GENERATED L-SUBGROUP OF AN L-GROUP

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Abstract. In this paper, we extend the construction of a fuzzy subgroup generated by a fuzzy subset to L-setting. This construction is illustrated by an example. We also prove that for an L-subset of a group, the subgroup generated by its level subset is the level subset of the subgroup generated by that L-subset provided the given L-subset possesses sup-property.

1. Introduction

After the introduction of the notion of fuzzy groups by Rosenfeld in 1971[22], the early development of fuzzy group theory is marked by the emergence of the notion of set product [19] which is an extension of the notion of product of complexes in classical group theory. The properties of set product are further discussed and explored in [1]. The results of this paper provided a formulation for the fuzzy subgroup generated by two given fuzzy subgroups of a group with identical tips. This leads to the formation of various types of lattices and sublattices of fuzzy subgroups in [2-3]. These results are already extended to L-setting [24]. However, no simplified construction of an L-subgroup generated by a given L-subset is available in the literature. Although, the construction of an L-subalgebra generated by a given L-subset of an algebra was provided by Biacino and Gerla [14] as early as in 1984. As this construction is not very much economical, it could not become popular among researchers. A particular case of Biacino and Gerlas version of generated L-subalgebra was later rediscovered by Ray [21] in case of fuzzy groups. Before this, an attempt was also made by Dixit et al in [15] to provide a construction for a generated fuzzy subgroup of a finite group. Later, Ajmal and Thomas modified, simplified and utilized the construction of generated fuzzy subalgebras in groups, rings and lattices in a series of papers [2-10]. The most economical construction of a generated fuzzy subgroup appeared so far, is due to Sultana and Ajmal [23] and later by Jain and Ajmal [13]. In the present paper, we extend these constructions to L-setting. We also prove that for an L-subset of a group, the subgroup generated by its level subset is the level subset of the subgroup generated by that L-subset provided the given L-subset possesses sup-property.

2. Preliminaries

Throughout this paper, the system $\langle L, \leq, \vee, \wedge \rangle$ denotes a completely distributive lattice where '$\leq$' denotes the partial ordering of L, the join (sup) and the meet (inf)

Received: July 2013; Accepted: February 2015
Key words and phrases: L-algebra, L-subgroup, Normal L-subgroup, Generated L-subgroup.

of the elements of L are denoted by '$\vee$' and '$\wedge$', respectively. Also, we write 1 and 0 for maximal and minimal elements of L, respectively. In this section, we first introduce some basic definitions and results which are used in the sequel. For details, we refer to [11, 12, 16, 19, 20, 24].

An L-subset of X is a function from X into L. The set of L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$, the set $\{\mu(x) : x \in X\}$ is called the image of μ and is denoted by $Im\ \mu$ and the tip of μ is defined as $\underset{x\in X}{\vee}\{\mu(x)\}$. If $\mu, \nu \in L^X$, then we say that μ is contained in ν if $\mu(x) \leq \nu(x)$ for every $x \in X$ and is denoted by $\mu \subseteq \nu$. For a family $\{\mu_i : i \in I\}$ of L-subsets of X, where I is a nonempty index set, the union $\underset{i\in I}{\cup}\mu_i$ and the intersection $\underset{i\in I}{\cap}\mu_i$ of the family $\{\mu_i : i \in I\}$ are, respectively, defined by

$$\left(\underset{i\in I}{\cup}\mu_i\right)(x) = \underset{i\in I}{\vee}\{\mu_i(x)\} \text{ and } \left(\underset{i\in I}{\cap}\mu_i\right)(x) = \underset{i\in I}{\wedge}\{\mu_i(x)\},$$

for each $x \in X$. If $\mu \in L^X$ and $a \in L$, then the level subset μ_a is defined as

$$\mu_a = \{x \in X : \mu(x) \geq a\}.$$

Proposition 2.1. *Let $\mu, \nu \in L^X$. Then,*

(i) *if $a, b \in L$ and $a \leq b$, then $\mu_b \subseteq \mu_a$,*
(ii) *if $\mu \subseteq \nu$, then $\mu_a \subseteq \nu_a$ for each $a \in L$.*

Throughout this paper G denotes an ordinary group with the identity element 'e', and I denotes a nonempty indexing set.

Definition 2.2. Let $\mu \in L^G$. Then, μ is called an L-subgroup of G if for each $x, y \in G$,

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(x^{-1}) = \mu(x)$.

