FUZZY RESOLVENT EQUATION WITH $H(\cdot,\cdot)$-ϕ-η-ACCRETIVE OPERATOR IN BANACH SPACES

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Abstract. In this paper, we introduce and study fuzzy variational-like inclusion, fuzzy resolvent equation and $H(\cdot,\cdot)$-ϕ-η-accretive operator in real uniformly smooth Banach spaces. It is established that fuzzy variational-like inclusion is equivalent to a fixed point problem as well as to a fuzzy resolvent equation. This equivalence is used to define an iterative algorithm for solving fuzzy resolvent equation. Some examples are given.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [15] in 1965. Fuzzy sets are distinguished from ordinary or crisp sets in that the degree of membership of an element in a fuzzy set can be any number in the unit interval [0,1] as opposed to a number from the binary pair $\{0,1\}$ for crisp sets. The theory of variational inequalities and inclusions have had a great impact and influence in the development of almost all branches of pure and applied sciences.

This theory is widely used as a mathematical programming tool in modeling, many optimization and decision making problems. However, facing uncertainty is a constant challenge for optimization and decision making. Treating uncertainty by fuzzy mathematics results in the study of fuzzy optimization and decision making.

Ansari [1], Chang and Zhu [3], separately, introduced and studied a class of variational inequalities for fuzzy mappings. Since then, many papers appeared on the variational inequalities (inclusions) for fuzzy mappings, see for example [2, 4–10, 12, 13] and references therein.

The resolvent operator technique for solving variational inequalities (variational inclusions) is interesting and important. The resolvent operator technique is used to establish an equivalence between variational inequalities and resolvent equations. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving various classes of variational inequalities (variational inclusions) and related optimization problems.

In this paper, we introduce and study fuzzy variational-like inclusion, fuzzy resolvent equation and $H(\cdot,\cdot)$-ϕ-η-accretive operator in real uniformly smooth Banach spaces. The resolvent operator associated with $H(\cdot,\cdot)$-ϕ-η-accretive operator is defined and its Lipschitz continuity is shown. It is also shown that the fuzzy

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variational-like inclusion is equivalent to a fixed point problem as well as to a fuzzy resolvent equation. An iterative algorithm is suggested by this equivalence for solving fuzzy resolvent equation.

2. **Preliminaries**

Let X be a real Banach space equipped with the norm $\|\cdot\|$ and $X^\star$ be its topological dual. Let $\langle\cdot,\cdot\rangle$ be the duality coupling between X and $X^\star$ and 2^X be the power set of X.

Definition 2.1. [14] For $q > 1$, a mapping $J_q : X \to 2^{X^\star}$ is said to be a generalized duality mapping, which is defined by

$$J_q(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^q, \|x\|^{q-1} = \|f\|\},\ \forall x \in X.$$

In particular, J_2 is the usual normalized duality mapping on X. It is well known (see e.g. [14]) that

$$J_q(x) = \|x\|^{q-2} J_2(x),\ \forall\ x(\neq 0) \in X.$$

Note that if $X = H$, a real Hilbert space, then J_2 becomes the identity mapping on H.

Definition 2.2. [14] A Banach space X is said to be smooth if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$, defined by

$$\rho_X(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = t\right\}.$$

The Banach space X is said to be uniformly smooth, if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

Note that, if X is uniformly smooth, J_q becomes single-valued.

A mapping P from a set X to the collection $\mathcal{F}(X) = \{B : X \to [0,1],\ \ \text{a function}\}$ of fuzzy sets over X is called a fuzzy mapping, which means that for each $x \in X$ a fuzzy set $P(x)$, denoted by P_x, is a function from X to $[0,1]$. For each $y \in X$, $P_x(y)$ denotes the membership-grade of y in P_x.

