to the pattern $\widetilde{A}_2$ based on each value of w. This result is similar to that of Liang and Shi [23].

Example 5.2. [28] Given five kinds of mineral fields, each is featured by the content of six minerals and contains one kind of typical hybrid mineral. The five kinds of typical hybrid mineral are represented by IVFSs $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4, \widetilde{C}_5$ in $X = \{x_1, x_2, ..., x_6\}$, respectively. Given another kind of hybrid mineral $\widetilde{B}$, to which field does this kind of mineral $\widetilde{B}$ most possibly belong to? These IVFSs are shown in Table 2.

The degrees of similarity measure are obtained as shown in Table 3. It is seen that the pattern $\widetilde{B}$ belongs to the class $\widetilde{C}_5$ according to the principle of the maximum degree of similarity between IVFSs (Definition 3.6). This result is the same as the result of Wang and Xin [28].

Example 5.3. [22] Let two patterns be represented by IVFSs

$$\widetilde{A}_1 = \{(x, [\mu_{\widetilde{A}_1}(x), 1 - \nu_{\widetilde{A}_1}(x)]) \mid x \in [1, 5]\},$$
$$\widetilde{A}_2 = \{(x, [\mu_{\widetilde{A}_2}(x), 1 - \nu_{\widetilde{A}_2}(x)]) \mid x \in [1, 5]\},$$

	$S_w(\widetilde{A}_1,\widetilde{B})$	$S_w(\widetilde{A}_2,\widetilde{B})$	$S_w(\widetilde{A}_3,\widetilde{B})$
$w=0.1$	0.8420	0.9309	0.6080
$w=0.2$	0.8269	0.9285	0.5909
$w=0.3$	0.8118	0.9261	0.5739
$w=0.4$	0.7966	0.9236	0.5568
$w=0.5$	0.7815	0.9212	0.5398
$w=0.6$	0.7664	0.9188	0.5227
$w=0.7$	0.7513	0.9164	0.5057
$w=0.8$	0.7361	0.9139	0.4886
$w=0.9$	0.7210	0.9115	0.4716

TABLE 1. Some Degrees of Similarity in Example 5.1

	x_1	x_2	x_3	x_4	x_5	x_6
	$[\mu(.),1-\nu(.)]$	$[\mu(.),1-\nu(.)]$	$[\mu(.),1-\nu(.)]$	$[\mu(.),1-\nu(.)]$	$[\mu(.),1-\nu(.)]$	$[\mu(.),1-\nu(.)]$
$\widetilde{C}_1$	$[0.739,0.875]$	$[0.033,0.182]$	$[0.188,0.374]$	$[0.492,0.642]$	$[0.020,0.372]$	$[0.739,0.875]$
$\widetilde{C}_2$	$[0.124,0.335]$	$[0.030,0.175]$	$[0.048,0.200]$	$[0.136,0.352]$	$[0.019,0.177]$	$[0.347,0.393]$
$\widetilde{C}_3$	$[0.449,0.613]$	$[0.662,0.702]$	$[1.000,1.000]$	$[1.000,1.000]$	$[1.000,1.000]$	$[1.000,1.000]$
$\widetilde{C}_4$	$[0.280,0.285]$	$[0.521,0.632]$	$[0.470,0.577]$	$[0.295,0.342]$	$[0.188,0.194]$	$[0.735,0.882]$
$\widetilde{C}_5$	$[0.326,0.548]$	$[1.000,1.000]$	$[0.182,0.275]$	$[0.156,0.235]$	$[0.049,0.104]$	$[0.675,0.737]$
$\widetilde{B}$	$[0.629,0.697]$	$[0.524,0.644]$	$[0.210,0.311]$	$[0.218,0.247]$	$[0.069,0.124]$	$[0.658,0.744]$

TABLE 2. Six Kinds of Materials represented by IVFSs in Example 5.2

	$S_w(\widetilde{C}_1,\widetilde{B})$	$S_w(\widetilde{C}_2,\widetilde{B})$	$S_w(\widetilde{C}_3,\widetilde{B})$	$S_w(\widetilde{C}_4,\widetilde{B})$	$S_w(\widetilde{C}_5,\widetilde{B})$
$w=\frac{1}{3}$	0.7625	0.6026	0.6339	0.8219	0.8674
$w=\frac{1}{2}$	0.7650	0.5688	0.6188	0.8203	0.8523
$w=\frac{2}{3}$	0.7676	0.5350	0.6038	0.8187	0.8373

