FUZZY TRANSPOSITION HYPERGROUPS

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Abstract. In this paper we introduce the notions of fuzzy transposition hypergroups and fuzzy regular relations and investigate their basic properties. We also study fuzzy quotient hypergroups of a fuzzy transposition hypergroup.

1. Introduction

Hyperstructure theory was born in 1934 [11], when F. Marty defined hypergroups and investigated their properties. Today hypergroups are widely studied from the theoretical point of view and for their application in to many fields. In 1965 [15] L. A. Zadeh introduced the notion of a fuzzy subset of a non-empty set X, as a function from X to $[0, 1]$ and in 1971 [13] A. Rosenfeld defined the concept of a fuzzy group. Since then many papers havebeen published in the field of fuzzy algebra. Recently fuzzy set theory has been developed in the context of hyperalgebraic structure theory [1, 2, 5, 6, 7, 9, 14, 16, 17]. In [14] we generalized the notion of a semihypergroup H to a fuzzy semihypergroup by replacing $P^{\star}(H)$, the set of all non-empty subsets of H, by $F(H)$, the set of all fuzzy subsets of H, and studied its basic properties. Moreover, this author introduced the notion of transposition hypergroups and studied its quotient space [8]. In this paper, we shall extend these ideas in the context of a fuzzy hyperalgebraic structure.

We first recall some definitions and theorems which will be used in the paper [3,4].

Definition 1.1. Let $H(\neq \emptyset)$ be a set and $P^{\star}(H)$ be the family of all nonempty subsets of H and $*$ a *hyperoperation* or a *join operation*, that is, $*$ is a mapping from $H \times H$ to $P^{\star}(H)$. If $(a, b) \in H \times H$, its image under $*$ is denoted by $a * b$ or ab.

Remark 1.2. The hyperoperation can be extended to subsets of H in a natural way, so that $A \cdot B$ or AB is given by

$$AB = \bigcup \{ab| \quad a \in A, b \in B\}.$$

The notations aA and Aa are used for $\{a\}A$ and $A\{a\}$, respectively. Generally, the singleton $\{a\}$ is identified with its element a.

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Definition 1.3. A hypergroupoid is a structure $(H, \cdot)$. If the hypergroupoid $(H, \cdot)$ is associative, i.e.

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad for \quad all \quad x, y, z \in H,$$

it is called a semihypergroup.

Definition 1.4. A *hypergroup* is a semihypergroup such that

$$x \cdot H = H = H \cdot x \quad for \quad all \quad x \in H \quad (reproduction\ axiom).$$

Definition 1.5. Let $(H, \cdot)$ be a quasihypergroup. We define two hyperoperations $/$, called right extension and $\backslash$, called left extension on H by
(i) $a/b = \{x \in H : a \in x \cdot b\}$ for all $a, b \in H$
and
(ii) $b\backslash a = \{x \in H : a \in b \cdot x\}$ for all $a, b \in H$.
If A and B are two subsets of H, then $A/B = \bigcup_{a \in A, b \in B} a/b$ and $A\backslash B = \bigcup_{a \in A, b \in B} a\backslash b$.

Definition 1.6. [8] A *transposition hypergroup* is a hypergroup such that

$$b\backslash a \cap c/d \neq \emptyset \Longrightarrow (a \cdot d) \cap (b \cdot c) \neq \emptyset \quad for \quad all \quad a, b, c, d \in H \quad (transposition \quad axiom).$$

Definition 1.7. [8] Let $(H, \cdot)$ be a hypergroup and K a nonempty subset of H. Then $(K, \cdot)$ is a *subhypergroup* of H if K is a hypergroup under the hyperoperation $\cdot$ restricted to K. It is clear that a subset K of H is a subhypergroup if and only if $a \cdot K = K \cdot a = K$ for all $a \in K$ under the hyperoperation on H. We write: $K <_P H$.

Definition 1.8. [8] Let $(H, \cdot)$ be a hypergroup. A nonempty subset N of H is said to be closed if

$$N/N \subseteq N \quad and \quad N\backslash N \subseteq N.$$

Definition 1.9. [8] Let $(H, \cdot)$ be a hypergroup. A nonempty subset N of H is said to be a subsemihypergroup of $(H, \cdot)$ if

$$N \cdot N \subseteq N.$$

Definition 1.10. [8] A subsemihypergroup $(N, \cdot)$ of a hypergroup $(H, \cdot)$ is said to be reflexive if

$$a\backslash N = N/a \quad for \quad all \quad a \in H.$$

2. **Fuzzy Hyperoperations**

Definition 2.1. [14] Let S be a non-empty set and $F(S)$ denote the set of all fuzzy subsets of S.
A fuzzy hyperoperation on S is a mapping $\circ : S \times S \mapsto F(S)$. The image of (a, b) is denoted by $a \circ b$.
The set S together with a fuzzy hyperoperation $\circ$ is called a fuzzy hypergroupoid.

Definition 2.2. [14] Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$, then we define $\mu \circ \nu$ as follows:

$$(\mu \circ \nu)(t) = \vee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q)) \quad for \quad all \quad t \in S.$$

Definition 2.3. [14] A fuzzy hypergroupoid $(S, \circ)$ is called a fuzzy semihypergroup if

$$(a \circ b) \circ c = a \circ (b \circ c) \quad for \quad all \quad a, b, c \in S.$$

Comment 2.4. [14] Let A and B be two subsets of a fuzzy hypergroupoid $(S, \circ)$, then

$$(A \circ B)(x) = \vee_{a \in A, b \in B}(a \circ b)(x) \quad for \quad all \quad x \in S.$$

Definition 2.5. [14] A fuzzy semihypergroup $(S, \circ)$ is called a fuzzy hypergroup if

$$x \circ S = S \circ x = \chi_S, \quad for \quad all \quad x \in S,$$

where χ_S is the characteristic function of the set S.

Example 2.6. [14] Define a fuzzy hyperoperation $\circ$ on a non-empty set S by $a\circ = \chi_{\{a,b\}}$ for all $a, b \in S$, where $\chi_{\{a,b\}}$ denotes the characteristic function of the set $\{a, b\}$. Then $(S, \circ)$ is both a fuzzy semihypergroup and a fuzzy hypergroup.

Example 2.7. [14] Let S be a semigroup and define a fuzzy hyperoperation $\circ$ on S by $a \circ b = \chi_{ab}$, for all $a, b \in S$, where χ_{ab} is the characteristic function of the element ab in S. Then $(S, \circ)$ is a fuzzy semihypergroup. Moreover, if S is a group then $(S, \circ)$ is a fuzzy hypergroup.

3. **Fuzzy Transposition Hypergroup**

In this section we introduce two fuzzy hyperoperations / , (right extension) and \ (left extension) on a fuzzy hyper structure $(H, \circ)$ and study their algebraic properties. Then we introduce the concepts of fuzzy transposition hypergroups and fuzzy subhypergroups.

Definition 3.1. Let $(H, \circ)$ be a fuzzy hypergroupoid. The fuzzy hyperoperations / (right extension) and \ (left extension) on H are defined by

$$(i) a/b(x) = (x \circ b)(a) \quad for \quad all \quad a, b, x \in H$$

and

$$(ii) b\backslash a(x) = (b \circ x)(a) \quad for \quad all \quad a, b, x \in H.$$

Let $(H, \circ)$ be a fuzzy hypergroupoid. If μ and ν are two fuzzy subsets of H, then
$a/\mu(x) = \bigvee_{b \in H}(a/b(x) \wedge \mu(b)) = \bigvee_{b \in H}(x \circ b)(a) \wedge \mu(b) = (x \circ \mu)(a)$,
$\mu\backslash a(x) = \bigvee_{b \in H}(b\backslash a(x) \wedge \mu(b)) = \bigvee_{b \in H}(b \circ x)(a) \wedge \mu(b) = (\mu \circ x)(a)$,
$\mu/a(x) = \bigvee_{b \in H}(b/a(x) \wedge \mu(b))$, $\quad a\backslash\mu(x) = \bigvee_{b \in H}(a\backslash b(x) \wedge \mu(b))$,
$\mu/\nu(x) = \bigvee_{a,b \in H}(a/b(x) \wedge \mu(a) \wedge \nu(b))$, $\quad \mu\backslash\nu(x) = \bigvee_{a,b \in H}(a\backslash b(x) \wedge \mu(a) \wedge \nu(b))$.