A fuzzy mapping $P : X \to \mathcal{F}(X)$ is said to be closed if for each $x \in X$, the function $y \to P_x(y)$ is upper semicontinuous, that is, for any given net $\{y_\alpha\} \subset X$, satisfying $y_\alpha \to y_0 \in X$, we have

$$\limsup_{\alpha} P_x(y_\alpha) \leq P_x(y_0).$$

For $B \in \mathcal{F}(X)$ and $\lambda \in [0,1]$, the set $(B)_\lambda = \{x \in X : B(x) \geq \lambda\}$ is called a λ-cut set of B. Let $P : X \to \mathcal{F}(X)$ be a closed fuzzy mapping satisfying the following condition:

Condition $(*)$: There exists a function $a : X \to [0,1]$ such that for each $x \in X$, the set $(P_x)_{a(x)} = \{y \in X : P_x(y) \geq a(x)\}$ is a nonempty bounded subset of X.

Remark 2.3. [10] Let X be a normed vector space. If P is a closed fuzzy mapping satisfying the condition $(*)$, then for each $x \in X$, the set $(P_x)_{a(x)}$ belongs to the collection $CB(X)$ of all nonempty closed and bounded subsets of X. In fact, let $\{y_\alpha\} \subset (P_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in X$, then $(P_x)(y_\alpha) \geq a(x)$ for each α. Since P is closed, we have

$$P_x(y_0) \geq \lim_{\alpha} P_x(y_\alpha) \geq a(x),$$

which implies that $y_0 \in (P_x)_{a(x)}$ and so $(P_x)_{a(x)} \in CB(X)$.

Definition 2.4. Let $A, B : X \to X$ and $H, \eta : X \times X \to X$ be the single-valued mappings.

(i) A is said to be η-accretive if,

$$\langle Ax - Ay,\ J_q(\eta(x,y))\rangle \geq 0,\ \forall\ x, y \in X;$$

(ii) A is said to be strictly η-accretive, if A is η-accretive and the equality holds if and only if x=y;

(iii) $H(A,\cdot)$ is said to be α-strongly η-accretive with respect to A if, there exists a constant $\alpha > 0$ such that

$$\langle H(Ax,u) - H(Ay,u),\ J_q(\eta(x,y))\rangle \geq \alpha\|x-y\|^q,\ \forall\ x,y,u \in X;$$

(iv) $H(\cdot,B)$ is said to be β-relaxed η-accretive with respect to B if, there exists a constant $\beta > 0$ such that

$$\langle H(u,Bx) - H(u,By),\ J_q(\eta(x,y))\rangle \geq (-\beta)\|x-y\|^q,\ \forall\ x,y,u \in X;$$

(v) $H(\cdot,\cdot)$ is said to be r_1-Lipschitz continuous with respect to A if, there exists a constant $r_1 > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \leq r_1\|x-y\|,\ \forall\ x,y,u \in X.$$

In a similar way, we can define the Lipschitz continuity of the mapping $H(\cdot,\cdot)$ with respect to B.

(vi) η is said to be τ-Lipschitz continuous if, there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x-y\|,\ \forall\ x,y \in X.$$

Definition 2.5. Let $N, \eta : X \times X \to X$ be the single-valued mappings. Let $M : X \times X \to 2^X$ be a multi-valued mapping.

(i) M is said to be η-accretive if, for each fixed $z \in X$

$$\langle u - v,\ J_q(\eta(x,y))\rangle \geq 0, \forall\ x,y \in X, u \in M(x,z), v \in M(y,z);$$

(ii) M is said to be strictly η-accretive if, M is η-accretive and the equality holds if and only if $x = y$;

(iii) N is said to be ξ-Lipschitz continuous in the first argument if, there exists a constant $\xi > 0$ such that

$$\|N(x,u) - N(y,u)\| \leq \xi\|x-y\|,\ \forall x,y,u \in X.$$

Similarly, we can define the Lipschitz continuity of N in the second argument.

3. $H(\cdot,\cdot)$-ϕ-η-accretive Operator

Definition 3.1. Let $A, B, \phi : X \to X$ and $H, \eta : X \times X \to X$ be the single-valued mappings. A multi-valued mapping $M : X \times X \to 2^X$ is called an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B if, for some fixed $z \in X$, $\phi \circ M(\cdot, z)$ is η-accretive in the first argument and

$$(H(A,B) + \phi \circ M(\cdot, z))(X) = X.$$

Example 3.2. Let $X = \mathbb{R}$. Let $Ax = x, Bx = \sin x, H(Ax, By) = Ax + By$ and $M(x,z) = x^2 + z^2, \forall x \in X$ and for each fixed $z \in X$. Let $\phi \circ M(x,z) = \frac{\partial}{\partial x}[M(x,z)] = 2x$ and $\eta(x,y) = x - y$. Then

