BEHAVIOR OF SOLUTIONS TO A FUZZY NONLINEAR DIFFERENCE EQUATION

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ABSTRACT. In this paper, we study the existence, asymptotic behavior of the positive solutions of a fuzzy nonlinear difference equation

$$x_{n+1} = \frac{Ax_n + x_{n-1}}{B + x_{n-1}}, \quad n = 0, 1, \cdots,$$

where (x_n) is a sequence of positive fuzzy number, A, B are positive fuzzy numbers and the initial conditions x_{-1}, x_0 are positive fuzzy numbers.

1. Introduction

It is known that difference equation appears naturally as discrete analogous and as numerical solutions of differential equations and delay differential equation having many applications in economics, biology, computer science, control engineering, etc (see, for example, [8, 9, 2, 5, 6, 15] and the references therein). Recently there has been a lot of work concerning the oscillatory behavior, the periodicity, and the boundedness of nonlinear difference equations. Moreover similar results in [12]have been derived for systems of two nonlinear difference equations. A fuzzy difference equation is a difference equation where constants and the initial values are fuzzy numbers, and its' solutions are sequences of fuzzy numbers. Recently there is an increasing interest in the study of fuzzy difference equation (see, for example [3, 4, 13, 10, 11, 16, 14, 1]).

Kulenvic et al. [8] studied the following difference equation

$$x_{n+1} = \frac{px_n + x_{n-1}}{q + x_{n-1}}, \quad n = 0, 1, \cdots, \tag{1}$$

where $p, q \in (0, \infty), x_{-1} \in [0, \infty), x_0 \in (0, \infty)$ and investigated the boundedness character, the periodic nature, and the global asymptotic stability of all positive solutions of equation (1).

Li and Sun [9] investigated the periodic character, invariant intervals, oscillation and global stability of all positive solutions of the equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, \cdots \tag{2}$$

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where $k \in \{1, 2, \cdots\}$, p, q and the initial conditions x_{-1}, x_0 are nonnegative real numbers.

Now in this paper we study the fuzzy analog of (1)

$$x_{n+1} = \frac{Ax_n + x_{n-1}}{B + x_{n-1}}, \quad n = 0, 1 \cdots, \tag{3}$$

where A, B, x_{-1}, x_0 are positive fuzzy numbers.

To be convenience, we need some definitions:

A is said to be a fuzzy number if $A : R \to [0, 1]$ satisfies the below (i)-(iv)

(i) A is normal, i.e., there exists a $x \in R$ such that $A(x) = 1$;

(ii) A is fuzzy convex, i.e., for all $t \in [0, 1]$ and $x_1, x_2 \in R$ such that

$$A(tx_1 + (1 - t)x_2) \geq \min\{A(x_1), A(x_2)\};$$

(iii) A is upper semicontinuous;

(iv) The support of A, supp$A = \overline{\bigcup_{\alpha \in (0,1]} [A]_\alpha} = \overline{\{x : A(x) > 0\}}$ is compact.

The $\alpha-$cuts of A are denoted by $[A]_\alpha = \{x \in R : A(x) \geq \alpha\}, \quad \alpha \in [0, 1]$, it is clear that the $[A]_\alpha$ are closed interval. We say that a fuzzy number is positive if supp$A \subset (0, \infty)$.

It is obvious that if A is a positive real number, then A is a fuzzy number and $[A]_\alpha = [A, A], \alpha \in (0, 1]$. In this case we say that A is a trivial fuzzy number.