Theorem 3.2. *Let $(H, \circ)$ be a fuzzy hypergroup. Then, for all $a, b, c \in H$, we have*
(i) $(a/b)/c = a/(c \circ b)$, $\quad (i')$ $c\backslash(b\backslash a) = (b \circ c)\backslash a$,
(ii) $(b\backslash a)/c = b\backslash(a/c)$,
(iii) $((a/b)\backslash a)(b) = 1$, $\quad (iii')$ $(a/(b\backslash a))(b) = 1$,

Proof. (*i*) $((a/b)/c)(x) = \bigvee_{p\in H}(a/b(p) \wedge p/c(x)) = \bigvee_{p\in H}((p \circ b)(a) \wedge (x \circ c)(p))$
$= ((x \circ c) \circ b)(a) = (x \circ (c \circ b))(a) = a/(c \circ b)(x)$, for all $x \in H$.

(*i'*) $((c\backslash(b\backslash a))(x) = \bigvee_{p\in H}(c\backslash p(x) \wedge b\backslash a(p)) = \bigvee_{p\in H}((c \circ x)(p) \wedge (b \circ p)(a))$
$= (b \circ (c \circ x))(a) = ((b \circ c) \circ x)(a) = (b \circ c)\backslash a(x)$, for all $x \in H$.

(*ii*) $((b\backslash a)/c)(x) = \bigvee_{p\in H}(b\backslash a(p) \wedge p/c(x)) = \bigvee_{p\in H}((b \circ p)(a) \wedge (x \circ c)(p))$
$= (b \circ (x \circ c))(a) = ((b \circ x) \circ c)(a) = \bigvee_{p\in H}((b \circ x)(p) \wedge (p \circ c)(a))$
$= \bigvee_{p\in H}(b\backslash p(x) \wedge a/c(p)) = b\backslash(a/c)(x)$, for all $x \in H$.

(*iii*) $(a/b)\backslash a(b) = \bigvee_{p\in H}(a/b(p) \wedge p\backslash a(b)) = \bigvee_{p\in H}((p \circ b)(a) \wedge (p \circ b)(a))$
$= \bigvee_{p\in H}(p \circ b)(a) = (H \circ b)(a) = \chi_H(a) = 1$.

(*iii'*) $(a/(b\backslash a)(b) = \bigvee_{p\in H}(a/p(b) \wedge b\backslash a(p)) = \bigvee_{p\in H}((b \circ p)(a) \wedge (b \circ p)(a))$
$= \bigvee_{p\in H}(b \circ p)(a) = (b \circ H)(a) = \chi_H(a) = 1$. □

We may generalize the above Theorem as follows:

Theorem 3.3. *Let $(H, \circ)$ be a fuzzy hypergroup. Then, for all fuzzy subsets α, β, γ of H, we have*
(*i*) $(\alpha/\beta)/\gamma = \alpha/(\gamma \circ \beta)$, (*i'*) $\gamma\backslash(\beta\backslash\alpha) = (\beta \circ \gamma)\backslash\alpha$,
(*ii*) $(\beta\backslash\alpha)/\gamma = \beta\backslash(\alpha/\gamma)$,
(*iii*) *If* $\bigvee_{x\in H}\beta(x) \leq \bigvee_{p\in H}\alpha(p)$, *then* $\beta \subseteq (\alpha/\beta)\backslash\alpha$ *and* $\beta \subseteq \alpha/(\beta\backslash\alpha)$,

Proof. (*i*) $((\alpha/\beta)/\gamma)(x) = \bigvee_{p,q,r\in H}(((p/q)/r)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r))$
$= \bigvee_{p,q,r\in H}((p/(r \circ q))(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r))$ (*by Theorem* 3.2)
$= (\alpha/(\gamma \circ \beta))(x)$, for all $x \in H$.

(*i'*) $(\gamma\backslash(\beta\backslash\alpha))(x) = \bigvee_{p,q,r\in H}(((r\backslash q)\backslash p)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r))$
$= \bigvee_{p,q,r\in H}(((q \circ r)\backslash p)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r)) = ((\beta \circ \gamma)\backslash\alpha)(x)$, for all $x \in H$.

(*ii*) $((\beta\backslash\alpha)/\gamma)(x) = \bigvee_{p,q,r\in H}((q\backslash p)/r)(x) \wedge \beta(q) \wedge \alpha(p) \wedge \gamma(r)$
$= \bigvee_{p,q,r\in H}(q\backslash(p/r))(x) \wedge \beta(q) \wedge \alpha(p) \wedge \gamma(r) = \beta\backslash(\alpha/\gamma)(x)$, for all $x \in H$.

(*iii*) $(\alpha/\beta)\backslash\alpha(x) = \bigvee_{p,q,r\in H}((p/q)\backslash r)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \alpha(r)$
$\geq \bigvee_{p,q\in H}((p/q)\backslash p)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \alpha(p) = \bigvee_{p,q\in H}((p/q)\backslash p)(x) \wedge \alpha(p) \wedge \beta(q)$
$\geq \bigvee_{p\in H}((p/x)\backslash p)(x) \wedge \alpha(p) \wedge \beta(x) = \bigvee_{p\in H}\alpha(p) \wedge \beta(x) = \beta(x)$, for all $x \in H$.

(*iii'*) $\alpha/(\beta\backslash\alpha)(x) = \bigvee_{p,q,r\in H}(p/(q\backslash r))(x) \wedge \alpha(p) \wedge \beta(q) \wedge \alpha(r)$
$\geq \bigvee_{p,q\in H}(p/(q\backslash p))(x) \wedge \alpha(p) \wedge \beta(q) \wedge \alpha(p) = \bigvee_{p,q\in H}(p/(q\backslash p))(x) \wedge \alpha(p) \wedge \beta(q)$
$\geq \bigvee_{p\in H}(p/(x\backslash p))(x) \wedge \alpha(p) \wedge \beta(x) = \bigvee_{p\in H}\alpha(p) \wedge \beta(x) = \beta(x)$, for all $x \in H$. □

Definition 3.4. Let $(H, \circ)$ be a fuzzy hypergroup. If for each $x \in H$ there exists $y \in H$ such that

$$(b\backslash a \wedge c/d)(x) \leq ((a \circ d) \wedge (b \circ c))(y),$$

then $(H, \circ)$ is called a fuzzy transposition hypergroup.

As we show below, this definition of a fuzzy transportation hypergroups is, in fact, a generalization of the definition of transportation hypergroups [*Definition*

1.6]. Suppose that $(H, \circ)$ is a transposition hypergroup and $a, b, c, d, x \in H$. Now, $(\chi_{b\backslash a} \wedge \chi_{c/d})(x) \neq 0 \Longleftrightarrow (\chi_{b\backslash a} \wedge \chi_{c/d})(x) = 1 \Longleftrightarrow x \in b\backslash a \cap c/d \Longrightarrow b\backslash a \cap c/d \neq \emptyset \Longrightarrow (a \circ d) \cap (b \circ c) \neq \emptyset \Longleftrightarrow$ *there exists* $y \in H$ *such that* $y \in (a \circ d) \cap (b \circ c) \Longleftrightarrow$ *there exists* $y \in H$ *such that* $(\chi_{(a \circ d)} \wedge \chi_{(b \circ c)})(y) = 1$. Therefore for each $x \in H$ there exists $y \in H$ such that $(\chi_{b\backslash a} \wedge \chi_{c/d})(x) \leq (\chi_{(a \circ d)} \wedge \chi_{(b \circ c)})(y)$.
In other words, the notion of fuzzy transposition hypergroup generalizes the concept of transposition hypergroup.

We now give an example of a fuzzy transposition hypergroup.

Example 3.5. [14] Let S be a group. Define a fuzzy hyperoperation $\circ$ on S by $a \circ b = \chi_{ab}$, for all $a, b \in S$, where χ_{ab} is the characteristic function of the element ab in S, then $(S, \circ)$ is a fuzzy hypergroup.
Let $a, b, c, d, x \in S$. Then $(b\backslash a \wedge c/d)(x) = ((b \circ x)(a) \wedge (x \circ d))(c) = \chi_{bx}(a) \wedge \chi_{xd}(c)$.
We have two cases:
Case I: $(b\backslash a \wedge c/d)(x) = 0$. Then $(b\backslash a \wedge c/d)(x) \leq ((a \circ d) \wedge (b \circ c))(y)$, for all $y \in S$.
Case II: $(b\backslash a \wedge c/d)(x) = 1$. Then $bx = a$ and $xd = c$. This implies that $ad = bc$.
Write $ad = bc = y$. Then $((a \circ d) \wedge (b \circ c))(y) = 1$.
Thus in either case, for every $x \in S$ there exists $y \in S$ such that
$(b\backslash a \wedge c/d)(x) \leq ((a \circ d) \wedge (b \circ c))(y)$. It follows that $(S, \circ)$ is a fuzzy transposition hypergroup.

Theorem 3.6. *If* $(H, \circ)$ *is a fuzzy transposition hypergroup, then*
(i) $(b\backslash a \wedge c/d)(x) \leq \bigvee_{y \in H} ((a \circ d) \wedge (b \circ c))(y)$, *for all* $x \in H$
and
(ii) $\bigvee_{x \in H} (b\backslash a \wedge c/d)(x) \leq \bigvee_{y \in H} ((a \circ d) \wedge (b \circ c))(y)$.