$$\begin{aligned}\langle \phi \circ M(x,z) - \phi \circ M(y,z), \eta(x,y)\rangle &= \langle 2x - 2y, x - y\rangle \\ &= 2(x-y)^2 \geq 0,\end{aligned}$$

which means that $\phi \circ M(\cdot, z)$ is η-accretive in the first argument. Also, for any $x \in X$, it follows from above that

$$(H(A,B) + \phi \circ M(\cdot,z)(x) = H(Ax,Bx) + \phi \circ M(x,z) = x + \sin x + 2x = 3x + \sin x,$$

which means that $(H(A,B) + \phi \circ M(\cdot, z))$ is surjective. Thus M is $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B.

Example 3.3. Let X, A, B, H and M are the same as in Example 3.1. Let $\phi \circ M(x,z) = e^{x^2+z^2}$ and $\eta(x,y) = y - x$. Then

$$\begin{aligned}\langle \phi \circ M(x,z) - \phi \circ M(y,z), \eta(x,y)\rangle &= \langle 2x - 2y, y - x\rangle \\ &= -2(x-y)^2 \leq 0,\end{aligned}$$

which means that $\phi \circ M(\cdot, z)$ is not η-accretive in the first argument. Also,

$$(H(A,B) + \phi \circ M(\cdot,z)(x) = H(Ax,Bx) + \phi \circ M(x,z) = x + \sin x + e^{x^2+z^2},$$

which shows that $0 \notin (H(A,B) + \phi \circ M(\cdot,z))(X)$, that is $(H(A,B) + \phi \circ M(\cdot,z))$ is not surjective, hence M is not $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings A and B.

Definition 3.4. Let $A, B, \phi : X \to X$ and $H, \eta : X \times X \to X$ be the single-valued mappings. Let $M : X \times X \to 2^X$ be an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B. The resolvent operator $R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}$ for some fixed $z \in X$ is defined by

$$R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(x) = (H(A,B) + \phi \circ M(\cdot,z))^{-1}(x).$$

Now, we show some properties of $H(\cdot,\cdot)$-ϕ-η-accretive operator.

Theorem 3.5. *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B for some fixed $z \in X$. Then the resolvent operator $(H(A,B) + \phi \circ M(\cdot,z))^{-1}$ is single-valued.*

Proof. Let $u \in X$ and $x, y \in (H(A,B) + \phi \circ M(\cdot,z))^{-1}(u)$. It follows that

$$u - H(Ax, Bx) \in \phi \circ M(x,z)$$

and

$$u - H(Ay, By) \in \phi \circ M(y,z).$$

Since $\phi \circ M(\cdot,z)$ is η-accretive in the first argument, we have

$$\begin{aligned}
0 &\le \langle u - H(Ax,Bx) - (u - H(Ay,By)), J_q(\eta(x,y)\rangle \\
&= -\langle H(Ax,Bx) - H(Ay,By), J_q(\eta(x,y))\rangle \\
&= -\langle H(Ax,Bx) - H(Ay,Bx), J_q(\eta(x,y))\rangle \\
&- \langle H(Ay,Bx) - H(Ay,By), J_q(\eta(x,y))\rangle \\
&\le -\alpha\|x-y\|^q + \beta\|x-y\|^q \\
&= -(\alpha-\beta)\|x-y\|^q \le 0.
\end{aligned}$$

Since $\alpha > \beta$, we have $x = y$ and so $(H(A,B) + \phi \circ M(\cdot,z))^{-1}$ is single-valued. □

Theorem 3.6. *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B, $\alpha > \beta$ and η is τ-Lipschitz continuous. Let $M : X \times X \to 2^X$ be an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B. Then the resolvent operator $R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)} : X \to X$ is $\frac{\tau^{q-1}}{\alpha-\beta}$-Lipschitz continuous, i.e.,*

$$\|R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(u) - R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(v)\| \le \frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|, \ \forall\, u, v \in X \text{ and each fixed } z \in X.$$