Let A, B be fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}], \alpha \in (0, 1]$. We define a norm on the fuzzy numbers space as follows:

$$\|A\| = \sup_{\alpha \in (0,1]} \max\{|A_{l,\alpha}|, |A_{r,\alpha}|\}.$$

We take the following metric :

$$D(A, B) = \sup_{\alpha \in (0,1]} \max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}$$

the fuzzy analog of the boundedness and persistence (see [4, 10]) as follows: we say that a sequence of positive fuzzy numbers x_n persists (resp. is bounded) if there exists a positive real number M (resp. N) such that

$$\text{supp}x_n \subset [M, \infty)(\text{resp. supp}x_n \subset (0, N]), \quad n = 1, 2, \cdots,$$

We say that (x_n) is bounded and persists if there exist positive real numbers $M, N > 0$ such that

$$\text{supp}x_n \subset [M, N], \quad n = 1, 2, \cdots.$$

We say $(x_n), n = 1, 2, \cdots$, is an unbounded sequence if the norm $\|x_n\|, n = 1, 2, \cdots$, is an unbounded sequence.

We say that x_n is a positive solution of (3) if (x_n) is a sequence of positive fuzzy numbers that satisfies (3). We say a positive fuzzy number x is a positive equilibrium for (3) if

$$x = \frac{Ax + x}{B + x}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x be a positive fuzzy number. Suppose that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n = 0, 1, 2, \cdots, \quad \alpha \in (0, 1] \tag{4}$$

and

$$[x]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0, 1] \tag{5}$$

We say that the sequence (x_n) converges to x with respect to D as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

Suppose that (3) has a unique positive equilibrium x. We say that the positive equilibrium x of (3) is stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every positive solution x_n of (3), that satisfies $D(x_{-i}, x) \leq \delta, i = 0, 1$ we have $D(x_n, x) \leq \varepsilon$ for all $n > 0$.

Moreover, we say that the positive equilibrium x of (3) is asymptotically stable, if it is stable and every positive solution of (3) tends to the positive equilibrium of (3) with respect to D as $n \to \infty$.

The purpose of this paper is to study the existence of positive solutions of (3). Furthermore, we find some conditions so that every positive solution of (3) is boundedness and persistence. Finally, under some conditions we prove that (3) has a unique positive equilibrium x that is stable.

2. **Main Results**

Firstly we study the existence of the positive solutions of (3). We need the following lemmas.

Lemma 2.1. [11] *Let $f : R^+ \times R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous, A, B, C, D, E are fuzzy numbers. Then*

$$[f(A, B, C, D, E)]_\alpha = f([A]_\alpha, [B]_\alpha, [C]_\alpha, [D]_\alpha, [E]_\alpha), \quad \alpha \in (0, 1] \tag{6}$$

Lemma 2.2. [17] *Let $u \in E^\sim$, write $[u]_\alpha = [u_-(\alpha), u_+(\alpha)], \alpha \in (0, 1]$. Then $u_-(\alpha)$ and $u_+(\alpha)$ can be regarded as functions on $(0, 1]$, that satisfy*

(i) $u_-(\alpha)$ *is nondecreasing and left continuous;*

(ii) $u_+(\alpha)$ *is nonincreasing and left continuous;*

(iii) $u_-(1) \leq u_+(1)$.

Conversely for any functions $a(\alpha)$ and $b(\alpha)$ defined on $(0, 1]$ that satisfy (i)-(iii) in the above, there exists a unique $u \in E^\sim$ such that $[u]_\alpha = [a(\alpha), b(\alpha)]$ for any $\alpha \in (0, 1]$.

Theorem 2.3. *Consider equation (3) where A, B are positive fuzzy numbers. Then for any positive fuzzy numbers x_{-1}, x_0 there exists a unique positive solution x_n of (3) with the initial conditions x_{-1}, x_0.*

Proof. The proof is similar to that of Proposition 2.1 [11]. Suppose that there exists a sequence of fuzzy numbers (x_n) satisfying (3) with the initial conditions x_{-1}, x_0. Consider the $\alpha-$cuts, $\alpha \in (0,1], n = 0,1,2,\cdots$,

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}],\ [A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}],\ \ [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]. \tag{7}$$

