BIFUZZY CORE OF FUZZY AUTOMATA

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Abstract. The purpose of the present work is to introduce the concept of bifuzzy core of a fuzzy automaton, which induces a bifuzzy topology on the state-set of this fuzzy automaton. This is shown that this bifuzzy topology can be used to characterize the concepts such as bifuzzy family of submachines, bifuzzy separable family and bifuzzy retrievable family of a fuzzy automaton.

1. Introduction

The concept of a finite automaton is well-known. Algebraic study of automata have been done by many authors in many forms (cf., eg., [1, 3, 6]). Among these studies, in [1], the concepts like separatedness, connectedness and retrievability of automata were introduced and studied. In [1], it has also been pointed out that the study of such concepts of automata naturally contributes toward a better understanding of the structure of automata and their applications. After the work of Bavel [1] on algebraic automata theory, the association of topology with an automaton was initiated firstly in [20, 21], and it was demonstrated there that several known topological concepts and ideas can often be used in automata theory to obtain certain results therein, pertaining particularly to their connectivity and separation properties.

The study of fuzzy automata was initiated by Wee [23] and Santos [19] in 1960's after the introduction of fuzzy set theory by Zadeh [25]. Much later, Malik, Mordeson and Sen [15] introduced considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) and initiated its algebraic study. Further, somewhat different notions of fuzzy automata and their algebraic studies were reported in [8, 9, 10, 11, 12, 13, 17, 18]. In order to enrich the algebraic study of fuzzy automata theory, recently, Zhang and Huang [26] studied the connectivity and separation properties of intuitionistic fuzzy automata based on complete residuated lattice-valued logic; Jin, Li and Li [7] established some basic concepts of fuzzy automata based on po-monoids and investigated their algebraic properties; while, Ignjatović, Ćiric and Simoović [5] studied the concepts of subsystems, reverse subsystems and double subsystems of a fuzzy automaton in terms of fuzzy relation inequalities and equations.

After the work of Malik, Mordeson and Sen [15] (cf. [16], for more details), on fuzzy finite state machine (which is almost identical to a fuzzy automaton) and its

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properties e.g., separatedness, connectedness and retrievability, in [22], it was shown that certain topological concepts can then be used in fuzzy automata theory also to obtain certain results related to their connectivity and separation properties. Also, the fuzzy topological characterization of a fuzzy automaton was presented in [2]. In [18], it was noted that the graded structure of some basic fuzzy concepts of a fuzzy automaton such as source, successor, subautomaton, retrievability and separability is not so satisfactory. In view of this, to repair this situation, in [18], such concepts in fuzzy automata theory were presented with bifuzzy property. Simultaneously, the bifuzzy topological concepts are also used in [18] to characterize such algebraic properties.

In the work of Bavel [1], the *core* of an automaton plays an important role, which was later fuzzified in [22]. The concept of core of a fuzzy automaton introduced in [22] depends upon the concept of source, introduced in [15]. As it is pointed out that the grade structure of source associated with a fuzzy automaton is not so satisfactory; the concept of core presented in [22] needs modification, which is the theme of this paper. Specifically, we introduce the concept of bifuzzy core of a fuzzy automaton and explore the relationship of the bifuzzy topology induced by it and the bifuzzy topology introduced in [18]. Finally, we characterize the bifuzzy concepts such as bifuzzy family of submachines of a fuzzy automaton, bifuzzy separable family of a fuzzy automaton and bifuzzy retrievable family of fuzzy automaton introduced in [18], in terms of this bifuzzy topology. In between, we try to explore some other related and interesting results.

2. **Preliminaries**

In this section, we recall the basic definitions and results for fuzzy automaton with bifuzzy property, which is to be used in the next section.

Throughout, X is a nonempty set, $\mathscr{F}(X)$ denote the set of all fuzzy subsets of X and I denotes an index set.