Proof. Let u, v be any points in X. It follows from the definition of resolvent operator that

$$R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(u) = (H(A,B) + \phi \circ M(\cdot,z))^{-1}(u),$$

and

$$R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(v) = (H(A,B) + \phi \circ M(\cdot,z))^{-1}(v).$$

This implies that

$$u - H(A(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(u)), B(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(u))) \in \phi \circ M(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(u), z),$$

and

$$v - H(A(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(v)), B(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(v))) \in \phi \circ M(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(v), z).$$

For our convenience, we denote $Pu = R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(u)$, and $Pv = R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(v)$.

Since $\phi \circ M(\cdot,z)$ is η-accretive in the first argument, we have

$$\langle u - H(A(Pu), B(Pu)) - (v - H(A(Pv), B(Pv))), J_q(\eta(Pu,Pv))\rangle \ge 0,$$

$$\langle u - v, J_q(\eta(Pu,Pv))\rangle \ge \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pv)), J_q(\eta(Pu,Pv))\rangle.$$

It follows that

$$\begin{aligned}\|u-v\|\|\eta(Pu,Pv)\|^{q-1} &\geq \langle u-v, J_q(\eta(Pu,Pv))\rangle \\ &\geq \langle H(A(Pu),B(Pu)) - H(A(Pv),B(Pv)), J_q(\eta(Pu,Pv))\rangle \\ &= \langle H(A(Pu),B(Pu)) - H(A(Pv),B(Pu)), J_q(\eta(Pu,Pv))\rangle \\ &+ \langle H(A(Pv),B(Pu)) - H(A(Pv),B(Pv)), J_q(\eta(Pu,Pv))\rangle \\ &\geq \alpha\|Pu-Pv\|^q - \beta\|Pu-Pv\|^q \\ &= (\alpha-\beta)\|Pu-Pv\|^q.\end{aligned}$$

Thus, by using the τ-Lipschitz continuity of η, we have

$$\|u-v\|\tau^{q-1}\|Pu-Pv\|^{q-1} \geq (\alpha-\beta)\|Pu-Pv\|^q$$

$$\|Pu-Pv\| \leq \frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|,$$

$$\text{i.e. } \|R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u) - R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)\| \leq \frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|.$$

This completes the proof. □

4. **Formulation of the Problem and Iterative Algorithm**

Let X be a real uniformly smooth Banach space and $\phi : X \to X$ be a mapping satisfying $\phi(x+y) = \phi(x) + \phi(y)$ and $\ker\phi = \{0\}$. Let $P, T, G : X \to \mathcal{F}(X)$ be closed fuzzy mappings satisfying condition $(*)$ with functions a, b, and $c : X \to [0,1]$, respectively. Let $A, B : X \to X$, $H, N, \eta : X \times X \to X$ be single-valued mappings. Suppose that a multi-valued mapping $M : X \times X \to 2^X$ is an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings A and B.

Problem 4.1. We consider the following problem of finding x, u, v and $z \in X$ such that

$$P_x(u) \geq a(x),\ T_x(v) \geq b(x),\ G_x(z) \geq c(x),$$

i.e.,

$$u \in (P_x)_{a(x)},\ v \in (T_x)_{b(x)},\ z \in (G_x)_{c(x)}$$

and

$$0 \in N(u,v) + M(x,z).$$

Problem 4.1 is called a fuzzy variational-like inclusion problem. Problem 4.1 includes many types of variational (-like) inequalities and variational inclusions (see [10]).

In connection with fuzzy variational-like inclusion Problem 4.1, we consider the following problem.

Problem 4.2. Find x, s, u, v and $z \in X$ such that

$$P_x(u) \geq a(x),\ T_x(v) \geq b(x),\ G_x(z) \geq c(x)$$

and that

$$\phi \circ N(u,v) + J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s) = 0,$$

and

$$J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot) = I - H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot))),$$

where

$$\begin{aligned} & H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) \\ = \ & (H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot))))(s), \end{aligned}$$

I is the identity mapping and $R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}$ is the resolvent operator.

Problem 4.2 is called fuzzy resolvent equation problem.