It follows from (3), (7) and Lemma 2.1 that

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{Ax_n + x_{n-1}}{B + x_{n-1}}\right]_\alpha = \frac{[A]_\alpha \times [x_n]_\alpha + [x_{n-1}]_\alpha}{[B]_\alpha + [x_{n-1}]_\alpha} \\ &= \frac{[A_{l,\alpha}, A_{r,\alpha}] \times [L_{n,\alpha}, R_{n,\alpha}] + [L_{n-1,\alpha}, R_{n-1,\alpha}]}{[B_{l,\alpha}, B_{r,\alpha}] + [L_{n-1,\alpha}, R_{n-1,\alpha}]} \\ &= \left[\frac{A_{l,\alpha}L_{n,\alpha} + L_{n-1,\alpha}}{B_{r,\alpha} + R_{n-1,\alpha}}, \frac{A_{r,\alpha}R_{n,\alpha} + R_{n-1,\alpha}}{B_{l,\alpha} + L_{n-1,\alpha}}\right] \end{aligned}$$

from which we have that for $n = 0,1,2,\cdots,\ \alpha \in (0,1]$

$$L_{n+1,\alpha} = \frac{A_{l,\alpha}L_{n,\alpha} + L_{n-1,\alpha}}{B_{r,\alpha} + R_{n-1,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{r,\alpha}R_{n,\alpha} + R_{n-1,\alpha}}{B_{l,\alpha} + L_{n-1,\alpha}}. \tag{8}$$

Then it is obvious that for any initial condition $(L_{-i,\alpha}, R_{-i,\alpha}), i = 0,1, \alpha \in (0,1]$ there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$. We now prove that $[L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0,1]$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of system (8) with the initial conditions $(L_{-i,\alpha}, R_{-i,\alpha}), i = 0,1$, determines the solution x_n of (3) with the initial conditions $x_{-i}, i = 0,1$ such that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0,1], \quad n = 0,1,2,\cdots. \tag{9}$$

From reference[15] and that A, B, x_{-1}, x_0 are positive fuzzy numbers, for any $\alpha_1, \alpha_2 \in (0,1], \alpha_1 \le \alpha_2$ we have

$$\begin{cases} 0 < A_{l,\alpha_1} \le A_{l,\alpha_2} \le A_{r,\alpha_2} \le A_{r,\alpha_1} \\ 0 < B_{l,\alpha_1} \le B_{l,\alpha_2} \le B_{r,\alpha_2} \le B_{r,\alpha_1} \\ 0 < L_{-1,\alpha_1} \le L_{-1,\alpha_2} \le R_{-1,\alpha_2} \le R_{-1,\alpha_1} \\ 0 < L_{0,\alpha_1} \le L_{0,\alpha_2} \le R_{0,\alpha_2} \le R_{0,\alpha_1} \end{cases} \tag{10}$$

We claim that

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}, \quad n = -1,0,1,2,\cdots. \tag{11}$$

We prove it by induction. It is obvious from (10) that (11) holds for $n = -1, 0$. Suppose that (11) are true for $n \le k.k \in \{1,2,\cdots\}$. Then from (8), (10) and (11), for $n \le k$ it follows that

$$\begin{aligned} L_{k+1,\alpha_1} &= \frac{A_{l,\alpha_1}L_{k,\alpha_1} + L_{k-1,\alpha_1}}{B_{r,\alpha_1} + R_{k-1,\alpha_1}} \le \frac{A_{l,\alpha_2}L_{k,\alpha_2} + L_{k-1,\alpha_2}}{B_{r,\alpha_2} + R_{k-1,\alpha_2}} = L_{k+1,\alpha_2} \\ &= \frac{A_{l,\alpha_2}L_{k,\alpha_2} + L_{k-1,\alpha_2}}{B_{r,\alpha_2} + R_{k-1,\alpha_2}} \le \frac{A_{r,\alpha_2}R_{k,\alpha_2} + R_{k-1,\alpha_2}}{B_{l,\alpha_2} + L_{k-1,\alpha_2}} = R_{k+1,\alpha_2} \\ &= \frac{A_{r,\alpha_2}R_{k,\alpha_2} + R_{k-1,\alpha_2}}{B_{l,\alpha_2} + L_{k-1,\alpha_2}} \le \frac{A_{r,\alpha_1}R_{k,\alpha_1} + R_{k-1,\alpha_1}}{B_{l,\alpha_1} + L_{k-1,\alpha_1}} = R_{k+1,\alpha_1} \end{aligned}$$