For $A, B, A_i \in \mathscr{F}(X), i \in I$, and for $x \in X$

(i) $A \leq B$ means that $A(x) \leq B(x)$;
(ii) $(A \cap B)(x) = \min\{A(x), B(x)\}$ and $\cap_{i\in I} A_i(x) = \wedge_{i\in I} A_i(x)$;
(iii) $(A \cup B)(x) = \max\{A(x), B(x)\}$ and $\cup_{i\in I} A_i(x) = \vee_{i\in I} A_i(x)$;
(iv) $(A - B)(x) = \max\{0, A(x) - B(x)\}$. In particular, $(X - A)(x) = 1 - A(x)$.

Now, we introduce the following useful definitions and notations, which are similar to the semantic expressions in logic (cf., [18]).

Let $A, B \in \mathscr{F}(X)$. Then

(i) $\nu(A \subseteq B) = \wedge_{x\in X} \min\{1, 1 - A(x) + B(x)\}$,
(ii) $\nu(A \equiv B) = \min\{\nu(A \subseteq B), \nu(B \subseteq A)\}$,
(iii) $a \odot b = \max\{0, a + b - 1\}, a, b \in [0, 1]$.

Proposition 2.1. [18] *Let* $A_i, B_i \in \mathscr{F}(Q), i \in I$. *Then* $\wedge_{i\in I}\nu(A_i \equiv B_i) \leq \min\{\nu(\cup_{i\in I} A_i \equiv \cup_{i\in I} B_i), \nu(\cap_{i\in I} B_i \equiv \cap_{i\in I} A_i)\}$.

Next, we recall the concept of bifuzzy topology in the sense of Lowen (cf., [14]).

Definition 2.2. [24] Let $\mathscr{T} \in \mathscr{F}(\mathscr{F}(X))$. Then $\mathscr{T}$ is called a bifuzzy topology on X, if

(i) $\mathscr{T}(X) = \mathscr{T}(\phi) = 1$,
(ii) $\forall A, B \in \mathscr{F}(X), \min\{\mathscr{T}(A), \mathscr{T}(B)\} \leq \mathscr{T}(A \cap B)$,
(iii) $\forall A_i \in \mathscr{F}(X), i \in I, \wedge_{i \in I} \mathscr{T}(A_i) \leq \mathscr{T}(\cup_{i \in I} A_i)$.

If $\mathscr{T}$ is a bifuzzy topology on X, then the pair $(X, \mathscr{T})$ is called a bifuzzy topological space. The members of $\mathscr{T}$ are called bifuzzy $\mathscr{T}$-open. Lastly, we recall the concept of a fuzzy automaton with some of its bifuzzy properties.

Definition 2.3. [16] A fuzzy automaton is a triple $M = (Q, X, \delta)$, where Q is a nonempty set (of states of M), X is a monoid (the input monoid of M), whose identity shall be denoted as e, and δ is a fuzzy subset of $Q \times X \times Q$, i.e. a map $\delta : Q \times X \times Q \longrightarrow [0, 1]$ such that $\forall q, p \in Q$ and $\forall x, y \in X$

$$\delta(q, e, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\text{and} \quad \delta(q, xy, p) = \vee_{r \in Q} \min\{\delta(q, x, r), \delta(r, y, p)\}.$$

Remark 2.4. As in [20], we take the view that a (crisp) automaton is a triple (Q, X, δ), where Q is a (possibly infinite) set, X is a monoid with identity e, and $\delta : Q \times X \to Q$ is a function such that $\delta(q, e) = q$ and $\delta(q, xy) = \delta(\delta(q, x), y), \forall q \in Q, \forall x, y \in X$. By identifying δ with the function $\hat{\delta} : Q \times X \times Q \to [0, 1]$, given by $\hat{\delta}(q, x, p) = 1$ or 0 according as $\delta(q, x) = p$ or $\delta(q, x) \neq p$, we see that the concept of a fuzzy automaton generalizes that of a (crisp) automaton.

Definition 2.5. [16] Let (Q, X, δ) be a fuzzy automaton and $A \subseteq Q$, the source and the successor of A are respectively the sets

$$\sigma(A) = \{q \in Q : \delta(q, x, p) > 0, \text{ for some } (x, p) \in X \times A\}, \text{ and}$$
$$s(A) = \{p \in Q : \delta(q, x, p) > 0, \text{ for some } (x, q) \in X \times A\}.$$

The bifuzzy generalization of the above is the following.