The set of L-subgroups of G is denoted by $L(G)$. Clearly, the tip of an L-subgroup is attained at the identity element of G.

Theorem 2.3. *Let $\mu \in L^G$ with tip a_0. Then, $\mu \in L(G)$ if and only if μ_a is a subgroup of G for each $a \leq a_0$.*

It is well known in literature that the intersection of an arbitrary family of L-subgroups of a group is an L-subgroup of the given group.

Definition 2.4. Let $\mu \in L^G$. Then, the L-subgroup of G generated by μ, denoted by $\langle\mu\rangle$, is defined as the smallest L-subgroup of G which contains μ. i.e.

$$\langle\mu\rangle = \bigcap\{\mu_i \in L(G)\ :\ \mu \subseteq \mu_i\}.$$

If $\mu, \eta \in L^X$ and $\eta \subseteq \mu$, then we say that η is an L-subset of μ. We denote the set of L-subsets of μ by L^μ. If $\mu, \eta \in L(G)$ and $\eta \subseteq \mu$, then we say that η is an L-subgroup of μ. The set of L-subgroups of an L-group μ is denoted by $L(\mu)$.

From now on μ denotes an L-subgroup of G and we shall call the parent L-subgroup μ simply an L-group.

3. Generated L-subgroup of an L-group

We begin this section by the following result for generating an L-subgroup by a given L-subset of an L-group and study its relationship with other notions of L-group theory.

Theorem 3.1. *Let $\eta \in L^{\mu}$. Let $a_0 = \vee_{x\in G} \{\eta(x)\}$ and define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \vee_{a\leq a_0} \{a : x \in \langle \eta_a \rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$.

Proof. Firstly we prove that $\hat{\eta}$ is an L-subgroup of μ. For this purpose, for any $z \in G$ define the following subset of L :

$$L_\eta(z) = \{c \in L : c \leq a_0 \text{ and } z \in \langle \eta_c \rangle\}.$$

Now for any $x, y \in G$, we claim that

$$\text{if } a \in L_\eta(x) \text{ and } b \in L_\eta(y), \text{ then } a \wedge b \in L_\eta(xy).$$

So let $a \in L_\eta(x)$ and $b \in L_\eta(y)$. Then $x \in \langle \eta_a \rangle, y \in \langle \eta_b \rangle$ and $a, b \leq a_0$. Thus

$$x = x_1 ... x_n \text{ where } x_i \text{ or } x_i^{-1} \in \eta_a \text{ for each } i = 1, 2, ...n,$$
$$y = y_1 ... y_m \text{ where } y_i \text{ or } y_i^{-1} \in \eta_b \text{ for each } j = 1, 2, ...m.$$

This implies

$$xy = x_1 ... x_n y_1 ... y_m, x_i \text{ or } x_i^{-1} \text{ and } y_i \text{ or } y_i^{-1} \in \eta_a \bigcup \eta_b \subseteq \eta_{a \bigwedge b},$$

for each $i = 1, 2, ...n$ and $j = 1, 2, ...m$. Hence $xy \in \langle \eta_{a\wedge b} \rangle$. Also as $a, b \leq a_0$, we have $a \wedge b \leq a_0$. Therefore $a \wedge b \in L_\eta(xy)$. Thus

$$\hat{\eta}(xy) \geq a \wedge b \text{ for each } a \in L_\eta(x) \text{ and } b \in L_\eta(y).$$

Accordingly

$$\begin{aligned}\hat{\eta}(xy) &\geq \vee\{a \wedge b : a \in L_\eta(x), b \in L_\eta(y)\} \\ &= \{\vee\{a : a \in L_\eta(x)\}\} \wedge \{\vee\{b : b \in L_\eta(y)\}\} \\ &\qquad\qquad \text{(as } L \text{ is a completely distributive lattice)} \\ &= \hat{\eta}(x) \wedge \hat{\eta}(y).\end{aligned}$$

Similarly, we can verify that $\hat{\eta}(x) = \hat{\eta}(x^{-1})$ for each $x \in G$. Thus $\hat{\eta} \in L(G)$. Further as $\eta \subseteq \mu$, by Proposition 2.1 $\eta_a \subseteq \mu_a$ for each $a \in L$. Also by Theorem 2.3, μ_a is a subgroup of G . Therefore $\langle \mu_a \rangle = \mu_a$. Moreover,

$$\vee_{a\leq\mu(e)} \{a : x \in \mu_a\} = \mu(x) \text{ for each } x \in G.$$

Thus if $x \in G$, then

$$\hat{\eta}(x) = \vee_{a\leq a_0} \{a : x \in \langle \eta_a \rangle\} \leq \vee_{a\leq\mu(e)} \{a : x \in \mu_a\} = \mu(x).$$