Therefore (11) is satisfied. Moreover from (8) we have

$$L_{1,\alpha} = \frac{A_{l,\alpha}L_{0,\alpha} + L_{-1,\alpha}}{B_{r,\alpha} + R_{-1,\alpha}}, \quad R_{1,\alpha} = \frac{A_{r,\alpha}R_{0,\alpha} + R_{-1,\alpha}}{B_{l,\alpha} + L_{-1,\alpha}}, \quad \alpha \in (0,1] \tag{12}$$

Since A, B, x_{-1}, x_0 are positive fuzzy numbers, $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}$, $B_{r,\alpha}, L_{-l,\alpha}, R_{-1,\alpha}$, $L_{0,\alpha}, R_{0,\alpha}$ are left continuous. So from (12) we have that $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By induction we can get that $L_{n,\alpha}, R_{n,\alpha}, n = 1, 2, \cdots$ are left continuous.

We now prove that the support of x_n, $\text{supp} x_n = \overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is sufficient to prove that $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]$ is bounded . Let $n = 1$, since A, B, x_{-1}, x_0 are positive fuzzy numbers , there exist constants $M_A > 0, N_A > 0, M_B > 0, N_B > 0, M_{-1} > 0, N_{-1} > 0, M_0 > 0, N_0 > 0$ such that for all $\alpha \in (0,1]$,

$$\begin{cases} [A_{l,\alpha}, A_{r,\alpha}] & \subset & \overline{\bigcup_{\alpha\in(0,1]}[A_{l,\alpha}, A_{r,\alpha}]} & \subset & [M_A, N_A], \\ [B_{l,\alpha}, B_{r,\alpha}] & \subset & \overline{\bigcup_{\alpha\in(0,1]}[B_{l,\alpha}, B_{r,\alpha}]} & \subset & [M_B, N_B], \\ [L_{-l,\alpha}, R_{-1,\alpha}] & \subset & \overline{\bigcup_{\alpha\in(0,1]}[L_{-l,\alpha}, R_{-1,\alpha}]} & \subset & [M_{-1}, N_{-1}], \\ [L_{0,\alpha}, R_{0,\alpha}] & \subset & \overline{\bigcup_{\alpha\in(0,1]}[L_{0,\alpha}, R_{0,\alpha}]} & \subset & [M_0, N_0] \end{cases} \tag{13}$$

Hence from (12) and (13) we can easily get

$$[L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_A M_0 + M_{-1}}{N_B + N_{-1}}, \frac{N_A N_0 + N_{-1}}{M_B + M_{-1}}\right], \quad \alpha \in (0,1]. \tag{14}$$

From which it is obvious that

$$\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_A M_0 + M_{-1}}{N_B + N_{-1}}, \frac{N_A N_0 + N_{-1}}{M_B + M_{-1}}\right], \quad \alpha \in (0,1] \tag{15}$$

Therefore (15) implies that $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty)$. Deducing inductively we can easily follow that $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact, and

$$\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0, \infty), \quad n = 1, 2, \cdots. \tag{16}$$

Therefore (11), (16) and that $L_{n,\alpha}, R_{n,\alpha}$ are left continuous we have that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers (x_n) such that (9) holds.