Definition 2.6. [18] Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. The bifuzzy source and bifuzzy successor of A are respectively defined as follows:

$$SO(A)(p) = \vee_{q \in Q} \vee_{x \in X} \delta(p, x, q) \odot A(q), \text{ and}$$
$$SU(A)(p) = \vee_{q \in Q} \vee_{x \in X} \delta(q, x, p) \odot A(q).$$

Remark 2.7. It is easy to see that for all $A \in \mathscr{F}(Q)$, $A \leq SO(A)$ and $A \leq SU(A)$.

Definition 2.8. [18] Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. The bifuzzy family $SM \in \mathscr{F}(\mathscr{F}(Q))$ is called bifuzzy family of submachines of M, and is defined as follows:

$$SM(A) = \wedge_{p,q \in Q} \wedge_{x \in X} \min\{1, 1 - A(q) \odot \delta(q, x, p) + A(p)\}.$$

Definition 2.9. [18] Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. The bifuzzy family $SP \in \mathscr{F}(\mathscr{F}(X))$ is called bifuzzy separated family, and is defined as follows:

$$SP(A) = \wedge_{q \in Q}\{1 - SU(Q - A)(q) \odot A(q)\}.$$

Proposition 2.10. [18] *Let (Q, X, δ) be a fuzzy automaton. Then for $A \in \mathscr{F}(Q)$, $\nu(SU(A) \equiv A) = \nu(SO(Q-A) \equiv (Q-A)) = SP(Q-A)$.*

Now, we recall the following concept of bifuzzy retrievable family of a fuzzy automaton.

Definition 2.11. [18] Let $M = (Q, X, \delta)$ be a fuzzy automaton and $A \in \mathscr{F}(Q)$. The fuzzy family $RE(M) \in \mathscr{F}(A)$ called bifuzzy retrievable family is defined as

$$RE(M) = \wedge_{q,p \in Q} \wedge_{x \in X} min\{1, 1 - \delta(q,x,p) + \vee_{y \in X} \delta(p,y,q)\}.$$

Proposition 2.12. [18] *Let (Q, X, δ) be a fuzzy automaton, $A \in \mathscr{F}(Q)$ and $\mathscr{T}_{SU}(A) = \nu(SU(A) \equiv A)$. Then $\mathscr{T}_{SU}$ is a saturated*[1] *bifuzzy topology on Q. Also, let $\mathscr{T}_{SO}(A) = \nu(SO(A) \equiv A)$. Then $\mathscr{T}_{SO}$ is also a saturated bifuzzy topology on Q such that $\mathscr{T}_{SO} = \mathscr{T}_{SU}^c$.*

Proposition 2.13. [18] *Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. Then $SM(Q-A) = \mathscr{T}_{SO}(A)$.*

Proposition 2.14. [18] *Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. Then $SP(A) = \nu(SO(A) \equiv A)$.*

The following is the characterization of bifuzzy retrievable family of a fuzzy automaton in terms of bifuzzy topology introduced in Proposition 2.12.

Proposition 2.15. [18] *Let (Q, X, δ) be a fuzzy automaton. Then*

$$\begin{aligned} RE(M) &= \wedge_{A \in \mathscr{F}(Q)} min\{1, 1 - SM(A) + \nu(SO(A) \equiv A)\} \\ &= \wedge_{A \in \mathscr{F}(Q)} min\{1, 1 - \mathscr{T}_{SU}(A) + \mathscr{T}_{SO}(A)\}. \end{aligned}$$

3. Bifuzzy Core Associated with a Fuzzy Automaton

In this section, we introduce the concept of bifuzzy core of a fuzzy automaton, which induces a bifuzzy topology on the state-set of a fuzzy automaton and explore its relationship with the bifuzzy topology introduced in [18]. Also, we characterize some of the bifuzzy concepts of a fuzzy automaton given in [18], in terms of this bifuzzy topology.

We begin with the following from [22].

Definition 3.1. The core of a subset R of the state-set Q of a fuzzy automaton (Q, X, δ) is the set

$$\mu(R) = \{q \in R : \sigma(q) \subseteq R\}.$$

We introduce the following concept of bifuzzy core, which is a natural generalization of the above.