Now, we show that fuzzy variational-like inclusion Problem 4.1 is equivalent to a fixed point problem as well as to fuzzy resolvent equation Problem 4.2.

Lemma 4.3. *(x, u, v, z) is a solution of Problem 4.1 if and only if it is a solution of the following equation:*

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}[H(Ax, Bx) - \phi \circ N(u, v)]. \tag{1}$$

Proof. Let (x, u, v, z), where $x \in X$, $u \in (P_x)_{a(x)}$, $v \in (T_x)_{b(x)}$, and $z \in (G_x)_{c(x)}$ satisfying the equation (1), i.e.,

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}[H(Ax, Bx) - \phi \circ N(u, v)].$$

Using the definition of resolvent operator, we have

$$\begin{aligned} & x = (H(A, B) + \phi \circ M(\cdot, z))^{-1}[H(Ax, Bx) - \phi \circ N(u, v)] \\ \Leftrightarrow \ & H(Ax, Bx) - \phi \circ N(u, v) \in H(Ax, Bx) + \phi \circ M(x, z) \\ \Leftrightarrow \ & 0 \in \phi \circ N(u, v) + \phi \circ M(x, z) \\ \Leftrightarrow \ & 0 \in \phi(N(u, v) + M(x, z)) \\ \Leftrightarrow \ & \phi^{-1}(0) \in N(u, v) + M(u, z) \\ \Leftrightarrow \ & 0 \in N(u, v) + M(x, z). \end{aligned}$$

This completes the proof. □

Lemma 4.4. *(x, u, v, z) is a solution of Problem 4.1 with $x \in X, u \in (P_x)_{a(x)}, v \in (T_x)_{b(x)}$ and $z \in (G_x)_{c(x)}$, if and only if (s, x, u, v, z) with $s, x \in X, u \in (P_x)_{a(x)}, v \in (T_x)_{b(x)}$ and $z \in (G_x)_{c(x)}$ is a solution of Problem 4.2, where*

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s), \tag{2}$$

and

$$s = [H(Ax, Bx) - \phi \circ N(u, v)].$$

Proof. If (x, u, v, z) is a solution of Problem 4.1, then by Lemma 4.3, it is a solution of the following equation

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}[H(Ax, Bx) - \phi \circ N(u, v)].$$

Using the fact that

$$J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot) = I - H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot))),$$

$$\begin{aligned} & H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) \\ = \ & (H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)))(s) \end{aligned}$$

and equation (2), we have

$$s = H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) - \phi \circ N(u,v),$$

$$s - H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) = -\phi \circ N(u,v),$$

$$[I - H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)))](s) = -\phi \circ N(u,v),$$

$$J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s) = -\phi \circ N(u,v),$$

which implies that

$$J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s) + \phi \circ N(u,v) = 0,$$

with $s = H(Ax, Bx) - \phi \circ N(u,v)$, i.e., (s, x, u, v, z) is a solution of fuzzy resolvent equation Problem 4.2.

Conversely, suppose that (s, x, u, v, z) is a solution of fuzzy resolvent equation Problem 4.2. Then

$$J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s) = -\phi \circ N(u,v), \tag{3}$$

$$[I - H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)))](s) = -\phi \circ N(u,v),$$

$$s = H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) - \phi \circ N(u,v),$$

$$s = H(Ax, Bx) - \phi \circ N(u,v).$$

It follows that

$$R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s) = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}[H(Ax, Bx) - \phi \circ N(u,v)],$$

$$\text{i.e., } x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}[H(Ax, Bx) - \phi \circ N(u,v)].$$

Thus by Lemma 4.3, (x, u, v, z) is a solution of fuzzy variational-like inclusion Problem 4.1. □

Alternative Proof. Let $s = H(Ax, Bx) - \phi \circ N(u,v)$. Then from (2), we have

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)$$

and

$$s = H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) - \phi \circ N(u,v).$$

By using the fact that

$$\begin{aligned} & H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s))) \\ = \quad & H(A(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)), B(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(\cdot)))(s), \end{aligned}$$

it follows that

$$J_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(s) + \phi \circ N(u,v) = 0,$$

is the required fuzzy resolvent equation Problem 4.2. □

We now invoke Lemmas 4.3 and 4.4 to suggest the following iterative algorithm for solving fuzzy resolvent equation Problem 4.2.