So $\hat{\eta} \subseteq \mu$. Hence $\eta \in L(\mu)$. Now we show that $\hat{\eta}$ is the least $L-$subgroup of μ containing η. Set $a = \eta(x)$ for $x \in G$ so that $a \leq a_0$ and $x \in \eta_a \subseteq \langle \eta_a \rangle$. Thus by the definition of $\hat{\eta}$, $\hat{\eta}(x) \geq \eta(x)$ for each $x \in G$. Hence $\eta \subseteq \hat{\eta}$. Next, let $\theta \in L(\mu)$ be such that $\eta \subseteq \theta$. So by Proposition 2.1,$\eta_a \subseteq \theta_a$. In view of Theorem 2.3, $\langle \eta_a \rangle \subseteq \theta_a$ for each $a \leq \theta(e)$. Therefore for each $x \in G$, it follows that

$$\hat{\eta}(x) = \underset{a\leq a_0}{\vee}\{a : x \in \langle\eta_a\rangle\} \leq \underset{a\leq\theta(e)}{\vee}\{a : x \in \theta_a\} = \theta(x).$$

So that $\hat{\eta} \subseteq \theta$. This completes the proof. □

An L-subset in general may not attain its tip. However, the following corollary shows that the tip of a given L-subset is attained by the L-subgroup generated by that L-subset.

Corollary 3.2. *Let $\eta \in L^\mu$. Then, $\langle\eta\rangle(e) = \underset{x\in G}{\vee}\{\eta(x)\}$.*

The research carried out so far in the area of fuzzy algebraic structures is mainly in two directions. In the first direction, the results obtained are generally simple extensions of their classical counterparts. In the other direction, the studies reflect some peculiarities of fuzzy setting. It is easy to observe that in the literature of fuzzy algebra at numerous places, the researchers imposed the condition of sup-property in order to extend the classical results. It is always in those cases where the definition of a property of fuzzy subgroups which is an extension of the corresponding classical property, involves supremum of certain subsets of reals. Also, it is due to sup-property that at number of places a classical property of subsets of ordinary algebra is carried over from level subsets of fuzzy subsets to its fuzzy version. This very idea was exploited by Professor Tom Head[17] for strong level subsets to formulate his well known metatheorem. For the application of strong level subsets, sup-property is just not required. Unfortunately, the metatheorem does not apply in the studies of L-algebraic substructures. The notion of sup-property remains significant in L-setting as it can be used to extend the results of classical algebra to L-setting. This concept can also be applied in the construction of various lattices and sublattices of L-algebraic substructures of L-algebras.

Definition 3.3. An L-subset $\eta \in L^\mu$ is said to have sup-property if for every nonempty subset A of G there exists $a_0 \in A$ such that $\underset{a\in A}{\vee}\{\eta(a)\} = \eta(a_0)$.

Proposition 3.4. *Let $\eta \in L^\mu$. Then η has sup-property if and only if $Im\eta$ is closed under arbitrary supremums.*

For the sake of completeness we prove the following theorem which is an extension of a result established in [13].

Theorem 3.5. *Let $\eta \in L^\mu$ and possesses sup-property. Then, define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \underset{a\in Im\eta}{\vee}\{a : x \in \langle\eta_a\rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle\eta\rangle$. Moreover, $\hat{\eta}$ possesses sup-property.

Proof. Firstly we prove that $\hat{\eta}$ is an L subgroup of μ. For this purpose, for any $z \in G$ define the following subset of L :

$$L_\eta(z) = \{c \in Im\ \eta : z \in \langle\eta_c\rangle\}.$$

Now for any $x, y \in G$, we claim that

$$\text{if } a \in L_\eta(x) \text{ and } b \in L_\eta(y), \text{ then } a \wedge b \in L_\eta(xy).$$