Definition 3.2. Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then the bifuzzy core of $A \in \mathscr{F}(Q)$ is defined as follows:

$$CO(A)(p) = \wedge_{q \in Q} \wedge_{x \in X} \min\{1, 1 - \delta(q,x,p) + A(q)\}, p \in Q.$$

[1]A bifuzzy topology $\mathscr{T}$ on X is being called here saturated if the (usual) requirement $\min\{\mathscr{T}(A), \mathscr{T}(B)\} \leq \mathscr{T}(A \cap B), \forall A, B \in \mathscr{F}(X)$ is replaced by $\wedge_{i \in I} \mathscr{T}(A_i) \leq \mathscr{T}(\cap_{i \in I} A_i), \forall A_i \in \mathscr{F}(X), i \in I$.

Example 3.3. Let $M = (Q, X, \delta)$ be a fuzzy automaton, where $Q = X = \{0, 1, 2, ...\}$ and $\delta : Q \times X \times Q \longrightarrow [0, 1]$ is given by

$$\delta(q, 0, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

with $\delta(q, x_0, p) = 0.8, \delta(q, x_0, q) = 0.66, \delta(p, x_0, p) = 0.5$ for fixed $x_0 \in X(x_0 \neq 0)$ and for fixed $p, q \in Q$. For other $p, q \in Q$ and $x \in X, \delta(p, x, q) = 0$. Also, let $A(p) = 1, A(q) = 0.75$, and for other $p \in Q, A(p) = 0$. Then $CO(A)(p) = \wedge_{q\in Q} \wedge_{x\in X} \min\{1, 1 - \delta(q, x, p) + A(q)\} = min\{1, 1, 0.95, 1\} = 0.95$. Similarly, $CO(A)(q) = 0.75$ and for other $p \in Q$, $CO(A)(p) = 0$.

Remark 3.4. For a fuzzy automaton $M = (Q, X, \delta)$ and $A \subseteq Q$, if $CO(A) = 1$ then CO is core in the sense of [22].

Proposition 3.5. *Let $A, B, A_i \in \mathscr{F}(Q), i \in I$. Then*

(i) $CO(A) \subseteq A$,
(ii) $A \leq B \Rightarrow CO(A) \subseteq CO(B)$,
(iii) $CO(\cap_{i\in I} A_i) = \cap_{i\in I} CO(A_i)$,
(iv) $CO(\cup_{i\in I} A_i) \geq \cup_{i\in I} CO(A_i)$

Proof. Let $q \in Q$. Then

$$(i) \quad \begin{aligned} CO(A)(q) &= \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1 - \delta(p, x, q) + A(p)\} \\ &\leq 1 - \delta(q, e, q) + A(q) \\ &= 1 - 1 + A(q) \\ &= A(q). \end{aligned}$$

$$(ii) \quad \begin{aligned} CO(A)(q) &= \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1 - \delta(p, x, q) + A(p)\} \\ &\leq \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1 - \delta(p, x, q) + B(p)\} \\ &= CO(B)(q). \end{aligned}$$

$$(iii) \quad \begin{aligned} CO(\cap\{A_i : i \in I\})(q) &= \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1 - \delta(p, x, q) + \wedge_{i\in I} A_i(p)\} \\ &= \wedge_{p\in Q} \wedge_{x\in X} \wedge_{i\in I} \min\{1, 1 - \delta(p, x, q) + A_i(p)\} \\ &= \wedge_{i\in I} \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1 - \delta(p, x, q) + A_i(p)\} \\ &= \cap\{CO(A_i) : i \in I\})(q). \end{aligned}$$

(iv) Similar to that of (iii). □

Proposition 3.6. *Let $A \in \mathscr{F}(Q), i \in I$. Then $CO(CO(A)) = CO(A)$.*

Proof. To prove $CO(CO(A)) = CO(A)$, it is enough to show that $\nu(CO(CO(A)) \equiv CO(A)) = 1$, i.e., $\min\{\nu(CO(CO(A)) \subseteq CO(A)), \nu(CO(A) \subseteq CO(CO(A)))\} = 1$. As, from Proposition 3.5 (i), $CO(A) \subseteq A, \forall A \in \mathscr{F}(Q)$, thus $\nu(CO(CO(A)) \subseteq CO(A)) = 1$. Therefore we have to only show that $\nu(CO(A) \subseteq CO(CO(A))) = 1$.