Algorithm 4.5. Step 1: Choose an arbitrary initial point $x_0 \in X$ such that $u_0 \in (P_{x_0})_{a(x_0)} \in CB(X)$, $v_0 \in (T_{x_0})_{b(x_0)} \in CB(X)$ and $z_0 \in (G_{x_0})_{c(x_0)} \in CB(X)$.

Step 2: Set $s_1 = H(Ax_0, Bx_0) - \phi \circ N(u_0, v_0)$, and take next iteration point $x_1 \in X$ such that

$$x_1 = R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_0)}(s_1).$$

Step 3: Using Nadler's Theorem [11], obtain $u_1 \in (P_{x_1})_{a(x_1)}$, $v_1 \in (T_{x_1})_{b(x_1)}$, and $z_1 \in (G_{x_1})_{c(x_1)}$ such that

$$\begin{aligned} \|u_0 - u_1\| &\le \mathcal{H}((P_{x_0})_{a(x_0)}, (P_{x_1})_{a(x_1)}), \\ \|v_0 - v_1\| &\le \mathcal{H}((T_{x_0})_{b(x_0)}, (T_{x_1})_{b(x_1)}), \\ \|z_0 - z_1\| &\le \mathcal{H}((G_{x_0})_{c(x_0)}, (G_{x_1})_{c(x_1)}), \end{aligned}$$

where $\mathcal{H}$ is Hausdorff metric on $CB(X)$.

Step 4: Set $s_2 = H(Ax_1, Bx_1) - \phi \circ N(u_1, v_1)$, and take next iteration point $x_2 \in X$ such that

$$x_2 = R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_1)}(s_2),$$

and continuing the above process inductively.

Step 5: Let the sequences $\{x_n\}, \{s_n\}, \{u_n\}, \{v_n\}$ and $\{z_n\}$ be generated by the following iterative procedure:

(i) $x_n = R^{H(\cdot,\cdot)-\phi-\eta}_{(M\cdot,z_{n-1})}(s_n)$,
(ii) $u_n \in (P_{x_n})_{a(x_n)}, \|u_n - u_{n-1}\| \le \mathcal{H}((P_{x_n})_{a(x_n)}, (P_{x_{n-1}})_{a(x_{n-1})})$,
(iii) $v_n \in (T_{x_n})_{b(x_n)}, \|v_n - v_{n-1}\| \le \mathcal{H}((T_{x_n})_{b(x_n)}, (T_{x_{n-1}})_{b(x_{n-1})})$,
(iv) $z_n \in (G_{x_n})_{c(x_n)}, \|z_n - z_{n-1}\| \le \mathcal{H}((G_{x_n})_{c(x_n)}, (G_{x_{n-1}})_{c(x_{n-1})})$,
(v) $s_{n+1} = H(Ax_n, Bx_n) - \phi \circ N(u_n, v_n)$,

where $n = 0, 1, 2, \cdots$.
If s_{n+1} satisfies a prescribed stopping rule, terminate. Otherwise, repeat *Step 5* with n replaced by $n + 1$.

5. Existence and Convergence Result

Theorem 5.1. *Let X be a real uniformly smooth Banach space and $A, B, \phi : X \to X$ be single-valued mappings. Let $P, T, G : X \to \mathcal{F}(X)$ be closed fuzzy mappings satisfying the condition $(*)$ with functions a, b and $c : X \to [0, 1]$, respectively. Let $H, N, \eta : X \times X \to X$ be single-valued mappings and a multi-valued mapping $M : X \times X \to 2^X$ be an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings A and B. Assume that $\phi(x + y) = \phi(x) + \phi(y)$ and $\ker \phi = \{0\}$, $H(A, B)$ is r_1-Lipschitz continuous in the first argument and r_2-Lipschitz continuous in the second argument and $\phi \circ N(\cdot,\cdot)$ is ξ_1-Lipschitz continuous in the first argument and ξ_2-Lipschitz continuous in second argument. Let η be τ-Lipschitz continuous, $H(A, B)$ is α-strongly η-accretive with respect to A and β-relaxed η-accretive with respect to B and P, T and G are $\mathcal{H}$-Lipschitz continuous mappings with constants λ_P, λ_T and λ_G, respectively. Suppose that the following conditions are satisfied:*