Proof. Straightforward. □

Theorem 3.7. *If* $(H, \circ)$ *is a fuzzy transposition hypergroup and* $\alpha, \beta, \gamma, \delta$ *are fuzzy subsets of* H*, then, for each* $x \in H$*, there exists* $y \in H$ *such that*

$$(\beta\backslash\alpha \wedge \gamma/\delta)(x) \leq ((\alpha \circ \delta) \wedge (\beta \circ \gamma))(y).$$

Proof. $(\beta\backslash\alpha \wedge \gamma/\delta)(x) = \bigvee_{a,b,c,d \in H} (b\backslash a \wedge c/d)(x) \wedge \beta(b) \wedge \alpha(a) \wedge \gamma(c) \wedge \delta(d)$
$\leq \bigvee_{a,b,c,d \in H} (a \circ d \wedge b \circ c)(y) \wedge \beta(b) \wedge \alpha(a) \wedge \gamma(c) \wedge \delta(d) \leq ((\alpha \circ \delta) \wedge (\beta \circ \gamma))(y)$. □

Theorem 3.8. *If* $(H, \circ)$ *is a fuzzy transposition hypergroup and* $\alpha, \beta, \gamma, \delta$ *are fuzzy subsets of* H*, then*
(i) $(\beta\backslash\alpha \wedge \gamma/\delta)(x) \leq \bigvee_{y \in H} ((\alpha \circ \delta) \wedge (\beta \circ \gamma))(y)$, *for all* $x \in H$
and
(ii) $\bigvee_{x \in H} (\beta\backslash\alpha \wedge \gamma/\delta)(x) \leq \bigvee_{y \in H} ((\alpha \circ \delta) \wedge (\beta \circ \gamma))(y)$.

Proof. Straightforward. □

Theorem 3.9. *In a fuzzy transposition hypergroup* $(H, \circ)$,
(i) $a \circ (b/c) \subseteq (a \circ b)/c$ *and (i′)* $(c\backslash b) \circ a \subseteq c\backslash(b \circ a)$,
(ii) $a/(b/c) \subseteq (a \circ c)/b$ *and (ii′)* $(c\backslash b)\backslash a \subseteq b\backslash(c \circ a)$, *for all* $a, b, c \in H$.

Proof. (*i*) $(a \circ (b/c))(x) = \bigvee_{p\in H}((a \circ p)(x) \wedge b/c(p)) = \bigvee_{p\in H}(a\backslash x(p) \wedge b/c(p))$
$\leq \bigvee_{q\in H}(x \circ c)(q) \wedge (a \circ b)(q) = \bigvee_{q\in H} q/c(x) \wedge (a \circ b)(q) = (a \circ b)/c(x)$, for all $x \in H$.

(*i'*) $((c\backslash b) \circ a)(x) = \bigvee_{p\in H}(c\backslash b(p) \wedge (p \circ a)(x)) = \bigvee_{p\in H}(c\backslash b(p) \wedge x/a(p))$
$\leq \bigvee_{q\in H}((b \circ a)(q) \wedge (c \circ x)(q)) = \bigvee_{q\in H}((b \circ a)(q) \wedge c\backslash q(x))$
$= c\backslash(b \circ a)(x)$, for all $x \in H$.

(*ii*) $(a/(b/c))(x) = \bigvee_{p\in H}(a/p(x) \wedge b/c(p)) = \bigvee_{p\in H}((x \circ p)(a) \wedge b/c(p))$
$= \bigvee_{p\in H}((x\backslash a(p) \wedge b/c(p)) \leq \bigvee_{q\in H}((a \circ c)(q) \wedge (x \circ b)(q))$
$= \bigvee_{q\in H}((a \circ c)(q) \wedge (q/b)(x)) = (a \circ c)/b(x)$, for all $x \in H$.

(*ii'*) $(c\backslash b)\backslash a(x) = \bigvee_{p\in H}(c\backslash b(p) \wedge p\backslash a(x)) = \bigvee_{p\in H}(c\backslash b(p) \wedge a/x(p))$
$\leq \bigvee_{q\in H}((b \circ x)(q) \wedge (c \circ a)(q)) = \bigvee_{q\in H}(b\backslash q)(x) \wedge (c \circ a)(q)$
$= b\backslash(c \circ a)(x)$, for all $x \in H$. □

Theorem 3.10. *Let $(H, \circ)$ be a fuzzy transposition hypergroup. Then, if α, β, and γ are arbitrary fuzzy subsets of H, we have:*
(i) $\alpha \circ (\beta/\gamma) \subseteq (\alpha \circ \beta)/\gamma$ *and (i')* $(\gamma\backslash\beta) \circ \alpha \subseteq \gamma\backslash(\beta \circ \alpha)$,
(ii) $\alpha/(\beta/\gamma) \subseteq (\alpha \circ \gamma)/\beta$ *and (ii')* $(\gamma\backslash\beta)\backslash\alpha \subseteq \beta\backslash(\gamma \circ \alpha)$,

Proof. Let $x \in H$.
(*i*) $(\alpha \circ (\beta/\gamma))(x) = \bigvee_{p,q,r\in H}(p \circ (q/r))(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r)$
$\leq \bigvee_{p,q,r\in H}((p \circ q)/r)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r)$ (*by Theorem* 3.9)
$= (\alpha \circ \beta)/\gamma(x)$.

(*i'*) $((\gamma\backslash\beta) \circ \alpha)(x) = \bigvee_{p,q,r\in H}((r\backslash q) \circ p)(x) \wedge \gamma(r) \wedge \beta(q) \wedge \alpha(p)$
$\leq \bigvee_{p,q,r\in H}(r\backslash(q \circ p))(x) \wedge \gamma(r) \wedge \beta(q) \wedge \alpha(p) = \gamma\backslash(\beta \circ \alpha)(x)$.

(*ii*) $(\alpha/(\beta/\gamma))(x) = \bigvee_{p,q,r\in H}(p/(q/r))(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r)$
$\leq \bigvee_{p,q,r\in H}((p \circ r)/q)(x) \wedge \alpha(p) \wedge \beta(q) \wedge \gamma(r) = (\alpha \circ \gamma)/\beta(x)$.

(*ii'*) $(\gamma\backslash\beta)\backslash\alpha(x) = \bigvee_{p,q,r\in H}((r\backslash q)\backslash p)(x) \wedge \gamma(r) \wedge \beta(q) \wedge \alpha(p)$
$\leq \bigvee_{p,q,r\in H}(q\backslash(r \circ p))(x) \wedge \gamma(r) \wedge \beta(q) \wedge \alpha(p) = \beta\backslash(\gamma \circ \alpha)(x)$. □

Definition 3.11. [14] Let $(H, \circ)$ be a fuzzy hypergroup. A fuzzy subset μ of H is said to be a fuzzy subsemihypergroup of $(H, \circ)$ if

$$\mu \circ \mu \subseteq \mu.$$

Theorem 3.12. *Let $(H, \circ)$ be a fuzzy hypergroup. A fuzzy subset μ of H is a fuzzy subsemihypergroup of $(H, \circ)$ if and only if*

$$\mu(a) \wedge \mu(b) \wedge (a \circ b)(x) \leq \mu(x) \quad for \quad all \quad a, b, x \in H.$$

Proof. $\mu(a) \wedge \mu(b) \wedge (a \circ b)(x) \leq \bigvee_{p,q\in H}(\mu(p) \wedge \mu(q) \wedge (p \circ q)(x)) = (\mu \circ \mu)(x)$.
This implies that μ is a fuzzy subsemihypergroup of $(H, \circ)$ if and only if
$\mu(a) \wedge \mu(b) \wedge (a \circ b)(x) \leq \mu(x)$, for all $a, b, x \in H$. □

Definition 3.13. A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is said to be a fuzzy subhypergroup of $(H, \circ)$ if

$$\mu(a) \wedge \mu(b) \leq (a \circ \mu)(b) \quad for \quad all \quad a, b \in H$$

and

$$\mu(a) \wedge \mu(b) \leq (\mu \circ a)(b) \quad for \quad all \quad a, b \in H.$$

Theorem 3.14. *If μ is a fuzzy subhypergroup of a fuzzy hypergroup $(H, \circ)$, then (i) $\mu \subseteq \mu \backslash \mu$, (ii) $\mu \subseteq \mu/\mu$, (iii) $\mu = \mu \circ \mu$.*

Proof. (i) We have $\mu(a) \wedge \mu(b) \leq (a \circ \mu)(b)$, for all $a, b \in H$.
Therefore $\mu(a) \wedge \mu(b) \leq \mu(b) \wedge (a \circ \mu)(b) \leq \bigvee_{r \in H} \mu(r) \wedge (a \circ \mu)(r)$
$= \bigvee_{r \in H} \mu(r) \wedge (r/\mu)(a) = \mu/\mu(a)$ and hence $\mu(a) \wedge \mu(b) \leq \mu/\mu(a)$, for all $a, b \in H$.
In particular, $\mu(a) \wedge \mu(a) \leq \mu/\mu(a)$, for all $a \in H$; i.e. $\mu(a) \leq \mu/\mu(a)$, for all $a \in H$.
Therefore $\mu \subseteq \mu/\mu$.