Let $CO(A)(q) > t, q \in Q, t \in [0, 1)$. Then

$$CO(A)(q) = \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1-\delta(p,x,q)+A(p)\} > t$$

or that,

$$1-\delta(p,x,q)+A(p) > t,\ p\in Q, x\in X,$$

i.e.,

$$A(p)-\delta(p,x,q) > t-1. \tag{1}$$

We claim that $CO(CO(A))(q) > t$; otherwise let $CO(CO(A))(q) \leq t$. Then

$$\wedge_{r\in Q} \wedge_{y\in X} \min\{1, 1-\delta(r,y,q)+CO(A)(r)\} \leq t,$$

or that,

$$1-\delta(r,y,q)+CO(A)(r) \leq t, \text{ for some } r\in Q \text{ and for some } y\in X,$$

i.e.,

$$1-\delta(r,y,q)+\wedge_{s\in Q} \wedge_{z\in X} \min\{1, 1-\delta(s,z,r)+A(s)\} \leq t$$

i.e.,

$$2-\delta(r,y,q)-\delta(s,z,r)+A(s) \leq t, \text{ for some } r,s\in Q \text{ and for some } y,z\in X,$$

i.e., for some $r,s\in Q$ and for some $y,z\in X$

$$\delta(r,y,q)+\delta(s,z,r)-A(s) \geq 2-t. \tag{2}$$

Now, let $p=s$ and $x=zy$ in inequality (1). Then

$$A(s)-\delta(s,zy,q) > t-1. \tag{3}$$

From inequalities (2) and (3), we obtain

$$\delta(r,y,q)+\delta(s,z,r)-\delta(s,zy,q) > 1. \tag{4}$$

However, $\delta(s,zy,q) = \vee_{r\in Q} \min\{\delta(s,z,r), \delta(r,y,q)\} \geq \min\{\delta(s,z,r), \delta(r,y,q)\}$ contradicts inequality (4). Thus the above claim holds, and so $\nu(CO(A) \subseteq CO(CO(A)))$ $= 1$. □

In the following proposition, we prove that the bifuzzy core CO induces a bifuzzy topology on Q.

Proposition 3.7. *Let (Q,X,δ) be a fuzzy automaton, $A\in \mathscr{F}(Q)$ and $\mathscr{T}_{CO}(A) = \nu(CO(A)\equiv A)$. Then $\mathscr{T}_{CO}$ is a saturated bifuzzy topology on Q.*

Proof. (i) Let $\alpha\in[0,1]$. Then for $q\in Q$,

$$\begin{aligned} CO(\alpha)(q) &= \wedge_{p\in Q} \wedge_{x\in X} \min\{1, 1-\delta(p,x,q)+\alpha(p)\} \\ &= \wedge_{p\in Q} \min\{1, 1-SO(q)(p)+\alpha(p)\} \\ &= \min\{1, 1-\vee_{p\in Q} SO(q)(p)+\alpha(p)\} \\ &= \min\{1, \alpha(q)\} \\ &= \alpha(q). \end{aligned}$$

Therefore $\mathscr{T}_{CO}(\alpha) = 1$ for all constant fuzzy sets α on Q, and thus $\mathscr{T}_{CO}(Q) = \mathscr{T}_{CO}(\phi) = 1$.
(ii) We have to show that $\wedge_{i\in I} \mathscr{T}_{CO}(A_i) \leq \mathscr{T}_{CO}(\cap_{i\in I} A_i)$.

