DIRECTLY INDECOMPOSABLE RESIDUATED LATTICES

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Abstract. The aim of this paper is to extend results established by H. Ono and T. Kowalski regarding directly indecomposable commutative residuated lattices to the non-commutative case. The main theorem states that a residuated lattice A is directly indecomposable if and only if its Boolean center $B(A)$ is $\{0, 1\}$. We also prove that any linearly ordered residuated lattice and any local residuated lattice are directly indecomposable. We apply these results to prove some properties of the Boolean center of a residuated lattice and also define the algebras on subintervals of residuated lattices.

1. Introduction

It is known that the study of classical logic can be reduced to studying Boolean algebras. Therefore, the discussion of any type of non-classical logic raises a question about the corresponding abstract algebra. There has been much research in the field of fuzzy logic when the conjunction of the truth values structure is not necessarily commutative. Developing algebraic models for non-commutative multiple-valued logics is a central topic in the research of fuzzy systems and one such algebraic structure is the non-commutative residuated lattice. In the last few years a corresponding fuzzy theory has developed in parallel with the classical theory [2, 3, 19, 13].

Commutative residuated lattices were first introduced by M. Ward and R.P. Dilworth as a generalization of ideal lattices of rings. Recently, these structures have been studied in [10] and [18]. Non-commutative residuated lattices, sometimes called pseudo-residuated lattices, biresiduated lattices or generalized residuated lattices, are the algebraic counterparts of substructural logics; i.e. logics which lack at least one of the three structural rules, namely contraction, weakening and exchange. Complete studies on non-commutative residuated lattices were developed in [1] and [17]. The aim of this paper is to extend results proved by H. Ono and T. Kowalski for the case of commutative residuated lattices to the non-commutative case. The main theorem states that a residuated lattice A is directly indecomposable if and only if its Boolean center $B(A)$ is $\{0, 1\}$. We also prove that any local residuated lattice is directly indecomposable and derive some properties of the Boolean center of a residuated lattice. As an application of the Boolean center of a residuated lattice we prove that any subinterval $[a, b]$ of a residuated lattice can be endowed with an algebraic structure of the same kind as the original one. These structures are

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very important, for studying the restriction of the existing states from the whole residuated lattice to subinterval algebras.

2. Preliminaires on Residuated Lattices

Definition 2.1. [17] A *residuated lattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 1)$ of the type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following axioms:
(A_1) $(A, \wedge, \vee)$ is a lattice;
(A_2) $(A, \odot, 1)$ is a monoid;
(A_3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$ (*pseudo-residuation*).

Note that, in general, 1 is not the top element of A. A residuated lattice with a constant 0 (which can denote any element) is called a *pointed residuated lattice* or a *full Lambek algebra* (*FL-algebra*). If $x \leq 1$ for all $x \in A$, then $\mathcal{A}$ is called an *integral residuated lattice*. A FL-algebra $\mathcal{A}$ which satisfies the condition $0 \leq x \leq 1$ for all $x \in A$ is called a FL_w-algebra. Clearly, if $\mathcal{A}$ is a FL_w-algebra, then $(A, \wedge, \vee, 0, 1)$ is a bounded lattice. A FL_w-algebra is sometimes called a *bounded integral residuated lattice*.
In this paper, a residuated lattice will be a FL_w-algebra.
$\mathcal{A}$ is called *commutative* if the operation $\odot$ is commutative (iff $\rightarrow = \rightsquigarrow$).
A totally ordered residuated lattice is called a *chain* or a *linearly ordered* residuated lattice.
In the sequel we assume that the operations $\wedge, \vee, \odot$ have higher priority than the operations $\rightarrow, \rightsquigarrow$ and refer to $\mathcal{A}$ by its universe A.

Example 2.2. [6] Let us consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a residuated lattice (more exactly, it is a pseudo-MTL chain).

It was proved in [17] that the class $\mathcal{RL}$ of all residuated lattices forms a variety.

In a residuated lattice $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ for all $x \in A$, we define:
$x^- = x \rightarrow 0$ and $x^\sim = x \rightsquigarrow 0$.

Properties of a residuated lattice were studied in [1], [7], [14], [17] and examples of these structures are given in [15] and [16]. In the following propositions we recall a few of these properties which are required in the sequel.

Proposition 2.3. *The following statements hold for any residuated lattice A:*
(c_1) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$ *and* $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;
(c_2) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$;
(c_3) $x \leq y \rightarrow x$ *and* $x \leq y \rightsquigarrow x$;

(c_4) $(x \to y) \odot x \leq x \wedge y$ *and* $x \odot (x \rightsquigarrow y) \leq x \wedge y$;
(c_5) *If* $x \leq y$ *then* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(c_6) *If* $x \leq y$ *then* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$.

