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FUZZY VECTOR EQUILIBRIUM PROBLEM

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ABSTRACT. In the present paper, we introduce and study a fuzzy vector equilibrium problem and prove some existence results with and without convexity assumptions by using some particular forms of results of *Kim* and *Lee* [W.K. Kim and K.H. Lee, Generalized fuzzy games and fuzzy equilibria, Fuzzy Sets and Systems, 122 (2001), 293-301] and *Tarafdar* [E. Tarafdar, Fixed point theorems in H-spaces and equilibrium points of abstract economies, J. Aust. Math. Soc.(Series A), 53(1992), 252-260]. An example is also constructed in support of fuzzy vector equilibrium problem.

1. Introduction

The fuzzy set theory which was initiated by *Zadeh* [15] in 1965 has emerged as an interesting and fascinating branch of pure and applied sciences. This theory has been applied in the areas of pattern recognition, artificial intelligence, optimization, decision theory, computer science and operations research [16].

The abstract economy defined by *Debreu* [5] generalized the Nash non- cooperative game and proved the existence of equilibrium in abstract economics with finitely many agents, finite dimensional strategy space and quasi-concave utility functions. Following the idea of *Borglin* and *Keiding* [2], many authors have generalized the existence of equilibria for generalized games, see e.g. [12, 6, 10]. In 1998, *Kim* and *Lee* [8] introduced the concept of a fuzzy game which was a fuzzy extension of a generalized game and proved the existence of equilibrium for 1-person fuzzy game. For related topics, we refer to [3, 9, 13].

This paper deals with the introduction and study of fuzzy vector equilibrium problem. By using some particular forms of results of *Kim* and *Lee* [8] and *Tarafdar* [11], we prove the existence of solutions for fuzzy vector equilibrium problem.

2. Preliminaries

Let X be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F}(X)$ be a collection of all fuzzy sets over X. A mapping $T : X \longrightarrow \mathcal{F}(X)$ is said to be a fuzzy mapping. For each $x \in X, T(x)$ (denote it by T_x, in the sequel) is a fuzzy set on X in $\mathcal{F}(X)$ and $T_x(y)$ is the degree of membership or membership function of point y in T_x.

A fuzzy mapping $T : X \longrightarrow \mathcal{F}(X)$ is said to be closed if for each $x \in X$, the function $y \longmapsto T_x(y)$ is upper semicontinuous i.e., for any given $\{y_\alpha\} \subset X$ satisfying

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$y_\alpha \to y_0 \in X$;

$$\limsup_{\alpha} T_x(y_\alpha) \le T_x(y_0).$$

For $B \subset \mathcal{F}(X)$ and $\lambda \in [0,1]$, the set $(B_\lambda) = \{x \in X : B(x) \ge \lambda\}$ is called a λ-cut set of B. A closed fuzzy mapping $T : X \longrightarrow \mathcal{F}(X)$ satisfying the following condition:

Condition $(*)$: If there exists a function $a : X \longrightarrow [0,1]$ such that for each $x \in X, (T_x)_{a(x)} = \{y \in X : T_x(y) \ge a(x)\}$ is nonempty bounded subset of X.

It is clear that if $T : X \longrightarrow \mathcal{F}(X)$ is closed fuzzy mapping satisfying *Condition* $(*)$, then for each $x \in X, (T_x)_{a(x)} \in CB(X)$, where $CB(X)$ denotes the family of all nonempty bounded closed subsets of X. In fact, let $\{y_\alpha\}_{\alpha\in\Gamma} \subset (T_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in X$; then $T_x(y_\alpha) \ge a(x)$, for each $\alpha \in \Gamma$.
Since T is closed, we have

$$\begin{aligned} & T_x(y_0) \ge \limsup_{\alpha\in\Gamma} T_x(y_\alpha) \ge a(x) \\ \Rightarrow\quad & T_x(y_0) \ge a(x) \\ \Rightarrow\quad & y_0 \in (T_x)_{a(x)} \\ \Rightarrow\quad & (T_x)_{a(x)} \in CB(X). \end{aligned}$$

Let $A : X \longrightarrow \mathcal{F}(X)$ be a closed fuzzy mapping satisfying *Condition* $(*)$, then there exists a function $a : X \longrightarrow [0,1]$ such that for each $x \in X, (A_x)_{a(x)} \in CB(X)$. Hence, we can define set-valued mapping $\widetilde{A} : X \longrightarrow CB(X)$ by

$$\widetilde{A}(x) = (A_x)_{a(x)}, \quad \forall x \in X.$$

In the sequel, $\widetilde{A}$ is called the set-valued mapping induced by the fuzzy mapping A.