Let $A_i \in \mathscr{F}(Q), i \in I$. Then, utilizing Propositions 2.1 and 3.5, part (ii), (iii), we have

$$\begin{aligned}
\wedge_{i\in I}\mathscr{T}_{CO}(A_i) &= \wedge_{i\in I}\nu(CO(A_i) \equiv A_i) \\
&\leq \min\{\nu(\cup_{i\in I}CO(A_i) \equiv \cup_{i\in I}A_i), \nu(\cap_{i\in I}CO(A_i) \equiv \cap_{i\in I}A_i)\} \\
&\leq \nu(\cap_{i\in I}A_i \subseteq CO(\cap_{i\in I}A_i)) \\
&= \nu(CO(\cap_{i\in I}A_i) \equiv \cap_{i\in I}A_i) \\
&= \mathscr{T}_{CO}(\cap_{i\in I}A_i).
\end{aligned}$$

(iii) We have to show that $\wedge_{i\in I}\mathscr{T}_{CO}(A_i) \leq \mathscr{T}_{CO}(\cup_{i\in I}A_i)$.
Let $A_i \in \mathscr{F}(Q), i \in I$. Then, utilizing Propositions 2.1 and 3.5, part (ii), (iv), we have

$$\begin{aligned}
\wedge_{i\in I}\mathscr{T}_{CO}(A_i) &= \wedge_{i\in I}\nu(CO(A_i) \equiv A_i) \\
&\leq \min\{\nu(\cup_{i\in I}CO(A_i) \equiv \cup_{i\in I}A_i), \nu(\cap_{i\in I}CO(A_i) \equiv \cap_{i\in I}A_i)\} \\
&\leq \nu(\cup_{i\in I}CO(A_i) \equiv \cup_{i\in I}A_i) \\
&\leq \nu(\cup_{i\in I}A_i \subseteq CO(\cup_{i\in I}A_i)) \\
&= \nu(\cup_{i\in I}A_i \equiv CO(\cup_{i\in I}A_i)) \\
&= \mathscr{T}_{CO}(\cup_{i\in I}A_i).
\end{aligned}$$

Hence $\mathscr{T}_{CO}$ is a saturated bifuzzy topology on Q. □

Proposition 3.8. *Let (Q, X, δ) be a fuzzy automaton. Then $\nu(CO(A) \equiv A) = \nu(SO(A) \equiv A), A \in \mathscr{F}(Q)$.*

Proof. Let $A \in \mathscr{F}(Q)$. Then as $CO(A) \subseteq A$ and $A \subseteq SO(A)$, $\nu(CO(A) \subseteq A) = \nu(A \subseteq SO(A)) = 1$.
Also,

$$\begin{aligned}
\nu(CO(A) \equiv A) &= \min\{\nu(CO(A) \subseteq A), \nu(A \subseteq CO(A))\} \\
&= \nu(A \subseteq CO(A)) \\
&= \wedge_{p\in Q}\min\{1, 1 - A(p) + CO(A)(p)\}, \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
\nu(SO(A) \equiv A) &= \min\{\nu(SO(A)) \subseteq A), \nu(A \subseteq SO(A))\} \\
&= \nu(SO(A) \subseteq A) \\
&= \wedge_{q\in Q}\min\{1, 1 - SO(A)(q) + A(q)\}. \qquad (6)
\end{aligned}$$

Firstly, we show that

$$\nu(CO(A) \equiv A) \leq \nu(SO(A) \equiv A). \qquad (7)$$

For this, let $\nu(CO(A) \equiv A) > t, t \in [0, 1)$. Then

$$\wedge_{p\in Q}\min\{1, 1 - A(p) + CO(A)(p)\} > t$$

or that,

$$\wedge_{r\in Q}\wedge_{x\in X}\min\{1, 1 - \delta(r, x, p) + A(r) - A(p)\} > t - 1, \quad p \in Q$$

i.e.,

$$1 - \delta(r, x, p) + A(r) - A(p) > t - 1,\ p, r \in Q \text{ and } x \in X,$$

i.e.,

$$A(r) - A(p) - \delta(r, x, p) > t - 2, p, r \in Q \text{ and } x \in X. \tag{8}$$

To verify inequality (7), we have to show that

$$\nu(SO(A) \equiv A) > t. \tag{9}$$

As,

$$\begin{aligned} SO(A)(q) &= \vee_{s\in Q} \vee_{y\in X}\, \delta(q, y, s) \odot A(s) \\ &= \vee_{s\in Q} \vee_{y\in X} \max\{0, \delta(q, y, s) + A(s) - 1\}, \end{aligned} \tag{10}$$

so (9) holds, if

$$A(q) - A(s) - \delta(q, y, s) > t - 2, \quad \text{for some } q, s \in Q \text{ and } y \in X. \tag{11}$$

Let $r = q, x = y$ and $s = p$ in inequality (8). Then $A(q) - A(s) - \delta(q, y, s) > t - 2$. Therefore (11) holds, and so (9) holds as well, whereby $\nu(CO(A) \equiv A) \leq \nu(SO(A) \equiv A)$.