Proposition 2.4. *The following statements hold for any residuated lattice A:*
(c_7) $x \odot (\vee_{i\in I} y_i) = \vee_{i\in I}(x \odot y_i)$ *and* $(\vee_{i\in I} y_i) \odot x = \vee_{i\in I}(y_i \odot x)$;
(c_8) $x \vee y \leq [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x]$ *and*
$x \vee y \leq [(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x]$;
(c_9) $y \to (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \to x_i)$ *and* $y \rightsquigarrow (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \rightsquigarrow x_i)$;
(c_{10}) $x \to y \leq (x \odot z) \to (y \odot z)$ *and* $x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y)$.

Let A be a residuated lattice. For any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. The *order* of $x \in A$, denoted by $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.
A residuated lattice A is *locally finite* if, for any $x \in A$, $x \neq 1$ implies $ord(x) < \infty$.

We now recall the definition and some properties of the filters of a residuated lattice ([5]).
Let A be a residuated lattice. A nonempty subset F of A is called a *filter* of A if the following conditions hold:
(F_1) If $x, y \in F$, then $x \odot y \in F$;
(F_2) If $x \in F$, $y \in A$, $x \leq y$, then $y \in F$.

We denote the set of all filters of A by $\mathcal{F}(A)$. Suppose F is a filter of $A \in \mathcal{F}(A)$, then:
(F_3) $1 \in F$;
(F_4) If $x \in F$, $y \in A$, then $y \to x \in F$, $y \rightsquigarrow x \in F$;
(F_5) If $x, y \in F$, then $x \wedge y \in F$.

Remark 2.5. For a subset F of A the following are equivalent:
(a) F is a filter;
(b) $1 \in F$ and if x, $x \to y \in F$, then $y \in F$;
(c) $1 \in F$ and if x, $x \rightsquigarrow y \in F$, then $y \in F$.
Indeed, since the filter F is nonempty, there exists $x \in F$ and from $x \leq 1$ we get $1 \in F$. If x, $x \to y \in F$, it follows that $(x \to y) \odot x \in F$. Since $(x \to y) \odot x \leq y$, by (F_2) hence $y \in F$.
Conversely, consider $F \subseteq A$ satisfying (b) and $x, y \in A$. Since $x \to (y \to x \odot y) = x \odot y \to x \odot y = 1 \in F$, hence $y \to x \odot y \in F$. It follows that $x \odot y \in F$, so condition (F_1) is satisfied.
If $x \in F$, $y \in A$ such that $x \leq y$, we have $x \to y = 1 \in F$, so $y \in F$, that is (F_2).
The equivalence (a) $\Leftrightarrow$ (c) can be proved similarly.
Thus, F is a filter of A.

For every subset $X \subseteq A$, the smallest filter of A containing X (i.e. the intersection of all filters $F \in \mathcal{F}(A)$ such that $X \subseteq F$) is called the filter *generated by* X and is denoted by $< X >$. If $X \subseteq A$, then
$$< X >= \{y \in A \mid y \geq x_1 \odot x_2 \odot \cdots \odot x_n \text{ for some } n \geq 1 \text{ and } x_1, x_2, \ldots, x_n \in X\}.$$

Remark 2.6. (1) If X is a filter of A, then $< X >= X$;
(2) If $X = \{x\}$ we write $< x >$ instead of $< \{x\} >$ and

$< x >= \{y \in A \mid y \geq x^n \text{ for some } n \geq 1\}$; $< x >$ is called the *principal* filter;
(3) If F_1, F_2 are filters of A, then

$$< F_1 \cup F_2 >= \{x \in A \mid x \geq (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n') \text{ for some } n \geq 1, f_1, f_2, \ldots, f_n \in F_1,\ f_1', f_2', \ldots, f_n' \in F_2\}.$$

If $F_1, F_2 \in \mathcal{F}(A)$, we define $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 =< F_1 \cup F_2 >$.
A filter H of A is called *normal* if for any $x, y \in A$

$$(N)\ x \to y \in H \text{ iff } \mathrm{x} \rightsquigarrow \mathrm{y} \in \mathrm{H}.$$

We denote the set of all normal filters of A by $\mathcal{F}_n(A)$.

For any residuated lattice A we have:
(1) $\{1\}, A \in \mathcal{F}_n(A)$; (2) $\mathcal{F}_n(A) \subseteq \mathcal{F}(A)$; (3) $(\mathcal{F}(A), \subseteq)$ is a lattice and $(\mathcal{F}_n(A), \subseteq)$ is a sublattice of $(\mathcal{F}(A), \subseteq)$.