Let K be a convex subset of X and $C : K \longrightarrow 2^X$ be a set-valued mapping such that for each $x \in K, C(x)$ is closed convex cone with $intC(x) \ne \emptyset$ where $intC(x)$ denotes the interior of $C(x)$.

Let $f : K \times K \longrightarrow X$ be a vector-valued bifunction and $A : K \longrightarrow \mathcal{F}(K)$ be a fuzzy mapping. Let $\widetilde{A} : K \longrightarrow CB(K)$ be the set-valued mapping induced by the fuzzy mapping A such that $\widetilde{A}(x) = (A_x)_{a(x)}$, for all $x \in K$ where $a : K \longrightarrow [0,1]$.

We consider the following fuzzy vector equilibrium problem (FVEP):

Find $x \in K$ such that

$$x \in cl\widetilde{A}(x) \quad and \quad f(x,y) \notin -intC(x); \quad \forall y \in \widetilde{A}(x).$$

If $A : K \longrightarrow CB(K)$ is a classical set-valued mapping, we can define the fuzzy mapping $A : K \longrightarrow \mathcal{F}(K)$ by

$$x \mapsto \chi_{A(x)}, \quad \forall x \in K,$$

where $\chi_{A(x)}$ is characteristic function of $A(x)$.
Taking $a(x) = 1$, for all $x \in X$, (FVEP) is equivalent to the following vector quasi-equilibrium problem considered by *Khaliq* and *Krishnan* [7]:

Find $x \in K$ such that

$$x \in clA(x) \quad and \quad f(x,y) \notin -intC(x); \quad \forall y \in A(x).$$

We remark that the excellent and noble work of *Witthayarat et al.* [14] and *Choudhury et al.* [4] can be extended for the vector case in fuzzy environment as we have solved fuzzy vector equilibrium problem (FVEP).

Definition 2.1. Let X and Y be two topological vector spaces and K be a nonempty convex subset of X. A multifunction $f : K \times K \longrightarrow 2^Y$ is called C_x-*quasiconvex-like* if, for all $x, y_1, y_2 \in K$, and $\lambda \in [0,1]$, we have either $f\big(x, \lambda y_1 + (1-\lambda y_2)\big) \subseteq f(x, y_1) - C(x)$ or $f\big(x, \lambda y_1 + (1-\lambda y_2)\big) \subseteq f(x, y_2) - C(x)$.

The following results are the special cases of results of [8] and can be proved easily.

Theorem 2.2. *Let $\Gamma = (K, A, P, a)$ be fuzzy game such that*

- (i) *K is a nonempty compact convex subset of a locally convex Hausdorff topological vector space X and $a : K \longrightarrow [0,1]$ is upper semicontinuous;*
- (ii) *the fuzzy mapping $A : K \longrightarrow \mathcal{F}(K)$ is such that $(A_x)_{a(x)}$ is nonempty convex for each $x \in K$ and $x \mapsto (A_x)_{a(x)}$ is upper semicontinuous and for each fixed $y \in K$, the mapping $x \mapsto (A_x)(y)$ is lower semicontinuous;*
- (iii) *the fuzzy mapping $P : K \longrightarrow \mathcal{F}(K)$ is convex such that for each $x \in K$, $x \mapsto (P_x)(y)$ is lower semicontinuous;*
- (iv) *for each $x \in K, x \notin (P_x)_{a(x)}$ i.e., $(P_x) \leq a(x)$;*

Then Γ has a fuzzy equilibrium $\bar{x} \in K$ i.e.,

$$\bar{x} \in cl(A_{\bar{x}})_{a(\bar{x})} \quad \text{and} \quad Co(A_{\bar{x}})_{a(\bar{x})} \cap (P_{\bar{x}})_{a(\bar{x})} = \emptyset,$$

where $Co(A_{\bar{x}})_{a(\bar{x})}$ denotes the convex hull of $(A_{\bar{x}})_{a(\bar{x})}$.

The following corollary is an easy consequence of Theorem 2.2.

Corollary 2.3. *Let $\Gamma = (K, A, P)$ be a game such that*

- (1) *K is a nonempty compact convex subset of a locally convex Hausdorff topological vector space X;*
- (2) *the mapping $A : K \longrightarrow 2^K$ is such that $A(x)$ is nonempty convex for each $x \in K$, A is upper semicontinuous;*
- (3) *for each $y \in K, A^{-1}(y)$ is open in K;*
- (4) *the mapping $P : K \longrightarrow 2^K$ is such that $P(x)$ is convex for each $x \in K$ and $P^{-1}(y)$ is open in K for each $y \in K$;*
- (5) *for each $x \in K, x \notin P(x)$.*

Then, Γ has an equilibrium choice $\bar{x} \in K$ i.e.,

$$\bar{x} \in clA(\bar{x}) \quad \text{and} \quad CoA(\bar{x}) \cap P(\bar{x}) = \emptyset.$$

3. **Existence Theory**

We begin this section by proving an existence result for fuzzy vector equilibrium problem (FVEP).