(ii) the prrof is similar to that of (i).

(iii) We have $\mu \circ \mu \subseteq \mu$ and $\mu(a) \wedge \mu(b) \leq (a \circ \mu)(b)$, for all $a, b \in H$. Therefore $\mu(a) \wedge \mu(b) \leq \mu(a) \wedge (a \circ \mu)(b) \leq \bigvee_{r \in H} \mu(r) \wedge (r \circ \mu)(b) = (\mu \circ \mu)(b)$. Therefore $\mu(a) \wedge \mu(b) \leq (\mu \circ \mu)(b)$, for all $a, b \in H$. In particular $\mu(b) \wedge \mu(b) \leq (\mu \circ \mu)(b)$, for all $b \in H$ i.e. $\mu(b) \leq (\mu \circ \mu)(b)$, for all $b \in H$. Therefore $\mu \subseteq \mu \circ \mu$.
Hence $\mu = \mu \circ \mu$. □

4. Closed and Reflexive Fuzzy Subsemihypergroups

In this section we introduce the notions of fuzzy closed sets, fuzzy normal subsemihypergroups and fuzzy reflexive subsemihypergroups and study some of their properties. We also investigate a fuzzy equivalence relation on a fuzzy transposition hypergroup $(H, \circ)$ induced by a fuzzy reflexive closed subsemihypergroup.

Definition 4.1. Let $(H, \circ)$ be a fuzzy hypergroup. A fuzzy subset μ of H is said to be closed if

$$\mu \backslash \mu \subseteq \mu \quad and \quad \mu/\mu \subseteq \mu.$$

Theorem 4.2. *Let $(H, \circ)$ be a fuzzy hypergroup. Let μ be a fuzzy subset of H. Then the following conditions are equivalent:*
(i) μ is closed.
(ii) For all $a, b, x, y \in H$, $\mu(a) \wedge \mu(b) \wedge (a \circ x)(b) \leq \mu(x)$ and $\mu(a) \wedge \mu(b) \wedge (y \circ a)(b) \leq \mu(y)$.
(iii) For all $a, b, x, y \in H$, $\mu(b) \wedge (\mu \circ x)(b) \leq \mu(x)$ and $\mu(b) \wedge (y \circ \mu)(b) \leq \mu(y)$, for all $b, x, y \in H$.

Proof. $(i) \Longrightarrow (ii)$.
$\mu(a) \wedge \mu(b) \wedge (a \circ x)(b) \leq \bigvee_{p,q \in H} \mu(p) \wedge \mu(q) \wedge (p \circ x)(q) = \mu \backslash \mu(x) \leq \mu(x)$ and $\mu(a) \wedge \mu(b) \wedge (y \circ a)(b) \leq \bigvee_{p,q \in H} \mu(p) \wedge \mu(q) \wedge (y \circ p)(q) = \mu/\mu(y) \leq \mu(x)$. This implies that $\mu(a) \wedge \mu(b) \wedge (a \circ x)(b) \leq \mu(x)$ and $\mu(a) \wedge \mu(b) \wedge (y \circ a)(b) \leq \mu(y)$, for all $a, b, x, y \in H$.

$(ii) \Longrightarrow (iii)$.

$\mu(b)\wedge(\mu\circ x)(b) = \bigvee_{a\in H}\mu(a)\wedge\mu(b)\wedge(a\circ x)(b) \le \bigvee_{a,b\in H}\mu(a)\wedge\mu(b)\wedge(a\circ x)(b) \le \mu(x)$ and $\mu(b)\wedge(y\circ\mu)(b) = \bigvee_{a\in H}\mu(a)\wedge\mu(b)\wedge(y\circ a)(b) \le \bigvee_{a,b\in H}\mu(a)\wedge\mu(b)\wedge(y\circ a)(b)$ $\le \mu(y)$. This implies that $\mu(b)\wedge(\mu\circ x)(b)\le\mu(x)$ and $\mu(b)\wedge(y\circ\mu)(b)\le\mu(y)$, for all $b,x,y\in H$.

$(iii)\Longrightarrow(i)$.
$\mu\backslash\mu(x) = \bigvee_{b\in H}\mu(b)\wedge(\mu\circ x)(b)\le\mu(x)$ and $\mu/\mu(y) = \bigvee_{b\in H}\mu(b)\wedge(y\circ\mu)(b)\le\mu(y)$. This implies that $\mu\backslash\mu\subseteq\mu$ and $\mu/\mu\subseteq\mu$. Therefore μ is fuzzy closed. □

Theorem 4.3. *Let $(H,\circ)$ be a fuzzy hypergroup. If a fuzzy subset μ of H is closed then*
(i) $\mu(a)\wedge\mu(b)\le(a\circ\mu)(b)$, for all $a,b\in H$,
(ii) $\mu(a)\wedge\mu(b)\le(\mu\circ a)(b)$, for all $a,b\in H$,
(iii) $\mu\subseteq\mu\circ\mu$
(iv) $\mu=\mu\backslash\mu=\mu/\mu$

Proof. (i) We have $\mu(a)\wedge\mu(b)\wedge(a\circ x)(b)\le\mu(x)$, for all $a,b,x,y\in H$. Therefore $\mu(a)\wedge\mu(b)\wedge(a\circ x)(b)\le\mu(x)\wedge(a\circ x)(b)\le\bigvee_{t\in H}\mu(t)\wedge(a\circ t)(b) = (a\circ\mu)(b)$. It follows that $\bigvee_{x\in H}(\mu(a)\wedge\mu(b)\wedge(a\circ x)(b))\le(a\circ\mu)(b)$ i.e. $\mu(a)\wedge\mu(b)\wedge(a\circ H)(b)\le(a\circ\mu)(b)$ i.e. $\mu(a)\wedge\mu(b)\le(a\circ\mu)(b)$.

(ii) The proof is similar to that of (i).

(iii) We have $\mu(a)\wedge\mu(b)\le(\mu\circ a)(b)$, for all $a,b\in H$. This implies that $\mu(a)\wedge\mu(b)$ $\le(\mu\circ a)(b)\wedge\mu(a)\le(\mu\circ\mu)(b)$. Therefore $\mu(a)\wedge\mu(b)\le(\mu\circ\mu)(b)$, for all $a,b\in H$. In particular $\mu(b)\le(\mu\circ\mu)(b)$, for all $b\in H$ i.e. $\mu\subseteq\mu\circ\mu$.

(iv) By (i) we have $\mu(a)\wedge\mu(b)\le(a\circ\mu)(b)$, for all $a,b\in H$.
Therefore $\mu(a)\wedge\mu(b)\le(a\circ\mu)(b)\wedge\mu(b) = b/\mu(a)\wedge\mu(b)\le\mu/\mu(a)$, for all $a,b\in H$. In particular $\mu(a)\le\mu/\mu(a)$, for all $a\in H$. It follows that $\mu\subseteq\mu/\mu$ and hence $\mu=\mu/\mu$. Similarly, $\mu=\mu\backslash\mu$. □

Corollary 4.4. *A fuzzy subset μ of a fuzzy hypergroup $(H,\circ)$ is closed if and only if*

$$\mu/\mu=\mu\backslash\mu=\mu.$$

We now give an example of a fuzzy closed subset which is not a fuzzy subsemihypergroup.