So let $a \in L_\eta(x)$ and $b \in L_\eta(y)$. One can verify as in the proof of the above theorem that

$$xy \in \langle \eta_{a\wedge b} \rangle.$$

Now as $a \in L_\eta(x)$ and $b \in L_\eta(y)$, we get $a,\ b \in Im\eta$. Since η possesses sup-property, by Proposition 3.4 $Im\eta$ is closed under arbitrary supremums. Thus

$$a \vee b = a \text{ or } b.$$

This implies

$$a \wedge b = b \text{ or } a \in Im\eta.$$

Hence $a \wedge b \in L_\eta(xy)$. Again by following the lines of the proof of the above theorem, we can establish that $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$. Next, we prove that $\hat{\eta}$ possesses sup-property. For this purpose, let X be a subset of G. Then,

$$\hat{\eta}(g) = \bigvee_{a \in Im\eta} \{a : g \in \langle \eta_a \rangle\} \text{ for } g \in X.$$

Again as η possesses sup-property, by Proposition 3.4 $Im\eta$ is closed under arbitrary supremums. Therefore, there exists $a_g \in Im\eta$ and $g \in \langle \eta_{a_g} \rangle$ such that $\hat{\eta}(g) = a_g$. Consequently,

$$\bigvee_{g\in X} \{\hat{\eta}(g)\} = \bigvee_{g\in X} \{a_g : a_g \in Im\eta \text{ and } g \in \langle \eta_{a_g} \rangle\}.$$

By repeating the above arguments, there exists $g_0 \in X$ such that

$$\bigvee_{g\in X} \{\hat{\eta}(g)\} = a_{g_0} = \hat{\eta}(g_0).$$

This establishes the result. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.6. *Let $\eta \in L^\mu$ and possesses sup-property. Then, $Im\langle \eta \rangle \subseteq Im\eta$.*

Theorem 3.7. *Let $\eta \in L^\mu$ and possesses sup-property. If $a_0 = \bigvee_{x\in G} \{\eta(x)\}$, then for each $b \le a_0$, $\langle \eta_b \rangle = \langle \eta \rangle_b$.*

Proof. Let $a_0 = \bigvee_{x\in G} \{\eta(x)\}$ and $b \le a_0$. Let $x \in \langle \eta \rangle_b$. Then ,$\langle \eta \rangle(x) \ge b$. By Theorem 3.5

$$\langle \eta \rangle (x) = \bigvee_{a \in Im\eta} \{a : x \in \langle \eta_a \rangle\} \ge b.$$

Since η has sup-property, by Proposition 3.4 $Im\ \eta$ is closed under arbitrary supremums . So there exists $b_0 \in Im\ \eta$ such that $x \in \langle \eta_{b_0} \rangle$ and $b_0 \ge b$. This implies $\eta_{b_0} \subseteq \eta_b$ and hence $x \in \langle \eta_b \rangle$. Thus $x \in \langle \eta_b \rangle$. So $\langle \eta \rangle_b \subseteq \langle \eta_b \rangle$. Also as $\eta \subseteq \langle \eta \rangle$, by Proposition 2.1 $\eta_b \subseteq \langle \eta \rangle_b$. As by Theorem 2.3 $\langle \eta \rangle_b$ is a subgroup of G, it follows that $\langle \eta_b \rangle \subseteq \langle \eta \rangle_b$. This establishes the result. □

The following example will demonstrate that the condition of sup-property is crucial and can not be removed from the above result.

Example 3.8. Let Z be the group of integers under addition, and let $\langle 2^n \rangle$ be the subgroup of Z generated by 2^n, where n is a fixed positive integer. The direct product $Z \times Z$ contains subgroups

$$\langle 2^r \rangle \times \langle 2^s \rangle \text{ for each } r, s = 0, 1, 2, ...$$

Define the following L-subset of $Z \times Z$ where L is the closed unit interval ordered by usual ordering of real numbers:

$$\mu(x) = \begin{cases} 0 & \text{if } x \in Z \times Z \sim \langle 2 \rangle \times Z, \\ \frac{3}{4} & \text{if } x \in \langle 2 \rangle \times Z. \end{cases}$$

$$\eta(x) = \begin{cases} 0 & \text{if } x \in x \in Z \times Z \sim \langle 2 \rangle \times Z, \\ \frac{1}{2}(1 - \frac{1}{2^n}) & \text{if } x \in \langle 2^n \rangle \times Z \sim \langle 2^{n+1} \rangle \times Z, \text{ where } n = 1, 2, 3, ... \\ 0 & \text{if } x \in \langle 0 \rangle \times Z \sim \langle 0 \rangle \times \langle 2 \rangle, \\ \frac{3}{4}(1 - \frac{1}{4^n}) & \text{if } x \in \langle 0 \rangle \times \langle 2^n \rangle \sim \langle 0 \rangle \times \langle 2^{n+1} \rangle, \text{ where } n = 1, 2, 3, ... \\ \frac{3}{4} & \text{if } x = (0, 0). \end{cases}$$