Theorem 3.1. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let $C : K \longrightarrow 2^X$ be such that for all $x \in K, C(x)$ is closed, convex and pointed cone in X such that $intC(x)$ is nonempty. Let $A : K \longrightarrow \mathcal{F}(K)$ be a closed fuzzy mapping satisfying Condition $(*)$, then there exists a function $a : K \longrightarrow [0,1]$ such that for each $x \in K, (A_x)_{a(x)} \in CB(K)$. Let $f : K \times K \longrightarrow X$ be a vector-valued bifunction and $\widetilde{A} : K \longrightarrow CB(K)$ be a set-valued mapping induced by the fuzzy mapping A such that $\widetilde{A}(x) = (A_x)_{a(x)}$, for all $x \in K$ and $\widetilde{A}(x)$ is* *nonempty convex for all $x \in K$, upper semicontinuous and for all $y \in K$, $\widetilde{A}^{-1}(y)$ is open in K. Suppose that the following assumptions hold:*

(i) $f(x,x) \notin -intC(x), \quad \forall x \in K$;

(ii) *f is continuous in the first argument and f is C_x-quasiconvex-like in K;*

(iii) *the multi-valued mapping $W : K \longrightarrow 2^X$ defined by $W(x) = X \setminus -intC(x), \forall x \in K$, is upper semicontinuous on K.*

Then, there exists $\bar{x} \in K$ such that

$$\bar{x} \in cl\widetilde{A}(\bar{x}) \quad and \quad f(\bar{x}, y) \notin -intC(\bar{x});\ \forall y \in \widetilde{A}(\bar{x}).$$

Proof. We define a multi-valued mapping $P : K \longrightarrow 2^K$ by

$$P(x) = \{y \in K : f(x,y) \in -intC(x)\}; \quad \forall x \in K.$$

Firstly, we show that for each $x \in K, x \notin P(x)$.
Suppose that $x \in P(x)$. Therefore, $f(x,x) \in -intC(x)$; which contradicts the assumption (i).

Now, we show that $P^{-1}(y)$ is open in K, which is equivalent to show that $[P^{-1}(y)]^c = K \setminus P^{-1}(y)$ is closed.
Here,

$$\begin{aligned} P^{-1}(y) &= \{x \in K : y \in P(x)\} \\ \Rightarrow P^{-1}(y) &= \{x \in K : f(x,y) \in -intC(x)\} \\ \Rightarrow [P^{-1}(y)]^c &= \{x \in K : f(x,y) \notin -intC(x)\}. \end{aligned}$$

Let $x_0 \in cl[P^{-1}(y)]^c$, the closure of $[P^{-1}(y)]^c$ in K. Then, we have to show that $x_0 \in [P^{-1}(y)]^c$.
Suppose that $\{x_\alpha\}$ be a net in $[P^{-1}(y)]^c$ such that $x_\alpha \to x_0$. Then, we have

$$\begin{aligned} (x_\alpha) &\in [P^{-1}(y)]^c. \\ \Rightarrow f(x_\alpha, y) &\notin -intC(x_\alpha);\ y \in K. \\ \Rightarrow f(x_\alpha, y) &\in W(x) = X \setminus \{-intC(x)\}. \end{aligned}$$

Since f is continuous in the first argument, thus we have $f(x_\alpha, y) \to f(x_0, y)$.
By upper semicontinuity of W, it follows that

$$\begin{aligned} f(x_0, y) &\in W(x_0) \\ \Rightarrow f(x_0, y) &\notin -intC(x_0), \end{aligned}$$

which implies that $x_0 \in [P^{-1}(y)]^c$. Hence, $[P^{-1}(y)]^c$ is closed.