Example 4.5. In example 2.7 we saw that if S is a group, then $(S,\circ)$ is a fuzzy hypergroup. We define a fuzzy hyperoperation $\circ$ on S by $a\circ b=\chi_{ab}$ for all $a,b\in S$, where χ_{ab} is the characteristic function of the element ab in S.
Since the set $S=\{1,-1,i,-i\}$, where i is the square root of -1, is a group under usual multiplication, hence S is a fuzzy hypergroup under the fuzzy hyperoperation $\circ$. We now define a fuzzy subset μ of S by $\mu(1)=1$, $\mu(-1)=\frac{1}{2}$, $\mu(i)=\frac{1}{3}$, $\mu(-i)=\frac{1}{4}$. It can be easily verified that $\mu/\mu(1)=\mu\backslash\mu(1)=1=\mu(1)$, $\mu/\mu(-1)=\mu\backslash\mu(-1)=\frac{1}{2}=\mu(-1)$, $\mu/\mu(i)=\mu\backslash\mu(i)=\frac{1}{3}=\mu(i)$, $\mu/\mu(-i)=\mu\backslash\mu(-i)=\frac{1}{4}=\mu(-i)$. It follows that μ is a fuzzy closed subset of the fuzzy hypergroup $(S,\circ)$.
On the other hand, $(\mu\circ\mu)(-i)=\bigvee_{r\in S}(\mu(r)\wedge(r\circ\mu)(-i)) = (\mu(1)\wedge(1\circ\mu)(-i))\vee$

$(\mu(-1) \wedge (-1 \circ \mu)(-i)) \vee (\mu(i) \wedge (i \circ \mu)(-i)) \vee (\mu(-i) \wedge (-i \circ \mu)(-i)) = (1 \wedge \frac{1}{4}) \vee (\frac{1}{2} \wedge \frac{1}{3}) \vee (\frac{1}{3} \wedge \frac{1}{2}) \vee (\frac{1}{4} \wedge 1) = \frac{1}{3} > \frac{1}{4} = \mu(-i)$. It follows that μ is not a fuzzy subsemihypergroup and hence not a fuzzy subhypergroup.

Theorem 4.6. *If a fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is closed then it is a fuzzy subhypergroup.*

Proof. The proof follows from Theorem 4.3. □

Definition 4.7. A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is said to be *normal* if

$$a \circ \mu = \mu \circ a \quad for \quad all \quad a \in H.$$

Theorem 4.8. *A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is normal if and only if*

$$\mu \backslash a = a/\mu \quad for \quad all \quad a \in H.$$

Proof. Let μ be *normal*.
Then $\mu \backslash a(x) = (\mu \circ x)(a) = (x \circ \mu)(a) = a/\mu(x)$ for all $x \in H$.

Conversely, assume that $\mu \backslash a = a/\mu$, for all $a \in H$.
Then $(\mu \circ a)(x) = \mu \backslash x(a) = x/\mu(a) = (a \circ \mu)(x)$ for all $x \in H$. □

Theorem 4.9. *A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is normal if and only if*

$$\mu \circ \nu = \nu \circ \mu \quad for \quad every \quad fuzzy \quad subset \quad \nu \quad of \quad H.$$

Proof. Let μ be *normal*. Then $(\mu \circ \nu)(x) = \bigvee_{p \in H}((\mu \circ p)(x) \wedge \nu(p))$
$= \bigvee_{p \in H}(\nu(p) \wedge (p \circ \mu)(x)) = (\nu \circ \mu)(x)$, for all $x \in H$.

Conversely, assume that $\mu \circ \nu = \nu \circ \mu$, for every fuzzy subset ν of H. Since χ_a is a fuzzy subset of H, hence $\mu \circ a = \mu \circ \chi_a = \chi_a \circ \mu = a \circ \mu$, for all $a \in H$.
It follows that μ is fuzzy normal. □

Theorem 4.10. *A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is normal if and only if*

$$\mu \backslash \nu = \nu/\mu \quad for \quad every \quad fuzzy \quad subset \quad \nu \quad of \quad H.$$

Proof. Let μ be *normal*. Then $(\mu \backslash \nu)(x) = \bigvee_{p \in H}(\mu \backslash p(x) \wedge \nu(p))$
$= \bigvee_{p \in H}(p/\mu(x) \wedge \nu(p)) = \nu/\mu(x)$ for all $x \in H$.

Conversely, assume that $\mu \backslash \nu = \nu/\mu$, for every fuzzy subset ν of H. Since χ_a is a fuzzy subset of H, then $\mu \backslash a(x) = (\mu \circ x)(a) = \bigvee_{p \in H}((\mu \circ x)(p) \wedge \chi_a(p)) = \mu \backslash \chi_a(x)$
$= \chi_a/\mu(x) = \bigvee_{p \in H}((x \circ \mu)(p) \wedge \chi_a(p)) = (x \circ \mu)(a) = a/\mu(x)$ for all $x \in H$.
It follows that μ is fuzzy normal. □

Definition 4.11. A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is said to be $reflexive$ if

$$a \backslash \mu = \mu/a \quad for \quad all \quad a \in H.$$

Theorem 4.12. *A fuzzy subsemihypergroup μ of a fuzzy hypergroup $(H, \circ)$ is reflexive if and only if*

$$\nu\backslash\mu = \mu/\nu \quad for \quad every \quad fuzzy \quad subset \quad \nu \quad of \quad H.$$

Proof. Let μ be fuzzy reflexive. Then $\nu\backslash\mu(x) = \bigvee_{p\in H}(p\backslash\mu(x) \wedge \nu(p))$
$= \bigvee_{p\in H}(\mu/p(x) \wedge \nu(p)) = \mu/\nu(x)$, for all $x \in H$.

Conversely assume that $\nu\backslash\mu = \mu/\nu$, for every fuzzy subset ν of H. Since χ_a is a fuzzy subset of H, hence $a\backslash\mu(x) = \bigvee_{p\in H}(\mu(p)\wedge(a\circ x)(p)) = \bigvee_{p\in H}(\mu(p)\wedge(\chi_a\circ x)(p))$
$= \chi_a\backslash\mu(x) = \mu/\chi_a(x) = \bigvee_{p\in H}(\mu/p(x) \wedge \chi_a(p)) = \mu/a(x)$, for all $a, x \in H$.
It follows that μ is fuzzy reflexive. □

Theorem 4.13. *A fuzzy normal closed subsemihypergroup μ of a fuzzy transposition hypergroup $(H, \circ)$ is fuzzy reflexive.*

Proof. $\mu/a(x) = \bigvee_{p\in H}((x \circ a)(p) \wedge \mu(p)) \leq \bigvee_{p\in H} \mu(p)$, for all $x \in H$.
By Theorems 3.3, 3.10, 4.10 we have
$\mu/a \subseteq (\mu/(\mu/a))\backslash\mu \subseteq (\mu \circ a/\mu)\backslash\mu = (\mu\backslash\mu \circ a)\backslash\mu \subseteq \mu \circ a\backslash\mu \circ \mu = \mu \circ a\backslash\mu = a\backslash(\mu\backslash\mu)$
$= a\backslash\mu$. Similarly, $a\backslash\mu \subseteq \mu/a$.
It follows that μ is fuzzy reflexive. □

Definition 4.14. [12] Let H be a non-empty set. A fuzzy subset ρ of $H \times H$ is said to be a fuzzy equivalence relation on H if
(i) $\rho(a, a) = 1$, for all $a \in H$ (fuzzy reflexive),
(ii) $\rho(a, b) = \rho(b, a)$, for all $a, b \in H$ (fuzzy symmetric).
and
(iii) $\rho(a, b) \wedge \rho(b, c) \leq \rho(a, c)$, for all $a, b, c \in H$ (fuzzy transitive).

The condition (iii) is equivalent to $\rho \circ \rho \subseteq \rho$, where $(\rho \circ \rho)(a, c) = \bigvee_{b\in H} \rho(a, b) \wedge \rho(b, c)$.
It is clear that if ρ is a fuzzy equivalence relation on H, then $\rho \circ \rho = \rho$.
Moreover, the fuzzy equivalence class ρ_a, corresponding to an element $a \in H$, is defined by

$$\rho_a(x) = \rho(a, x) \quad for \quad all \quad x \in H.$$

We may generalize the above definition as follows.

Definition 4.15. Let ρ be a fuzzy relation on a non-empty set H and μ be a fuzzy subset of H, then the fuzzy subset ρ_μ of H is defined by

$$\rho_\mu(x) = \bigvee_{r\in H} (\mu(r) \wedge \rho_r(x)) \quad for \quad all \quad x \in H.$$

Example 4.16. Let χ_{ab} denote the characteristic function of the element ab in S. Then we know that $(S, \circ)$, where $S = \{1, -1, i, -i\}$ and the fuzzy hyperoperation $\circ$ is defined by $a \circ b = \chi_{ab}$, for all $a, b \in S$, is a fuzzy hypergroup.
We now define a fuzzy subset μ of S by $\mu(1) = 1$, $\mu(-1) = \frac{1}{2}$, $\mu(i) = \frac{1}{4}$, $\mu(-i) = \frac{1}{4}$.
It can be easily verified that $\mu/\mu = \mu\backslash\mu = \mu$. Hence μ is a fuzzy closed subset of the fuzzy hypergroup $(S, \circ)$.

Moreover, we can show that $\mu \circ \mu = \mu$ and hence μ is a fuzzy subsemihypergroup. Also, $\mu/a(x) = \bigvee_{r\in S} \mu(r) \wedge (x \circ a)(r) = \bigvee_{r\in S} \mu(r) \wedge \chi_{xa}(r) = \bigvee_{r\in S} \mu(r) \wedge \chi_{ax}(r)$ $= \bigvee_{r\in S} \mu(r) \wedge (a \circ x)(r) = a\backslash\mu(x)$ for all $x \in S$.
Therefore $\mu/a = a\backslash\mu$, for all $a \in S$ and hence μ is fuzzy reflexive.
Now, $\bigvee_{p\in S} \mu(p) = 1$.
Therefore μ is a fuzzy reflexive closed subsemihypergroup of the fuzzy transposition hypergroup $(S, \circ)$ such that $\bigvee_{p\in S} \mu(p) = 1$.