We now prove that x_n is the solution of (3) with the initial conditions x_{-1}, x_0. Since for all $\alpha \in (0,1]$,

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{A_{l,\alpha}L_{n,\alpha} + L_{n-1,\alpha}}{B_{r,\alpha} + R_{n-1,\alpha}}, \frac{A_{r,\alpha}R_{n,\alpha} + R_{n-1,\alpha}}{B_{l,\alpha} + R_{n-1,\alpha}}\right] \\ &= \left[\frac{Ax_n + x_{n-1}}{B + x_{n-1}}\right]_\alpha, \end{aligned}$$

we have that x_n is the solution of (3) with initial condition x_{-1}, x_0.

Suppose that there exists another solution $\overline{x}_n$ of (3) with the initial conditions x_{-1}, x_0. Then from arguing as above we can easily prove that

$$[\overline{x}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0,1], \quad n = 0,1,2,\cdots. \tag{17}$$

Then from (9) and (17) we have $[x_n]_\alpha = [\overline{x}_n]_\alpha, \alpha \in (0,1], n = 0,1,2,\cdots$ from which it follows that $x_n = \overline{x}_n, n = 0,1,\cdots$. Thus the proof of the Theorem 2.3 is completed. □

In the following we will study the property of the fuzzy positive solution of (3). We need the following lemma.

Lemma 2.4. *Consider the system of the difference equations*

$$y_{n+1} = \frac{py_n + y_{n-1}}{V + z_{n-1}}, \quad z_{n+1} = \frac{qz_n + z_{n-1}}{U + y_{n-1}}, \ n = 0,1,\cdots \tag{18}$$

where p, q, U, V are positive real numbers and the initial values $y_{-i}, z_{-i}, i = 0,1$ are positive real numbers. Then the following statements are true.

(i) *For any $h > p + 1 - V$ and $k > q + 1 - U$, system (18) has no solution that is eventually in $[k, \infty) \times [h, \infty)$.*

(ii) *For any $l < p + 1 - V$ and $m < q + 1 - U$, if the following condition holds*

$$p + 1 > V, \qquad q + 1 > U. \tag{19}$$

then system (18) has no solution that is eventually in $[0, m] \times [0, l]$.

(iii) *If $p + 1 < V, q + 1 < U$, then the equilibrium $(0,0)$ is locally asymptotically stable.*

(iv) *If (19) holds, then system (18) has a unique positive equilibrium $(q+1-U, p+1-V)$.*

Proof. (i) Suppose otherwise that there exists a solution $(y_n, z_n)_{n=-1}^{\infty}$ of system (18) such that

$$y_n \geq k, \qquad z_n \geq h, \quad n \geq N.$$

for some $h > p+1-V$ and $k > q+1-U$. Without loss of generality we can assume that $N = 0$. Then system (18) implies that, $n \geq 0$,

$$\begin{cases} y_{n+1} \leq \frac{py_n + y_{n-1}}{V+h} = \frac{p}{V+h} y_n + \frac{1}{V+h} y_{n-1} \\ z_{n+1} \leq \frac{qz_n + z_{n-1}}{U+k} = \frac{q}{U+k} z_n + \frac{1}{U+k} z_{n-1} \end{cases} \tag{20}$$

Using difference inequality result (see [16]), (20) implies that

$$y_n \leq u_n, \quad z_n \leq v_n, \quad y_0 = u_0, \quad z_0 = v_0, \quad y_{-1} = u_{-1}, \quad z_{-1} = v_{-1}.$$

where (u_n) and (v_n) satisfy

$$u_{n+1} = \frac{p}{V+h} u_n + \frac{1}{V+h} u_{n-1}, \quad v_{n+1} = \frac{q}{U+k} v_n + \frac{1}{U+k} v_{n-1}. \tag{21}$$

system (21) is linear homogenous equations with constant coefficients, and all the roots of characteristic equation

$$\lambda^2 - \frac{p}{V+h}\lambda - \frac{1}{V+h} = 0, \quad \lambda^2 - \frac{q}{U+k}\lambda - \frac{1}{U+k} = 0. \tag{22}$$

lie inside the unit disk. Thus

$$\lim_{n\to\infty} u_n = 0, \quad \lim_{n\to\infty} v_n = 0.$$

for all solutions of system (21). Consequently

$$\lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty} z_n = 0.$$

which is a contradiction.