Here $A \sim B$ means usual set difference. Clearly, $\eta \subseteq \mu$, $\eta \neq \mu$ and $\mu \in L(G)$. Observe that η does not possess sup-property and for $t = \frac{1}{2}$,

$$\langle 0 \rangle \times \langle 2 \rangle = \langle \eta_{\frac{1}{2}} \rangle \subset \langle \eta \rangle_{\frac{1}{2}} = \langle 0 \rangle \times Z.$$

In [18], the author while constructing the fuzzy subgroup $\langle \eta \rangle$ of a group G claimed that for each $t \in Im\ \langle \eta \rangle$, that the level subset $\langle \eta \rangle_t$ is the ordinary subgroup of G generated by some level subset $\langle \eta_\alpha \rangle$ i.e.

$$\langle \eta \rangle_t = \langle \eta_\alpha \rangle \text{ for some } \alpha \in [0, 1].$$

This argument is also incorrect as is demonstrated in [23].

The following example of a generated L-subgroup shows that the condition of sup-property is not necessary in Corollary 3.5 and Theorem 3.6 :

Example 3.9. Let

$$L = \{l,\ f,\ a,\ b,\ c,\ d,\ u\}$$

be a lattice given by Figure 1:

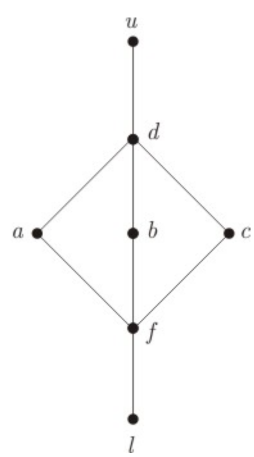

FIGURE 1

Let $D_8 = \{\langle x, y\rangle \ : \ x^2 = \ e \ = y^8, \ xy = y^{-1}x\}$. If

$$D_4 = \{\langle x, y^2\rangle \ : \ x^2 = \ e \ = (y^2)^4, \ xy^2 = y^{-2}x\}$$

is a dihedral subgroup of D_8.Then, define the following L-subsets of D_8:

$$\mu(z) = \begin{cases} u & \text{if } z \in D_4, \\ f & \text{if } z \in D_8 \sim D_4. \end{cases}$$

$$\eta(z) = \begin{cases} d & \text{if } z \in \{e\}, \\ a & \text{if } z \in \{x, xy^4\}, \\ b & \text{if } z \in \{xy^2, xy^6\}, \\ c & \text{if } z \in \{y^2, y^4, y^6\}, \\ l & \text{if } z \in D_8 \sim D_4. \end{cases}$$

Here $A \sim B$ means usual set difference. Clearly, $\eta \subseteq \mu$, $\eta \neq \mu$ and $\eta, \mu \in L(G)$. Further, the following is easy to verify:

$$\begin{aligned} &\eta_d = \{e\}, \\ &\langle \eta_a \rangle = \langle \{e, x, xy^4\}\rangle = \{e, x, y^4, xy^4\}, \\ &\langle \eta_b \rangle = \langle \{e, xy^2, xy^6\}\rangle = \{e, y^4, xy^2, xy^6\}, \\ &\langle \eta_c \rangle = \langle \{y^2, y^4, y^6\}\rangle = \{e, y^2, y^4, y^6\}, \\ &\eta_l = D_8. \end{aligned}$$

Now by applying the construction of Theorem 3.6 on the L-subset η of μ and after necessary calculations, we get the L-subgroup $\langle \eta \rangle$ as follows:

$$\begin{aligned} &\langle \eta \rangle_d = \{e\}, \\ &\langle \eta \rangle_a = \{e, y^4, x, xy^4\}, \\ &\langle \eta \rangle_b = \{e, x, y^4, xy^2, xy^6\}, \\ &\langle \eta \rangle_c = \{e, y^2, y^4, y^6\}, \\ &\langle \eta \rangle_l = D_8. \end{aligned}$$

Next, note that in this example the L-subset η does not possess sup-property, but

$$\langle \eta_r \rangle = \langle \eta \rangle_r$$

for each $r \leq \eta(e)$ and also

$$Im\langle \eta \rangle \subseteq Im\eta.$$

Acknowledgements. The authors are highly indebted to the unknown referees who have worked hard to evaluate this work.

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