Theorem 4.17. *Let μ be a fuzzy reflexive closed subsemihypergroup of a fuzzy transposition hypergroup $(H, \circ)$ such that $\bigvee_{p\in H} \mu(p) = 1$. Then the fuzzy relation ρ on H defined by $\rho(a, b) = \bigvee_{x\in H} (a \circ \mu)(x) \wedge (\mu \circ b)(x)$, for all $a, b \in H$ is a fuzzy equivalence relation and for any element $a \in H$ the fuzzy equivalence class ρ_a is given by*

$$\rho_a = (\mu \circ a)/\mu = \mu/(\mu/a) = \mu\backslash(a \circ \mu) = (a\backslash\mu)\backslash\mu.$$

Proof. By Theorem 3.8 we have $\rho(a, a) = \bigvee_{x\in H} (a \circ \mu)(x) \wedge (\mu \circ a)(x)$
$\geq \bigvee_{p\in H} a\backslash\mu(p) \wedge \mu/a(p) = \bigvee_{p\in H} a\backslash\mu(p) = \bigvee_{p,q\in H} \mu(q) \wedge (a \circ p)(q) = \bigvee_{q\in H} \mu(q) = 1$.
It follows that $\rho(a, a) = 1$, for all $a \in H$. Hence ρ is fuzzy reflexive.
By Theorems 3.8, 3.2, 4.12 we have $\rho(b, a) = \bigvee_{x\in H} (b \circ \mu)(x) \wedge (\mu \circ a)(x)$
$\geq \bigvee_{p\in H} b\backslash\mu(p) \wedge \mu/a(p) \geq \bigvee_{p\in H} b\backslash(\mu\backslash\mu)(p) \wedge (\mu/\mu)/a(p)$
$= \bigvee_{p\in H} (\mu \circ b)\backslash\mu(p) \wedge \mu/(a \circ \mu)(p) = \bigvee_{p\in H} \mu/(\mu \circ b)(p) \wedge \mu/(a \circ \mu)(p)$
$= \bigvee_{p,q,r\in H} \mu(q) \wedge (p \circ \mu \circ b)(q) \wedge \mu(r) \wedge (p \circ a \circ \mu)(r)$
$\geq \bigvee_{p,q\in H} \mu(q) \wedge (p \circ \mu \circ b)(q) \wedge (p \circ a \circ \mu)(q)$
$= \bigvee_{p,q,r,s\in H} \mu(q) \wedge (p \circ r)(q) \wedge (\mu \circ b)(r) \wedge (p \circ s)(q) \wedge (a \circ \mu)(s)$
$\geq \bigvee_{p,q,r\in H} \mu(q) \wedge (p \circ r)(q) \wedge (\mu \circ b)(r) \wedge (p \circ r)(q) \wedge (a \circ \mu)(r)$
$= \bigvee_{q,r\in H} \mu(q) \wedge (a \circ \mu)(r) \wedge (\mu \circ b)(r) = \bigvee_{q\in H} \mu(q) \wedge \bigvee_{r\in H} (a \circ \mu)(r) \wedge (\mu \circ b)(r)$
$= \bigvee_{r\in H} (a \circ \mu)(r) \wedge (\mu \circ b)(r) = \rho(a, b)$.
Similarly, $\rho(a, b) \geq \rho(b, a)$. Therefore $\rho(a, b) = \rho(b, a)$.
Hence ρ is fuzzy symmetric.
Now, for all $a, b, c \in H$,$\rho(a, b) \wedge \rho(b, c) = \bigvee_{x\in H} ((a \circ \mu)(x) \wedge (\mu \circ b)(x)) \wedge \bigvee_{y\in H} ((b \circ \mu)(y) \wedge (\mu \circ c)(y))$
$= \bigvee_{x\in H} ((a \circ \mu)(x) \wedge (\mu\backslash x)(b) \wedge y/\mu(b) \wedge (\mu \circ c)(y)) = \mu\backslash(a \circ \mu)(b) \wedge (\mu \circ c)/\mu(b)$
$\leq \bigvee_{p\in H} ((\mu \circ \mu \circ c)(p) \wedge (a \circ \mu \circ \mu)(p)) = \bigvee_{p\in H} ((a \circ \mu)(p) \wedge (\mu \circ c)(p)) = \rho(a, c)$.
Hence ρ is fuzzy transitive.
It follows that ρ is a fuzzy equivalence relation and hence.
$\rho_a(x) = \rho(a, x) = \rho(x, a) = \bigvee_{p\in H} ((x \circ \mu)(p) \wedge (\mu \circ a)(p))$
$= \bigvee_{p\in H} (p/\mu(x) \wedge (\mu \circ a)(p)) = (\mu \circ a)/\mu(x)$.
$\rho_a(x) = \rho(a, x) = \bigvee_{p\in H} ((a \circ \mu)(p) \wedge (\mu \circ x)(p)) = \bigvee_{p\in H} ((a \circ \mu)(p) \wedge \mu\backslash p(x))$
$= \mu\backslash(a \circ \mu)(x)$.
By Theorems 3.10, 4.12, 3.2 we have
$\mu/(\mu/a) \subseteq (\mu \circ a)/\mu \subseteq (\mu/((\mu \circ a)/\mu))\backslash\mu \subseteq ((\mu \circ \mu)/(\mu \circ a))\backslash\mu = (\mu/(\mu \circ a))\backslash\mu$
$= \mu/(\mu/(\mu \circ a)) = \mu/((\mu/a)/\mu) \subseteq (\mu \circ \mu)/(\mu/a) = \mu/(\mu/a)$.
It follows that $(\mu \circ a)/\mu = \mu/(\mu/a)$.
Similarly, $\mu\backslash(a \circ \mu) = (a\backslash\mu)\backslash\mu$. □

Theorem 4.18. *If μ is a fuzzy reflexive closed subsemihypergroup of a fuzzy transposition hypergroup $(H, \circ)$, then*

$$\bigvee_{p\in H}(a\circ b)(p)\wedge\mu(p)=\bigvee_{p\in H}(b\circ a)(p)\wedge\mu(p)=\mu/a(b)=\mu/b(a).$$

Proof. $\bigvee_{p\in H}(a\circ b)(p)\wedge\mu(p)=\bigvee_{p\in H}p/b(a)\wedge\mu(p)=\mu/b(a)=b\backslash\mu(a)$
$=\bigvee_{p\in H}(b\circ a)(p)\wedge\mu(p)=\bigvee_{p\in H}p/a(p)\wedge\mu(p)=\mu/a(b)$. □

Theorem 4.19. *If μ is a fuzzy reflexive closed subsemihypergroup of a fuzzy transposition hypergroup $(H, \circ)$ such that $\bigvee_{p\in H}\mu(p)=1$ and ρ is the corresponding fuzzy equivalence relation, then*

$$(\rho_a\circ\rho_b)(x)\leq\bigvee_{r\in H}(a\circ b)(r)\wedge\rho_r(p)=\rho_{a\circ b}(x)\quad for\quad all\quad x\in H.$$

Proof. By Theorem 4.17 we have
$\rho_a\circ\rho_b=(\mu\backslash(a\circ\mu)\circ(\mu/(\mu/b))\subseteq(\mu\backslash(a\circ\mu)\circ\mu)/(\mu/b)\subseteq(\mu\backslash(a\circ\mu\circ\mu)/(\mu/b)$
$=(\mu\backslash(a\circ\mu)/(\mu/b)=((\mu\circ a)/\mu)/(\mu/b)=(\mu\circ a)/((\mu/b)/\mu)=(\mu\circ a)/((b\backslash\mu/\mu)$
$\subseteq(\mu\circ a)/((b\backslash(\mu\circ\mu))=(\mu\circ a)/(b\backslash\mu)=(\mu\circ a)/(\mu/b)\subseteq(\mu\circ a\circ b)/\mu$.
It follows that $(\rho_a\circ\rho_b)(x)\leq\bigvee_{r\in H}(a\circ b)(r)\wedge\rho_r(x)=\rho_{a\circ b}(x)$. □

5. **Fuzzy Regular Relation and Fuzzy Quotient Hypergroup**

In this section we introduce the notion of a fuzzy regular relation on a fuzzy hypergroupid $(H, \circ)$ and study the fuzzy quotient hypergroups of a fuzzy hypergroup and a fuzzy transposition hypergroup. We also investigate the scalar identity of a fuzzy regular relation.