(ii) Suppose otherwise that there exists a solution $(y_n, z_n)_{n=-1}^{\infty}$ such that

$$y_n \le m, \quad z_n \le l, \quad n \ge N.$$

for some $l < p+1-V$ and $m < q+1-U$. Without loss of generality we can assume that $N = 0$. System (18) implies that, $n \ge 0$,

$$\begin{cases} y_{n+1} \ge \frac{py_n + y_{n-1}}{V+l} = \frac{p}{V+l} y_n + \frac{1}{V+l} y_{n-1} \\ z_{n+1} \ge \frac{qz_n + z_{n-1}}{U+m} = \frac{q}{U+m} z_n + \frac{1}{U+m} z_{n-1} \end{cases} \tag{23}$$

Using difference inequalities result (see [16]), (23) implies that

$$y_n \ge \mu_n, \quad z_n \ge \omega_n, \quad y_0 = \mu_0, \quad z_0 = \omega_0, \quad y_{-1} = \mu_{-1}, \quad z_{-1} = \omega_{-1}.$$

where μ_n and ω_n satisfy

$$\mu_{n+1} = \frac{p}{V+l}\mu_n + \frac{1}{V+l}\mu_{n-1}, \quad \omega_{n+1} = \frac{q}{U+m}\omega_n + \frac{1}{U+m}\omega_{n-1}. \tag{24}$$

We now show that $\lim_{n\to\infty} \mu_n = \infty$, $\lim_{n\to\infty} \omega_n = \infty$, which is a contradiction. Indeed, the characteristic polynomials

$$f(\lambda) = \lambda^2 - \frac{p}{V+l}\lambda - \frac{1}{V+l}, \quad g(\lambda) = \lambda^2 - \frac{q}{U+m}\lambda - \frac{1}{U+m}.$$

satisfy $f(1) = \frac{V+l-p-1}{V+l} < 0$, $g(1) = \frac{U+m-q-1}{U+m} < 0$. So they have a root in $(1, \infty)$. Therefore we can choose $y_i, z_i, i = -1, 0$ such that $\lim_{n\to\infty} \mu_n = \infty, \lim_{n\to\infty} \omega_n = \infty$.

(iii) The linearized equation of system (18) about the equilibrium $(0,0)$ is

$$\Psi_{n+1} = D\Psi_n, \quad n = 0, 1, \cdots, \tag{25}$$

where

$$\Psi_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}, \quad D = \begin{pmatrix} \frac{p}{V} & \frac{1}{V} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{q}{U} & \frac{1}{U} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Its characteristic equation is

$$\left(\lambda^2 - \frac{p}{V}\lambda - \frac{1}{V}\right)\left(\lambda^2 - \frac{q}{U}\lambda - \frac{1}{U}\right) = 0 \tag{26}$$

It is obvious that all roots of equation (26) lie inside unit disk. So the equilibrium $(0,0)$ is locally asymptotically stable.

(iv) Suppose relation (19) is satisfied . The equilibrium (x,y) of equation (18) satisfies the system

$$y = \frac{py+y}{V+z}, \qquad z = \frac{qz+z}{U+y}. \tag{27}$$

From (27) it is clear that there exists a unique positive equilibrium $(x,y) = (q+1-U, p+1-V)$. □

Theorem 2.5. *Consider fuzzy difference equation (3), where $A, B, x_{-i}, i = 0,1$ are positive fuzzy numbers. Then the following statements are true.*

(i) *If for every $\alpha \in (0,1]$,*

$$A_{r,\alpha} + 1 < B_{l,\alpha} \tag{28}$$

then every positive solution of (3) is bounded and persists.