To show that $P(x)$ is convex, let $y_1, y_2 \in P(x)$. Then, for each $x \in K$,

$$f(x, y_1) \in -intC(x) \quad and \quad f(x, y_2) \in -intC(x).$$

Since f is C_x-quasiconvex-like, we have for all $\lambda \in [0, 1)$
either,

$$\begin{aligned} f\big(x, \lambda y_1 + (1-\lambda) y_2\big) &\subseteq f(x, y_1) - C(x) \\ &\subseteq -intC(x) - C(x) \\ &\subseteq -intC(x), \end{aligned}$$

or,

$$\begin{aligned} f\big(x, \lambda y_1 + (1-\lambda) y_2\big) &\subseteq f(x, y_2) - C(x) \\ &\subseteq -intC(x) - C(x) \\ &\subseteq -intC(x). \end{aligned}$$

In both cases, $f\big(x, \lambda y_1 + (1-\lambda y_2)\big) \in -intC(x)$. Therefore, $\lambda y_1 + (1-\lambda)y_2 \in P(x)$. Hence, $P(x)$ is convex.

Thus, all the hypothesis of Corollary 2.3 are satisfied. Hence, there exists $\bar{x} \in K$ such that

$$\bar{x} \in cl\widetilde{A}(\bar{x}) \quad and \quad Co\widetilde{A}(\bar{x}) \cap P(\bar{x}) = \emptyset,$$

which implies that there exists $\bar{x} \in K$ such that

$$\bar{x} \in cl\widetilde{A}(\bar{x}) \quad and \quad f(\bar{x}, y) \notin -intC(\bar{x}); \quad \forall y \in \widetilde{A}(\bar{x}).$$

This completes the proof. □

Example 3.2. Let $K = [0, 1]$ be a non-empty compact convex set. Define the fuzzy mapping $A : K \longrightarrow \mathcal{F}(K)$ as follows:

$$A_x(y) = \begin{cases} 2xy & ; if\ x \in [0, \frac{1}{2}), y \in [0, 1] \\ 2(1-x)y & ; if\ x \in [\frac{1}{2}, 1], y \in [0, 1]. \end{cases}$$

The fuzzy mapping $P : K \longrightarrow \mathcal{F}(K)$ is defined by

$$P_x(y) = \begin{cases} 0 & ; if\ x \in [0, \frac{1}{2}), y \in [0, 1] \\ (1-x)y & ; if\ x \in [\frac{1}{2}, 1], y \in [0, 1]. \end{cases}$$

The mapping $a : K \longrightarrow [0, 1]$ is defined by

$$a(x) = \begin{cases} 0 & ; if\ x \in [0, \frac{1}{2}) \\ (1-x)x & ; if\ x \in [\frac{1}{2}, 1]. \end{cases}$$

Also, the mapping $f : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is defined by

$$f(x, y) = x - y;\ \forall x, y \in [0, 1].$$

Obviously $P_x(x) \leq a(x)$ and all the conditions of Theorem 2.2 are satisfied and $x = \frac{1}{2}$ is the equilibrium point, such that

$$\frac{1}{2} \in cl(A_x)_{a(x)} \quad and \quad Co(A_x)_{a(x)} \cap (P_x)_{a(x)} = \emptyset.$$

Hence, there exists $x^* = \frac{1}{2} \in X$, such that

$$\frac{1}{2} = x^* \in cl\widetilde{A}(x^*)\ and\ f(x^*, y) \notin -intC(x^*).$$

3.1. **Existence Theory Without Convexity.** In this section, we prove an existence theorem for fuzzy vector equilibrium problem (FVEP) by replacing convexity assumptions with merely topological properties. The following definitions can be found in [1].

Definition 3.3. [1] Let X be a topological space; and let $\{\Gamma_A\}$ be a given family of nonempty and contractible subsets of X, indexed by finite subsets of X.

1. A pair $(X, \{\Gamma_A\})$ is said to be a H-space iff $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.
2. A subset $D \subset X$ is called H-convex iff $\Gamma_A \subset D$ holds for every finite subset $A \subset D$.

(3) A subset $D \subset X$ is called weakly H-convex iff $\Gamma_A \cap D$ is nonempty and contractible for every finite subset $A \subset D$. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space.
(4) A subset $K \subset X$ is called H-compact iff there exists a compact and weakly H-convex set $D \subset X$ such that $K \cup A \subset D$ for every finite subset $A \subset X$.
(5) For a nonempty subset K of an H-space, H-convex hull of K, denoted by $H - coK$, is defined by

$$H - coK = \cap\{D \subset X : D \text{ } is \text{ } H - convex \text{ } and \text{ } K \subset D\}.$$

The following result is a special case of a result of [11] and can be proved easily.