TABLE 3. Some Degrees of Similarity in Example 5.2

where

$$\mu_{\widetilde{A}_1}(x)=\begin{cases}0.8(x-1) & 1\le x<2,\\ 4(5-x)/15 & 2\le x\le 5,\end{cases}\quad \nu_{\widetilde{A}_1}(x)=\begin{cases}1.9-0.9x & 1\le x<2,\\ 0.3x-0.5 & 2\le x\le 5,\end{cases}$$

$$\mu_{\widetilde{A}_2}(x)=\begin{cases}0.2(x-1) & 1\le x<4,\\ 0.6(5-x) & 4\le x\le 5,\end{cases}\quad \nu_{\widetilde{A}_2}(x)=\begin{cases}1.3-0.3x & 1\le x<4,\\ 0.9x-3.5 & 4\le x\le 5.\end{cases}$$

Now, assume that the sample $\widetilde{B}=\{(x,[\mu_{\widetilde{B}}(x),1-\nu_{\widetilde{B}}(x)])\mid x\in[1,5]\}$ is given, where

$$\mu_{\widetilde{B}}(x)=\begin{cases}0.3(x-1) & 1\le x<3,\\ 0.3(5-x) & 3\le x\le 5,\end{cases}\quad \nu_{\widetilde{B}}(x)=\begin{cases}1.4-0.4x & 1\le x<3,\\ 0.4x-1 & 3\le x\le 5.\end{cases}$$

The weighted similarity measures between $\widetilde{B}$ and $\widetilde{A}_i$, $i=1,2$, are obtained as follows

$$\begin{aligned} S_w(\widetilde{A}_1,\widetilde{B}) &= 1-w\cdot\int_{-\infty}^{\infty}\varphi_\mu(x)dx-(1-w)\cdot\int_{-\infty}^{\infty}\varphi_\nu(x)dx\\ &= 1-w\times\frac{23}{119}-(1-w)\times\frac{23}{119}\\ &= \frac{96}{119},\\ S_w(\widetilde{A}_2,\widetilde{B}) &= 1-w\cdot\int_{-\infty}^{\infty}\varphi_\mu(x)dx-(1-w)\cdot\int_{-\infty}^{\infty}\varphi_\nu(x)dx\\ &= 1-w\times\frac{1}{5}-(1-w)\times\frac{23}{119}. \end{aligned}$$

Since, for each $w\in[0,1]$, $S_w(\widetilde{A}_2,\widetilde{B})<S_w(\widetilde{A}_1,\widetilde{B})$, then, according to the principle of the maximum degree of similarity between IVFSs (Definition 3.6), the pattern

$\widetilde{B}$ belongs to the class $\widetilde{A}_1$. This result is different from the result of Li and Cheng [22].

6. Conclusion

Similarity measure is a term that describes the difference between two objects (e.g. two fuzzy sets or two interval-valued fuzzy sets), and can be considered as a dual concept of distance measure. As an important content in fuzzy mathematics, the similarity measures between interval-valued fuzzy sets have also gained much attentions for their wide applications in real world, such as pattern recognition, machine learning, decision making and market prediction.

In this paper, we introduced a new similarity measure for interval-valued fuzzy data. This measure, by using a weight parameter, employs two similarity measure between the lower functions and between upper functions. The weight parameter makes the measure more applicable so that a user can take attention on, for example, the lower functions by considering a great value for the weight.

The properties of the introduced measure were investigated. It was shown that, having a more useful property, the proposed measure works much better than a lot of commonly used measures, specially in some pattern recognition problems.

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References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.
[2] K. Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Springer, Heidelberg, 1999.
[3] K. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61** (1994), 137-142.
[4] K. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **31** (1989), 343-349.
[5] H. Bustince and P. Burillo, *Vague sets are intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **79** (1996), 403-405.
[6] J. Chachi and S. M. Taheri, *A unified approach to similarity measures between intuitionistic fuzzy sets*, International Journal of Intelligent Systems, **28** (2013), 669-685.
[7] S. M. Chen, *Measures of similarity between vague sets*, Fuzzy Sets and Systems, **74** (1995), 217-223.
[8] S. M. Chen, *Similarity measure between vague sets and between elements.*, IEEE Trans. Systems Man Cybernt., **27** (1997), 153-158.
[9] Y. Chen and B. Li, *Dynamic multi-attribute decision making model based on triangular intuitionistic fuzzy numbers*, Scientia Iranica, **18** (2011), 268-274.
[10] S. M. Chen and J. M. Tan, *Handling multicriteria fuzzy decision-making problems based on vague set theory*, Fuzzy Sets and Systems, **67** (1994), 163-172.
[11] S. K. De, R. Biswas and A. R. Roy, *An application of intuitionistic fuzzy sets in medical diagnosis*, Fuzzy Sets and Systems, **117** (2001), 209-213.
[12] D. S. Dinagar and A. Anbalagan, *A new similarity measure between type-2 fuzzy numbers and fuzzy risk analysis*, Iranian Journal of Fuzzy Systems, **10** (2013), 79-95.
[13] B. Farhadinia, *An efficient similarity measure for intuitionistic fuzzy sets*, Soft Computing, **18** (2014), 85-94.
[14] W. L. Gau and D. J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man, and Cybernetics, **23** (1993), 610-614.
[15] M. B. Gorzalczany, *An Interval-valued fuzzy inference method - some basic properties*, Fuzzy Sets and Systems, **31** (1989), 243-251.