Definition 5.1. A fuzzy equivalence relation ρ on a fuzzy hypergroupoid $(H, \circ)$ is said to be right regular if
$\bigvee_{a\in H}(\rho(a,b)\wedge(a\circ c)(t))\leq\bigvee_{r\in H}((b\circ c)(r)\wedge\rho(t,r))$, for all $b, c, t\in H$.
and
$\bigvee_{b\in H}(\rho(a,b)\wedge(b\circ c)(t))\leq\bigvee_{r\in H}(a\circ c)(r)\wedge\rho(r,t)$, for all $a, c, t\in H$.
We can similarly define the left regularity of a fuzzy equivalence relation ρ on a fuzzy hypergroupoid $(H, \circ)$.

Definition 5.2. A fuzzy equivalence relation ρ on a fuzzy hypergroupoid $(H, \circ)$ is said to be regular if it is both right and left regular.

Theorem 5.3. *A fuzzy equivalence relation ρ on a fuzzy hypergroupoid $(H, \circ)$ is regular if and only if*
$\bigvee_{a,c\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(a\circ c)(t))\leq\bigvee_{r\in H}((b\circ d)(r)\wedge\rho(t,r))$, *for all* $b, d, t\in H$
and
$\bigvee_{b,d\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(b\circ d)(t))\leq\bigvee_{r\in H}(a\circ c)(r)\wedge\rho(r,t)$, *for all* $a, c, t\in H$.

Proof. Let ρ be regular. Then it is both right and left regular.
Now $\bigvee_{a,c\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(a\circ c)(t))\leq\bigvee_{u,c\in H}((b\circ c)(u)\wedge\rho(t,u)\wedge\rho(c,d))$
$\leq\bigvee_{u,r\in H}((b\circ d)(r)\wedge\rho(t,u)\wedge\rho(u,r)=\bigvee_{r\in H}((b\circ d)(r)\wedge\rho(t,r)$, for all $b, d, t\in H$.
Similarly, $\bigvee_{b,d\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(b\circ d)(t))$

$\leq \bigvee_{r\in H}(a\circ c)(r)\wedge\rho(r,t)$, for all $a,c,t\in H$.
Conversely, assume that $\bigvee_{a,c\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(a\circ c)(t))\leq\bigvee_{r\in H}((b\circ d)(r)\wedge\rho(t,r))$, for all $b,d,t\in H$ and
$\bigvee_{b,d\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(b\circ d)(t))\leq\bigvee_{r\in H}(a\circ c)(r)\wedge\rho(r,t)$, for all $a,c,t\in H$.
Now $\bigvee_{a\in H}(\rho(a,b)\wedge(a\circ c)(t))\leq\bigvee_{a,p\in H}(\rho(a,b)\wedge\rho(p,c)\wedge(a\circ p)(t))$
$\leq\bigvee_{r\in H}((b\circ c)(r)\wedge\rho(t,r))$, for all $b,c,t\in H$
and
$\bigvee_{b\in H}(\rho(a,b)\wedge(b\circ c)(t))\leq\bigvee_{b,d\in H}(\rho(a,b)\wedge\rho(c,d)\wedge(b\circ d)(t))\leq\bigvee_{r\in H}(a\circ c)(r)\wedge\rho(r,t)$, for all $a,c,t\in H$.
It follows that ρ is right regular. Similarly, ρ is left regular and hence ρ is regular. □

Theorem 5.4. *Let ρ be a fuzzy regular relation on a fuzzy hypergroup $(H,\circ)$. Let ρ_a be the fuzzy equivalence class corresponding to an element $a\in H$. We consider the set $H_\rho=\{\rho_a : a\in H\}$ and define a fuzzy hyperoperation $\star$ on H_ρ by*

$$(\rho_a\star\rho_b)(\rho_x)=\bigvee_{r\in H}(a\circ b)(r)\wedge\rho_x(r),\quad for\quad all\quad a,b,x\in H.$$

Then $(H_\rho,\star)$ is a fuzzy hypergroup.

Proof. First we show that the fuzzy hyperoperation $\star$ on H_ρ given by

$$(\rho_a\star\rho_b)(\rho_x)=\bigvee_{r\in H}(a\circ b)(r)\wedge\rho_x(r),\quad for\quad all\quad a,b,x\in H.$$

is well-defined.
Let $\rho_a=\rho_{a'}$, $\rho_b=\rho_{b'}$, $\rho_x=\rho_{x'}$, then $\rho(a,a')=\rho(b,b')=\rho(x,x')=1$.
$(\rho_a\star\rho_b)(\rho_x)=\bigvee_{r\in H}(a\circ b)(r)\wedge\rho_x(r)=\bigvee_{r\in H}(a\circ b)(r)\wedge\rho(a,a')\wedge\rho(b,b')\wedge\rho(x,r)$
$\leq\bigvee_{r,p,q\in H}(p\circ q)(r)\wedge\rho(p,a')\wedge\rho(q,b')\wedge\rho(x,r)$
$\leq\bigvee_{r,r'\in H}(a'\circ b')(r')\wedge\rho(r,r')\wedge\rho(x,r)\wedge\rho(x,x')\leq\bigvee_{r'\in H}(a'\circ b')(r')\wedge\rho(x',r')$
$=(\rho_{a'}\star\rho_{b'})(\rho_{x'})$. It follows that $(\rho_a\star\rho_b)(\rho_x)\leq(\rho_{a'}\star\rho_{b'})(\rho_{x'})$.
Similarly, $(\rho_{a'}\star\rho_{b'})(\rho_{x'})\leq(\rho_a\star\rho_b)(\rho_x)$.
Therefore $(\rho_a\star\rho_b)(\rho_x)=(\rho_{a'}\star\rho_{b'})(\rho_{x'})$ and hence the fuzzy hyperoperation $\star$ on H_ρ is well-defined.
We also note that $(a\circ b)(x)\leq(\rho_a\star\rho_b)(\rho_x)$, for all $a,b,x\in H$.
$((\rho_a\star\rho_b)\star\rho_c)(\rho_x)=\bigvee_{\rho_r\in H_\rho}(\rho_a\star\rho_b)(\rho_r)\wedge(\rho_r\star\rho_c)(\rho_x)$
$=\bigvee_{r,s,t\in H}((a\circ b)(s)\wedge\rho(r,s)\wedge(r\circ c)(t)\wedge\rho(x,t))$
$\leq\bigvee_{s,t,t'\in H}((a\circ b)(s)\wedge(s\circ c)(t')\wedge\rho(t,t')\wedge\rho(x,t))$
$\leq\bigvee_{s,t'\in H}((a\circ b)(s)\wedge(s\circ c)(t')\wedge\rho(x,t'))=\bigvee_{t'\in H}((a\circ b)\circ c)(t')\wedge\rho(x,t'))$
$=\bigvee_{t'\in H}((a\circ(b\circ c))(t')\wedge\rho(x,t'))=\bigvee_{r,t'\in H}((a\circ r)(t')\wedge(b\circ c)(r)\wedge\rho(x,t'))$
$=\bigvee_{r\in H}((\rho_a\star\rho_r)(\rho_x)\wedge(b\circ c)(r))\leq\bigvee_{r\in H}((\rho_a\star\rho_r)(\rho_x)\wedge(\rho_b\star\rho_c)(\rho_r))$
$=(\rho_a\star(\rho_b\star\rho_c))(\rho_x)$.
Similarly, $(\rho_a\star(\rho_b\star\rho_c))(\rho_x)\leq((\rho_a\star\rho_b)\star\rho_c)(\rho_x)$.
Therefore $((\rho_a\star\rho_b)\star\rho_c)(\rho_x)=(\rho_a\star(\rho_b\star\rho_c))(\rho_x)$, for all $\rho_a,\rho_b,\rho_c,\rho_x\in H_\rho$. Hence $(H_\rho,\star)$ is a fuzzy semihypergroup.