$$\|R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_n)}(x) - R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_{n-1})}(x)\| \le \delta\|z_n - z_{n-1}\|, \forall x, z_n, z_{n-1} \in X \text{ and } \delta > 0. \quad (4)$$

$$(r_1 + r_2) + (\xi_1\lambda_P + \xi_2\lambda_T) < \frac{(\alpha - \beta)(1 - \delta\lambda_G)}{\tau^{q-1}}, \quad (5)$$
$$\alpha > \beta,\ 0 < \delta\lambda_G < 1.$$

Then (x, s, u, v, z) *with* $x, s \in X$, $u \in (P_x)_{a(x)}$, $v \in (T_x)_{b(x)}$ *and* $z \in (G_x)_{c(x)}$ *is a solution of fuzzy resolvent equation Problem* 4.2 *and the sequences* $\{x_n\}$, $\{s_n\}$, $\{u_n\}$, $\{v_n\}$ *and* $\{z_n\}$ *generated by Algorithm* 4.5 *converge strongly to* x, s, u, v *and* z, *respectively.*

Proof. From Algorithm 4.5, we have

$$\begin{aligned} & \|s_{n+1} - s_n\| \\ = \ & \|H(Ax_n, Bx_n) - \phi \circ N(u_n, v_n) - [H(Ax_{n-1}, Bx_{n-1}) - \phi \circ N(u_{n-1}, v_{n-1})]\| \\ \leq \ & \|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\| + \|\phi \circ N(u_n, v_n) - \phi \circ N(u_{n-1}, v_{n-1})\|. \end{aligned} \quad (6)$$

Since $H(A, B)$ is r_1-Lipschitz continuous in the first argument and r_2-Lipschitz continuous in the second argument, we have

$$\begin{aligned} \|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\| &= \|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_n) \\ &\quad + H(Ax_{n-1}, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\| \\ &\leq r_1\|x_n - x_{n-1}\| + r_2\|x_n - x_{n-1}\| \\ &= (r_1 + r_2)\|x_n - x_{n-1}\|. \end{aligned} \quad (7)$$

Since $\phi \circ N(\cdot, \cdot)$ is ξ_1-Lipschitz continuous in the first argument and ξ_2-Lipschitz continuous in the second argument, by using the $\mathcal{H}$-Lipschitz continuities of P, T and Algorithm 4.5, we have

$$\begin{aligned} \|\phi \circ N(u_n, v_n) - \phi \circ N(u_{n-1}, v_{n-1})\| &= \|\phi \circ N(u_n, v_n) - \phi \circ N(u_{n-1}, v_n) \\ &\quad + \phi \circ N(u_{n-1}, v_n) - \phi \circ N(u_{n-1}, v_{n-1})\| \\ &\leq \|\phi \circ N(u_n, v_n) - \phi \circ N(u_{n-1}, v_n)\| \\ &\quad + \|\phi \circ N(u_{n-1}, v_n) - \phi \circ N(u_{n-1}, v_{n-1})\| \\ &\leq \xi_1\|u_n - u_{n-1}\| + \xi_2\|v_n - v_{n-1}\| \\ &\leq \xi_1(\mathcal{H}((P_{x_n})_{a(x_n)}, (P_{x_{n-1}})_{a(x_{n-1})})) \\ &\quad + \xi_2(\mathcal{H}((T_{x_n})_{b(x_n)}, (T_{x_{n-1}})_{b(x_{n-1})})) \\ &\leq \xi_1\lambda_P\|x_n - x_{n-1}\| + \xi_2\lambda_T\|x_n - x_{n-1}\| \\ &= (\xi_1\lambda_P + \xi_2\lambda_T)\|x_n - x_{n-1}\|. \end{aligned} \quad (8)$$