(ii) *If relation (28) holds, then every positive solution x_n of equation (3) converges to the equilibrium $x = 0$ with respect to D as $n \to \infty$.*

Proof. (i) If x_n is the unique positive solution of (3) with the initial values x_{-1}, x_0, such that (4) holds. We consider the system

$$H_{n+1} = \frac{M_A H_n + H_{n-1}}{N_B + E_{n-1}}, \quad E_{n+1} = \frac{N_A E_n + E_{n-1}}{M_B + H_{n-1}} \tag{29}$$

where M_A, N_A, M_B, N_B are defined in (13).

Let (H_n, E_n) be a solution of (29) with initial values

$$H_i = M_i, \quad E_i = N_i, \quad i = -1, 0 \tag{30}$$

where $M_i, N_i, (i = -1, 0)$ are defined in (13).

From (8), (13),(29), and (30) we have

$$H_1 = \frac{M_A H_0 + H_{-1}}{N_B + E_{-1}} \le L_{1,\alpha}, \quad R_{1,\alpha} \le \frac{N_A E_1 + E_{-1}}{M_B + H_{-1}} = E_1 \tag{31}$$

Using (8), (29), (30), and (31),inductively we can prove that

$$H_n \le L_{n,\alpha}, \quad R_{n,\alpha} \le E_n, \quad n = 1, 2, \cdots, \tag{32}$$

Since relation (28) is satisfied, it is clear that

$$N_A + 1 < M_B, \quad M_A + 1 < N_B. \tag{33}$$

From (i) and (ii) of Lemma 2.4, it follows that the solution (H_n, E_n) of (29) is bounded and persists. From (4) and (32) we also have that the positive solution x_n of (3) is bounded and persists.

(ii) Let x_n be a positive solution of (3) such that (7) holds. Since (28) is satisfied we can apply (i) and (iii) of Lemma 2.4, so we have

$$\lim_{n\to\infty} L_{n,\alpha} = 0, \quad \lim_{n\to\infty} R_{n,\alpha} = 0. \tag{34}$$

Therefore from (34) we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - 0|, |R_{n,\alpha} - 0|\}\} = 0.$$

This completes the proof of (ii) of Theorem 2.5.

If for the unique positive equilibrium $\overline{x}$ of (3), relation (7) holds, we have the following relations:

$$L_\alpha = \frac{A_{l,\alpha}L_\alpha + L_\alpha}{B_{r,\alpha} + R_\alpha}, \quad R_\alpha = \frac{A_{r,\alpha}R_\alpha + R_\alpha}{B_{l,\alpha} + L_\alpha} \tag{35}$$

It follows from (35) that

$$L_\alpha = A_{r,\alpha} + 1 - B_{l,\alpha}, \quad R_\alpha = A_{l,\alpha} + 1 - B_{r,\alpha}, \quad \alpha \in (0, 1]. \tag{36}$$

It is obvious that $\overline{x} = [L_\alpha, R_\alpha]$ is positive equilibrium point if and only if $A_{l,\alpha} = A_{r,\alpha}$ and $B_{l,\alpha} = B_{r,\alpha}, \alpha \in (0, 1]$, namely A, B are positive real numbers. So the positive equilibrium $\overline{x}$ is a positive trivial fuzzy number, i.e.,

$$[\overline{x}]_\alpha = [L_\alpha, R_\alpha], \quad L_\alpha = R_\alpha = A + 1 - B. \tag{37}$$

□

Theorem 2.6. *Consider fuzzy difference equation (3), where A, B are positive trivial fuzzy numbers, i.e., positive real numbers, Moreover if relation*

$$B < A + 1 \tag{38}$$

holds, then the unique positive equilibrium $\overline{x}$ of (3) is stable.