Conversely, we have to show that $\nu(SO(A) \equiv A) \leq \nu(CO(A) \equiv A)$, i.e., if $\nu(SO(A) \equiv A) > t, t \in [0, 1)$, then $\nu(CO(A) \equiv A) > t, t \in (0, 1]$. Let $\nu(SO(A) \equiv A) > t, t \in [0, 1)$. Then from inequalities (9) and (11), $A(r)-A(p)-\delta(r, x, p) > t-2$, $\forall p, r \in Q$ and $\forall x \in X$, or that $\wedge_{r\in Q} \min\{1, 1-A(r)+CO(A)(r)\} > t$ (follows from (8)), and thus $\nu(A \equiv CO(A)) > t$. □

Proposition 3.9. $\mathscr{T}_{SO} = \mathscr{T}_{CO}$.

Proof. Follows from Propositions 2.12 and 3.8. □

In the next two propositions, we characterize the bifuzzy submachine and the bifuzzy separated submachine of a fuzzy automaton in terms of bifuzzy core.

Proposition 3.10. *Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. Then $SM(Q - A) = \mathscr{T}_{CO}(A)$.*

Proof. Follows from Propositions 2.13 and 3.9. □

Proposition 3.11. *Let (Q, X, δ) be a fuzzy automaton and $A \in \mathscr{F}(Q)$. Then $SP(A) = \mathscr{T}_{CO}(A)$.*

Proof. Follows from Propositions 2.14 and 3.8. □

Proposition 3.12. *Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then $SP(CO(A)) = \nu(SO(CO(A)) \equiv CO(A)), A \in \mathscr{F}(Q)$.*

Proof. From Proposition 2.10, $SP(CO(A)) = \nu(SU(Q-CO(A)) \subseteq (Q-CO(A))) = \nu(SU(Q - CO(A)) \equiv (Q - CO(A))) = \nu(SO(CO(A)) \equiv CO(A))$. □

Proposition 3.13. *Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then $SM(Q - CO(A)) = 1, A \in \mathscr{F}(Q)$.*

Proof. From Propositions 3.6, 3.7 and 3.10, $SM(Q - CO(A)) = \mathscr{T}_{CO}(CO(A)) = \nu(CO(CO(A)) \equiv CO(A)) = 1$. Thus $SM(Q - CO(A)) = 1$. □

Proposition 3.14. *Let* (Q, X, δ) *be a fuzzy automaton. Then* $\nu(SO(CO(A)) \equiv CO(A)) = 1, A \in \mathscr{F}(Q)$.

Proof. Let $A \in \mathscr{F}(Q)$. Then from Proposition 3.12, $SP(CO(A)) = \nu(SO(CO(A)) \equiv CO(A)) = \nu(SU(Q - CO(A)) \equiv (Q - CO(A)))$. Thus to prove $SP(CO(A)) = 1$, it is enough to show that $SU(Q - CO(A))(p) \leq 1 - CO(A)(p)$, $\forall p \in Q$. For this, let $SU(Q - CO(A))(p) > t, t \in [0, 1)$. Then

$$\vee_{q \in Q} \vee_{x \in X} \delta(q, x, p) \odot (1 - CO(A)(q)) > t$$

i.e.,

$$\delta(q, x, p) + 1 - CO(A)(q) - 1 > t, \text{ for some } p, q \in Q \text{ and for some } x \in X,$$

i.e.,

$$\delta(q, x, p) - \wedge_{s \in Q} \wedge_{z \in X} \min\{1, 1 - \delta(s, z, q) + A(s)\} > t, \text{ for some } p, q \in Q \text{ and for some } x \in X,$$

or that,

$$\delta(q, x, p) + \delta(s, z, q) - A(s) > t + 1, \tag{12}$$

for some $p, q, s \in Q$ and for some $x, z \in X$.