From (7) and (8), (6) becomes

$$\begin{aligned} \|s_{n+1} - s_n\| &\leq (r_1 + r_2)\|x_n - x_{n-1}\| + (\xi_1\lambda_P + \xi_2\lambda_T)\|x_n - x_{n-1}\| \\ &= [(r_1 + r_2) + (\xi_1\lambda_P + \xi_2\lambda_T)]\|x_n - x_{n-1}\|. \end{aligned} \quad (9)$$

Using Theorem 3.6, Algorithm 4.5, condition (4) and $\mathcal{H}$-Lipschitz continuity of G, we have

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_{n-1})}(s_n) - R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_{n-2})}(s_n) + R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_{n-2})}(s_n) - \\ &\quad R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z_{n-2})}(s_{n-1})\| \\ &\leq \delta\|z_{n-1} - z_{n-2}\| + \frac{\tau^{q-1}}{\alpha - \beta}\|s_n - s_{n-1}\| \\ &\leq \delta(\mathcal{H}((Gx_{n-1})_{c(x_{n-1})}, (Gx_{n-2})_{c(x_{n-2})})) + \frac{\tau^{q-1}}{\alpha - \beta}\|s_n - s_{n-1}\| \\ &\leq \delta\lambda_G\|x_{n-1} - x_{n-2}\| + \frac{\tau^{q-1}}{\alpha - \beta}\|s_n - s_{n-1}\|. \end{aligned} \quad (10)$$

From (9), (10) becomes

$$\begin{aligned} & \|x_n - x_{n-1}\| \\ \leq\ & \delta\lambda_G \|x_{n-1} - x_{n-2}\| + \frac{\tau^{q-1}}{\alpha-\beta}[(r_1+r_2) + (\xi_1\lambda_P + \xi_2\lambda_T)]\|x_{n-1} - x_{n-2}\| \\ =\ & \delta\lambda_G + \frac{\tau^{q-1}}{\alpha-\beta}[(r_1+r_2) + (\xi_1\lambda_P + \xi_2\lambda_T)]\|x_{n-1} - x_{n-2}\|. \end{aligned} \tag{11}$$

It follows that

$$\|x_n - x_{n-1}\| \leq \theta \|x_{n-1} - x_{n-2}\|,$$

where

$$\theta = \delta\lambda_G + \frac{\tau^{q-1}}{\alpha-\beta}[(r_1+r_2) + (\xi_1\lambda_P + \xi_2\lambda_G)]. \tag{12}$$

It is easy to prove by condition (5) that $\theta < 1$ and so it follows that $\{x_n\}$ is a Cauchy sequence, and then from (9) that $\{s_n\}$ is also a Cauchy sequence. Let $x_n \to x$ and $s_n \to s$ (as $n \to \infty$) . By the $\mathcal{H}$-Lipschitz continuities of P, T and G and by (ii)-(iv) of Algorithm 4.5, it follows that $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ are all Cauchy sequences. Let $u_n \to u$, $v_n \to v$ and $z_n \to z$ (as $n \to \infty$). Now, using the continuities of the mappings $H, A, B, \phi \circ N, P, T, G, M$ and by (v) of Algorithm 4.5, we have

$$s = H(Ax, Bx) - \phi \circ N(u,v) = H(A(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(s)), B(R^{H(\cdot,\cdot)-\phi-\eta}_{M(\cdot,z)}(s))) - \phi \circ N(u,v).$$

Finally, we prove that $u \in (P_x)_{a(x)}$, $v \in (T_x)_{b(x)}$ and $z \in (G_x)_{c(x)}$. Since $u \in (P_{x_n})_{a(x_n)}$, we have

$$\begin{aligned} dist(u, (P_x)_{a(x)}) &\leq \|u - u_n\| + dist(u_n, (P_x)_{a(x)}) \\ &\leq \|u - u_n\| + dist(u_n, (P_{x_n})_{a(x_n)}) + \mathcal{H}((P_{x_n})_{a(x_n)}, (P_x)_{a(x)}) \\ &\leq \|u - u_n\| + 0 + \lambda_P \|x_n - x\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

It follows that $u \in (P_x)_{a(x)}$ due to the closedness of $(P_x)_{a(x)}$. Similarly, we can show that $v \in (T_x)_{b(x)}$ and $z \in (G_x)_{c(x)}$. By Lemma 4.4, the required result follows. □

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