Proof. Let $\overline{x}$ be a positive equilibrium of equation (3) and $\varepsilon(< 1)$ be a positive real number. Since (38) holds, we consider the positive real number δ as follows

$$0 < \delta \leq \frac{(A + 1)\varepsilon}{2A + 2 - B + \varepsilon} \tag{39}$$

Let x_n be a positive solution of (3) such that

$$D(x_{-i}, \overline{x}) \leq \delta < \varepsilon, \quad i = 0, 1 \tag{40}$$

from (40) we have

$$|L_{-i,\alpha} - L_\alpha| \leq \delta, \quad |R_{-i,\alpha} - R_\alpha| \leq \delta, \quad i = 0, 1, \quad \alpha \in (0, 1] \tag{41}$$

From (8), (37), and (41) we get

$$\begin{aligned} L_{1,\alpha} - L_\alpha &= \frac{AL_{0,\alpha} + L_{-1,\alpha}}{B + R_{-1,\alpha}} - L_\alpha \le \frac{A(L_\alpha + \delta) + (L_\alpha + \delta)}{B + R_\alpha - \delta} - L_\alpha \\ &= \delta \frac{A + L_\alpha + 1}{B + R_\alpha - \delta} \end{aligned} \quad (42)$$

$$\begin{aligned} L_{1,\alpha} - L_\alpha &= \frac{AL_{0,\alpha} + L_{-1,\alpha}}{B + R_{-1,\alpha}} - L_\alpha \ge \frac{A(L_\alpha - \delta) + (L_\alpha - \delta)}{B + R_\alpha + \delta} - L_\alpha \\ &= -\delta \frac{A + L_\alpha + 1}{B + R_\alpha + \delta} \end{aligned} \quad (43)$$

from(39), (42), and (43) it is obvious that

$$|L_{1,\alpha} - L_\alpha| \le \delta < \varepsilon \quad (44)$$

Moreover, from (8), (37), and (41) we have that

$$\begin{aligned} R_{1,\alpha} - R_\alpha &= \frac{AR_{0,\alpha} + R_{-1,\alpha}}{B + L_{-1,\alpha}} - R_\alpha \le \frac{A(R_\alpha + \delta) + (R_\alpha + \delta)}{B + L_\alpha - \delta} - R_\alpha \\ &= \delta \frac{A + R_\alpha + 1}{B + L_\alpha - \delta} \end{aligned} \quad (45)$$

$$\begin{aligned} R_{1,\alpha} - R_\alpha &= \frac{AR_{0,\alpha} + R_{-1,\alpha}}{B + L_{-1,\alpha}} - R_\alpha \ge \frac{A(R_\alpha - \delta) + (R_\alpha - \delta)}{B + L_\alpha + \delta} - R_\alpha \\ &= -\delta \frac{A + R_\alpha + 1}{B + L_\alpha + \delta} \end{aligned} \quad (46)$$

from (2.34), (2.40) and (2.41) we get

$$|R_{1,\alpha} - R_\alpha| \le \delta < \varepsilon$$

From (44), (47) and working inductively we can prove that

$$|L_{n,\alpha} - L_\alpha| < \varepsilon, \quad |R_{n,\alpha} - R_\alpha| < \varepsilon, \quad \alpha \in (0,1], \quad n = 0, 1, \cdots \quad (47)$$

and so we have $D(x_n, \overline{x}) < \varepsilon, n \ge 0$. Therefore, the positive equilibrium $\overline{x}$ is stable. □

3. Conclusion

In this paper, we study the fuzzy nonlinear difference equation $x_{n+1} = \frac{Ax_n + x_{n-1}}{B + x_{n-1}}$, $n = 0, 1, \cdots$. Firstly the existence of positive fuzzy solutions is proved. Secondly, we find that under the condition $A_{r,\alpha} + 1 < B_{l,\alpha}$, the positive solutions of the fuzzy difference equation are bounded and persistence. When the parameters A, B are positive trivial fuzzy numbers, the unique positive equilibrium $\overline{x}$ of (3) is stable.

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