Theorem 3.4. *Let $\Gamma = (X, P, A)$ be an abstract economy such that the following conditions are hold:*

(i) *X is compact;*
(ii) *for each $x \in X, A(x)$ is nonempty and H-convex valued;*
(iii) *the set $G = \{x \in X : A(x) \cap P(x) \neq \emptyset\}$ is a closed subset of X;*
(iv) *for each $y \in X, P^{-1}(y)$ is an open subset of X and $A^{-1}(y)$ is an open subset of X;*
(v) *for each $x \in X, x \notin H - coP(x)$.*

Then, Γ has an equilibrium point $\bar{x} \in X$ such that

$$\bar{x} \in A(\bar{x}) \text{ } and \text{ } P(\bar{x}) \cap A(\bar{x}) = \emptyset.$$

Theorem 3.5. *Let (X, Γ) be a compact Hausdorff locally convex H-space, $\widetilde{A} : X \longrightarrow 2^X$ be the set-valued mappings induced by the fuzzy mapping $A : X \longrightarrow \mathcal{F}(X)$ such that $\widetilde{A}^{-1}(y)$ is an open subset of X for each $y \in X$. Let $f : X \times X \longrightarrow X$ be a vector-valued bifunction, $C : X \longrightarrow 2^X$ be set-valued mapping such that $C(x)$ is pointed closed convex cone X and intC(x) is closed. Suppose that the following conditions holds:*

(i) *for each $x \in X, f(x, x) \notin -intC(x)$;*
(ii) *f is continuous in the first argument;*
(iii) *$\widetilde{A}$ is continuous mapping with nonempty H-convex value;*
(iv) *$W : X \longrightarrow 2^X$ defined by $W(x) = X \setminus (-intC(x))$ is upper semicontinuous.*

Then, there exists $\bar{x} \in X$ such that

$$\bar{x} \in \widetilde{A}(\bar{x}) \text{ } and \text{ } f(\bar{x}, y) \notin -intC(\bar{x}); \ \forall y \in \widetilde{A}(\bar{x}).$$

Proof. We define the multifunction $P : X \longrightarrow 2^X$ by

$$P(x) = \{y \in X : f(x, y) \in -intC(x)\}; \text{ } for \text{ } each \text{ } x \in X.$$

First, we show that for each $x \in X, x \notin H - coP(x)$. On contrary, Suppose that $x \in H - coP(x) = \cap\{D \subset X : D \text{ } is \text{ } H - convex \text{ } and \text{ } P(x) \subset D\}$. Therefore, $x \in D$ such that $P(x) \subset D$. As $x \in P(x) \subset D$, we have $f(x, x) \in -intC(x)$, which is a contradiction to the assumption (i). Hence, $x \notin H - coP(x)$.

Now, we have to show that the set $G = \{x \in X : \widetilde{A}(x) \cap P(x) \neq \emptyset\}$ is closed. For this, Suppose that $\{x_\lambda\}$ be a net in G such that $x_\lambda \to x_0$.

As $x_\lambda \in G$, therefore $\widetilde{A}(x_\lambda) \cap P(x_\lambda) \neq \emptyset$. Then, there exists $z \in X$ such that $z \in \widetilde{A}(x_\lambda) \cap P(x_\lambda)$, which implies $z \in \widetilde{A}(x_\lambda)$ and $z \in P(x_\lambda)$. Since $\widetilde{A}$ is continuous, therefore $\widetilde{A}(x_\lambda) \to \widetilde{A}(x_0)$. Hence $z \in \widetilde{A}(x_0)$.
Also, $z \in P(x_\lambda) \Rightarrow f(x_\lambda, z) \in -intC(x_\lambda)$. Since f is continuous in the first argument, therefore $f(x_\lambda, z) \to f(x_0, z)$.
So, $f(x_0, z) \in -intC(x_\lambda)$. As $intC(x)$ is closed, we have $f(x_0, z) \in -intC(x_0)$. Therefore, $z \in P(x_0)$.

Altogether, we have $z \in \widetilde{A}(x_0) \cap P(x_0)$, which implies $\widetilde{A}(x_0) \cap P(x_0) \neq \emptyset$, and hence $x_0 \in G$. Therefore, G is a closed subset of X.

To show that, for each $y \in X, P^{-1}(y)$ is an open subset of X, we use a similar arguments as in the proof of Theorem 3.1.
Hence, all the conditions of Theorem 3.4 are satisfied. Thus, there exists $\bar{x} \in X$ such that

$$\bar{x} \in \widetilde{A}(\bar{x}) \; and \; P(\bar{x}) \cap \widetilde{A}(\bar{x}) = \emptyset.$$

This implies that, there exists $\bar{x} \in X$ such that

$$\bar{x} \in \widetilde{A}(\bar{x}) \; and \; f(\bar{x}, y) \notin -intC(\bar{x}); \; \forall y \in \widetilde{A}(\bar{x}).$$

This completes the proof. □

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