We claim that $1 - CO(A)(p) > t$; for if $1 - CO(A)(p) \leq t$. Then

$$1 - \wedge_{r \in Q} \wedge_{y \in X} \min\{1, 1 - \delta(r, y, p) + A(r)\} \leq t$$

i.e.,

$$A(r) - \delta(r, y, p) \geq -t, r \in Q, x \in X. \tag{13}$$

Now, let $s = r$ and $y = zx$ in inequality (13). Then

$$A(s) - \delta(s, zx, p) > -t. \tag{14}$$

Combining inequalities (12) and (14), we have

$$\delta(q, x, p) + \delta(s, z, q) - \delta(s, zx, p) > 1. \tag{15}$$

But, as $\delta(s, zx, p) = \vee_{r \in Q} \min\{\delta(s, z, r), \delta(r, x, p)\} \geq \min\{\delta(s, z, q), \delta(q, x, p)\}$, which contradicts inequality (15), whereby $1 - CO(A)(p) > t$. Hence $\nu(SO(CO(A)) \equiv CO(A)) = 1, A \in \mathscr{F}(Q)$. □

The characterization of bifuzzy retrievable family of a fuzzy automaton in terms of the bifuzzy topology induced by bifuzzy core is the following.

Proposition 3.15. *Let* (Q, X, δ) *be a fuzzy automaton. Then*

$$RE(M) = \wedge_{A \in \mathscr{F}(Q)} min\{1, 1 - \mathscr{T}_{SU}(A) + \mathscr{T}_{CO}(A)\}.$$

Proof. : Follows from Propositions 2.15 and 3.10. □

Lastly, the following is the characterization of bifuzzy retrievable family of a fuzzy automaton in terms of the bifuzzy core.

Proposition 3.16. *Let* $M = (Q, X, \delta)$ *be a fuzzy automaton. Then*

(i) $RE(M) = \wedge_{q, p \in Q} min\{1, 1 - CO(\{p\})(q) + CO(\{q\})(p)\}$,

(ii) *M is bifuzzy retrievable.*

Proof. $(i) \Rightarrow (ii)$: $RE(M) = \wedge_{q,p\in Q} min\{1, 1 - CO(\{p\})(q) + CO(\{q\})(p)\} = \wedge_{q,p\in Q} min\{1, 1 - \wedge_{y\in X} min(1, 2 - \delta(p,y,q)) + \wedge_{x\in X} min(1, 2 - \delta(q,x,p))\} = \wedge_{q,p\in Q} \wedge_{x\in X} min\{1, 1 + \vee_{y\in X}\delta(p,y,q) + (-\delta(q,x,p))\} = \wedge_{q,p\in Q} \wedge_{x\in X} min\{1, 1 + \vee_{y\in X}\delta(p,y,q) - \delta(q,x,p)\} = \wedge_{q,p\in Q} \wedge_{x\in X} min\{1, 1 - \delta(q,x,p) + \vee_{y\in X}\delta(p,y,q)\}$.

$(ii) \Rightarrow (i)$: The proof is similar to the above. □

4. **Conclusions**

In this paper, we tried to generalize the concept of core of a fuzzy automaton and presented the concept of bifuzzy core of a fuzzy automaton. Also, we introduced a bifuzzy topology on the state-set of a fuzzy automaton by using the concept of bifuzzy core, and characterized some of the bifuzzy concepts of a fuzzy automaton in terms of this induced bifuzzy topology. Much more can be further done by introducing the concepts of primaries and decomposition (as done e.g. in [22]) of fuzzy automata in this new set up. Of course, we need to overcome more difficulties due to the more complicated situations. We plan to carry out such a study in the near future.

Finite automata have significant applications [4]. Fuzzy finite automata also have important applications such as clinical monitoring [16]. In [18], it is pointed out that fuzzy automata with bifuzzy property may be used in understanding natural languages, which is a pivotal field in artificial intelligence. We hope that the work done in this paper may be further supplement in this direction.

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