Again, $\bigvee_{\rho_r \in H_\rho} (\rho_a \star \rho_r)(\rho_x) = \bigvee_{r,t \in H} ((a \circ r)(t) \wedge \rho(x,t)) = \bigvee_{t \in H} \rho(x,t) = 1$. Similarly, $\bigvee_{\rho_r \in H_\rho} (\rho_r \star \rho_a)(\rho_x) = 1$. Therefore $\rho_a \star H_\rho = H_\rho \star \rho_a = \chi_{H_\rho}$, for all $\rho_a \in H_\rho$. It follows that $(H_\rho, \star)$ is a fuzzy hypergroup.
The fuzzy hypergroup $(H_\rho, \star)$ is called a *fuzzy factor* or *fuzzy quotient hypergroup* of H modulo ρ. □

Theorem 5.5. *If ρ is a fuzzy regular relation on a fuzzy transposition hypergroup $(H, \circ)$, then the fuzzy quotient hypergroup $(H_\rho, \star)$ is a fuzzy transposition hypergroup.*

Proof. Let $\rho_a, \rho_b, \rho_c, \rho_d, \rho_x \in H_\rho$.
$(\rho_b \backslash \rho_a \wedge \rho_c / \rho_d)(\rho_x) = \rho_b \backslash \rho_a(\rho_x) \wedge \rho_c/\rho_d(\rho_x) = (\rho_b \star \rho_x)(\rho_a) \wedge (\rho_x \star \rho_d)(\rho_c)$
$= \bigvee_{r,t \in H} ((b \circ x)(r) \wedge \rho(a,r) \wedge (x \circ d)(t) \wedge \rho(c,t))$
$= \bigvee_{r,t \in H} (b \backslash r(x) \wedge t/d(x) \wedge \rho(a,r) \wedge \rho(c,t))$
$\leq \bigvee_{r,t \in H} ((b \circ t)(y) \wedge (r \circ d)(y) \wedge \rho(a,r) \wedge \rho(c,t))$
$\leq \bigvee_{z,w \in H} ((b \circ c)(z) \wedge \rho(z,y) \wedge (a \circ d)(w) \wedge \rho(w,y))$
$= (\rho_b \star \rho_c)(\rho_y) \wedge (\rho_a \star \rho_d)(\rho_y)$.
Therefore, for each $\rho_x \in H_\rho$, there exists $\rho_y \in H_\rho$, such that

$$\rho_b \backslash \rho_a(\rho_x) \wedge \rho_c/\rho_d(\rho_x) \leq (\rho_b \star \rho_c)(\rho_y) \wedge (\rho_a \star \rho_d)(\rho_y).$$

Hence it follows that $(H_\rho, \star)$ is a fuzzy transposition hypergroup. □

Theorem 5.6. *Let μ be a fuzzy reflexive closed subsemihypergroup of a fuzzy transposition hypergroup $(H, \circ)$ such that $\bigvee_{p \in H} \mu(p) = 1$. Then the fuzzy relation ρ on H defined by $\rho(a,b) = \bigvee_{x \in H} ((a \circ \mu)(x) \wedge (\mu \circ b)(x)$, for all $a, b \in H$, is a fuzzy regular relation.*

Proof. We have proved that ρ is a fuzzy equivalence relation.
$\bigvee_{b \in H} ((b \circ d)(t) \wedge \rho(a,b)) = \bigvee_{b \in H} ((b \circ d)(t) \wedge \rho(a,b) \wedge \rho(d,d))$
$\leq \bigvee_{b,c \in H} ((b \circ c)(t) \wedge \rho_a(b) \wedge \rho_d(c))$
$= (\rho_a \circ \rho_d)(t)$.
Therefore $\bigvee_{b \in H} ((b \circ d)(t) \wedge \rho(a,b)) \leq (\rho_a \circ \rho_d)(t)$. Now by Theorem 4.19 we have $\bigvee_{b \in H} ((b \circ d)(t) \wedge \rho(a,b)) \leq \bigvee_{r \in H} ((a \circ d)(r) \wedge \rho(r,t))$, for all $a, d, t \in H$. Similarly, $\bigvee_{a \in H} ((a \circ d)(t) \wedge \rho(a,b)) \leq \bigvee_{r \in H} ((b \circ d)(r) \wedge \rho(t,r))$, for all $b, d, t \in H$. Therefore ρ is right regular. Similarly, ρ is left regular. It follows that ρ is regular. □

Definition 5.7. Let ρ be a fuzzy regular relation on a fuzzy hypergroup $(H, \circ)$. An element $n \in H$ is said to be a fuzzy scalar identity for ρ if $n \circ a = a \circ n = \rho_a$, for all $a \in H$.

Theorem 5.8. *Let ρ be a fuzzy regular relation on a fuzzy transposition hypergroup $(H, \circ)$ that has a fuzzy scalar identity n, then*
(i) $\rho_{a \circ n} = \rho_{n \circ a} = \rho_a$, *for all* $a \in H$,
(ii) $a \circ \rho_n = \rho_n \circ a = \rho_a$, *for all* $a \in H$,
(iii) $\rho_a \circ \rho_n = \rho_n \circ \rho_a = \rho_a$, *for all* $a \in H$.

Proof. (*i*) $\rho_{a\circ n}(x) = \bigvee_{r\in H}(a\circ n)(r)\wedge\rho_r(x) = \bigvee_{r\in H}\rho_a(r)\wedge\rho_r(x)$
$= \bigvee_{r\in H}\rho(a,r)\wedge\rho(r,x) = (\rho\circ\rho)(a,x) = \rho(a,x) = \rho_a(x)$.
Similarly, $\rho_{n\circ a}(x) = \rho_a(x)$.
Therefore $\rho_{a\circ n} = \rho_{n\circ a} = \rho_a$, for all $a \in H$.

(*ii*) $(a\circ\rho_n)(x) = \bigvee_{r\in H}(a\circ r)(x)\wedge\rho_n(r) = \bigvee_{r\in H}\rho(n,r)\wedge(a\circ r)(x)$
$\leq \bigvee_{t\in H}(a\circ n)(t)\wedge\rho(t,x) = \bigvee_{t\in H}\rho(a,t)\wedge\rho(t,x) = (\rho\circ\rho)(a,x) = \rho(a,x) = \rho_a(x)$.
Also $(a\circ\rho_n)(x) = \bigvee_{r\in H}(a\circ r)(x)\wedge\rho_n(r) \geq (a\circ n)(x) = \rho_a(x)$.
It follows that $(a\circ\rho_n)(x) = \rho_a(x)$.
Similarly, $(\rho_n\circ a)(x) = \rho_a(x)$. Therefore $a\circ\rho_n = \rho_n\circ a = \rho_a$, for all $a \in H$.

(*iii*) $(\rho_a\circ\rho_n)(x) = \bigvee_{r\in H}\rho_a(r)\wedge(r\circ\rho_n)(x) = \bigvee_{r\in H}\rho(a,r)\wedge\rho(r,x) = (\rho\circ\rho)(a,x)$
$= \rho(a,x) = \rho_a(x)$. Similarly, $(\rho_n\circ\rho_a)(x) = \rho_a(x)$.
Therefore $\rho_a\circ\rho_n = \rho_n\circ\rho_a = \rho_a$, for all $a \in H$. □

Theorem 5.9. *Let ρ be a fuzzy regular relation on a fuzzy transposition hypergroup $(H,\circ)$ that has a fuzzy scalar identity n. Then ρ_n is a fuzzy reflexive closed subsemihypergroup of $(H,\circ)$ such that $\bigvee_{p\in H}\rho_n(p) = 1$ and ρ is the relation equivalence modulo ρ_n.*

Proof. $\bigvee_{p\in H}\rho_n(p) = \bigvee_{p\in H}\rho(n,p) = 1$.
$(\rho_n\circ\rho_n)(x) = \bigvee_{r\in H}\rho_n(r)\wedge(r\circ\rho_n)(x) = \bigvee_{r\in H}\rho_n(r)\wedge\rho_r(x)$
$= \bigvee_{r\in H}\rho(n,r)\wedge\rho(r,x) = (\rho\circ\rho)(n,x) = \rho(n,x) = \rho_n(x)$, for all $x \in H$.
It follows that ρ_n is a fuzzy subsemihypergroup of H.
$\rho_n/\rho_n(x) = \bigvee_{r\in H}\rho_n(r)\wedge(x\circ\rho_n)(r) = \bigvee_{r\in H}\rho_n(r)\wedge\rho_x(r) = \bigvee_{r\in H}\rho(n,r)\wedge\rho(r,x)$ $= (\rho\circ\rho)(n,x) = \rho(n,x) = \rho_n(x)$, for all $x \in H$. Similarly, $\rho_n\backslash\rho_n(x) = \rho_n(x)$, for all $x \in H$. It follows that ρ_n is fuzzy closed.
By Theorem 5.8, $a\circ\rho_n = \rho_n\circ a$, for all $a \in H$. It follows that ρ_n is fuzzy normal.
Therefore ρ_n is a fuzzy normal and closed subsemihypergroup of the fuzzy transposition hypergroup $(H,\circ)$and hence, by Theorem 4.13, ρ_n is fuzzy reflexive.
$\bigvee_{r\in H}((a\circ\rho_n)(r)\wedge(\rho_n\circ b)(r)) = \bigvee_{r\in H}\rho_a(r)\wedge\rho_b(r) = (\rho\circ\rho)(a,b) = \rho(a,b)$.
Therefore $\rho(a,b) = \bigvee_{r\in H}((a\circ\rho_n)(r)\wedge(\rho_n\circ b)(r))$, for all $a, b \in H$. It follows that ρ is equivalence modulo ρ_n. □

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