[16] M. B. Gorzalczany, *Approximate inference with interval-valued fuzzy sets - an outline*, In: Proceeding Polish. Symp. on Interval and Fuzzy Mathematics, Poznan, (1983), 89-95.
[17] K. W. Hipel, D. M. Kilgour and M. Abul Bashar, *Fuzzy preferences in multiple participant decision making*, Scientia Iranica, **18** (2011), 627-638.
[18] W. L. Hung and M. Sh. Yang, *Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance*, Pattern Recognition Letters, **25** (2004), 1603-1611.
[19] W. L. Hung and M. Sh. Yang, *Similarity measures of intuitionistic fuzzy sets based on L_p metric*, International Journal of Approximate Reasoning, **46** (2007), 120-136.
[20] Ch. M. Hwang, M. Sh. Yang, W. L. Hung and M. G. Lee, *A similarity measure of intuitionistic fuzzy sets based on the Sugeno integral with its application to pattern recognition*, Information Sciences, **189** (2012), 93-109.
[21] P. Julian, K. Ch. Hung and Sh. J. Lin, *On the Mitchell similarity measure and its application to pattern recognition,* Pattern Recognition Letters, **33** (2012), 1219-1223.
[22] D. Li and C. Cheng, *New similarity measures of intuitionistic fuzzy sets and application to pattern recognition*, Pattern Recognition Letters, **23** (2002), 221-225.
[23] Z. Liang and P. Shi, *Similarity measures on intuitionistic fuzzy sets*, Pattern Recognition Letters, **24** (2003), 2687-2693.
[24] H. B. Mitchell, *On the Dengfeng-Chuitian similarity measure and its application to pattern recognition*, Pattern Recognition Letters, **24** (2003), 3101-3104.
[25] M. R. Rabiei, N. R. Arghami, S. M. Taheri and B. S. Gildeh, *Least-squares approach to regression modeling in full interval-valued fuzzy environment*, Soft Computing, **18** (2014), 2043-2059.
[26] M. K. Roy and R. Biswas, *I-V fuzzy relations and Sanchez's approach for medical diagnosis*, Fuzzy Sets and Systems, **47** (1992), 35-38.
[27] B. Turksen, *Interval valued fuzzy sets based on normal forms*, Fuzzy Sets and Systems, **20** (1986), 191-210.
[28] W. Wang and X. Xin, *Distance measure between intuitionistic fuzzy sets*, Pattern Recognition Letters, **26** (2005), 2063-2069.
[29] C. P. Wei, P. Wang and Y. Zh. Zhang, *Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications*, Information Sciences, **181** (2011), 4273-4286.
[30] G. W. Wei and J. M. Merigó, *Methods for strategic decision-making problems with immediate probabilities in intuitionistic fuzzy setting*, Scientia Iranica, **19** (2012), 1936-1946.
[31] Z. Xu, *A method based on distance measure for interval-valued intuitionistic fuzzy group decision making*, Information Sciences, **180** (2010), 181-190.
[32] J. Ye, *Cosine similarity measures for intuitionistic fuzzy sets and their applications*, Mathematical and Computer Modelling, **53** (2011), 91-97.
[33] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-356.
[34] W. Zeng and P. Guo, *Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship*, Information Sciences, **178** (2008), 1334-1342.
[35] W. Zeng and H. Li, *Relationship between similarity measure and entropy of interval valued fuzzy sets*, Fuzzy Sets and Systems, **157** (2006), 1477-1484.
[36] Q. Zhang, F. Liu, L. Wu and Sh. Luo, *Information entropy, similarity measure and inclusion measure of intuitionistic fuzzy sets*, Communications in Computer and Information Science, **307** (2012), 392-398.

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