A filter F of A is *proper* if $F \neq A$. A proper filter of A is called a *maximal filter* or an *ultrafilter* if it is not strictly contained in any other proper filter of A.
We write $Max(A) = \{F \mid F \text{ is maximal filter of } A\}$ and $Max_n(A) = \{F \mid F$ is maximal and normal filter of $A\}$.
A residuated lattice is called *local* if and only if it has a unique maximal filter.
We write $D(A) = \{x \in A \mid ord(x) = \infty\}$.

Proposition 2.7. [5] *A residuated lattice A is local iff $D(A)$ is a filter of A.*

Proposition 2.8. [5] *If A is a local residuated lattice, then for any $x \in A$ we have $ord(x) < \infty$ or $[ord(x^-) < \infty$ and $ord(x^\sim) < \infty]$.*

A *congruence* on a residuated lattice A is an equivalence relation compatible with the operations of A. The set of all congruences of A is denoted by $Con(A)$ and $(Con(A), \subseteq)$ is a lattice. More precisely, $(Con(A), \subseteq)$ is a complete sublattice of $(Eq(A), \subseteq)$, where $Eq(A)$ is the lattice of all equivalence relations on A (see [4]).
The next two results can be proved as in [11] for the case of pseudo-hoops.

Proposition 2.9. *If H is a normal filter of A, then the relation θ_H defined by: $(x, y) \in \theta_H$ iff $x \to y, y \to x \in H$ iff $x \rightsquigarrow y, y \rightsquigarrow x \in H$ is a congruence on A. Conversely, if $\theta \in Con(A)$, then $H_\theta = \{x \in A \mid (x, 1) \in \theta\}$ is a normal filter of A.*

Theorem 2.10. *The map $H \longmapsto \theta_H$ is an isomorphism between the lattices $\mathcal{F}_n(A)$ and $Con(A)$.*

3. Boolean Center of a Residuated Lattice

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall [12] that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; in this case, b is called a *complement* of a. We write $b = a'$ and denote the set of all complemented elements in L by $B(L)$. In the case of a residuated lattice A, the set $B(A)$ is called the *Boolean center* of A.
Complements are generally not unique unless the lattice is distributive. In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.
The next proposition extends a result proved in [18] for the case of commutative residuated lattices.

Proposition 3.1. *Let A be a residuated lattice and suppose that $a \in A$ has the complement $b \in A$. Then we have:*
(1) *If c is a complement of a in A, then $c = b$;*
(2) *$a^- = a^\sim = b$ and $b^- = b^\sim = a$;*
(3) *$a^2 = a$.*

Proof. (1) Since b and c are complements of a, it follows that $a \vee b = 1 = a \vee c$ and $a \wedge b = 0 = a \wedge c$. Hence, $c = c \odot 1 = c \odot (a \vee b) = (c \odot a) \vee (c \odot b) = c \odot b$ (since $c \odot a \leq c \wedge a = 0$, we have $c \odot a = 0$).
Similarly, $b = 1 \odot b = (a \vee c) \odot b = (a \odot b) \vee (c \odot b) = c \odot b$. Thus, $b = c$.
(2) We have:
$a \odot b \leq a \wedge b = 0$, so $a \leq b \to 0 = b^-$ and $b \leq a \rightsquigarrow 0 = a^\sim$;
$b \odot a \leq b \wedge a = 0$, so $b \leq a \to 0 = a^-$ and $a \leq b \rightsquigarrow 0 = b^\sim$.
On the other hand, $a^- = a^- \odot 1 = a^- \odot (a \vee b) = (a^- \odot a) \vee (a^- \odot b) = a^- \odot b$.
Hence, $a^- \odot b = a^- \geq a^- \wedge b$. Now since $a^- \odot b \leq a^- \wedge b$, we obtain
$a^- = a^- \odot b = a^- \wedge b$, so $b \geq a^- \wedge b = a^-$.
Thus, $a^- = b$ and similarly $a^\sim = b$.
(3) $a = a \odot (a \vee b) = a^2 \vee (a \odot b) = a^2$ (since $a \odot b \leq a \wedge b = 0$). □

Corollary 3.2. *Let A be a residuated lattice. Then, the following statements are equivalent:*
(a) $a \in B(A)$;
(b) $a \vee a^- = 1$ and $a \wedge a^- = 0$;
(c) $a \vee a^\sim = 1$ and $a \vee a^\sim = 0$.

Corollary 3.3. *If $a \in B(A)$, then:*
(1) *$(a^-)^2 = a^-$ and $(a^\sim)^2 = a^\sim$;*
(2) *$a \to a^- = a \rightsquigarrow a^- = a^-$ and $a \to a^\sim = a \rightsquigarrow a^\sim = a^\sim$;*
(3) *$a^- \to a = a^- \rightsquigarrow a = a^\sim \to a = a^\sim \rightsquigarrow a = a$.*

Proof. We have:
(1) $a^- = a^- \odot 1 = a^- \odot (a^- \vee a) = (a^-)^2 \vee (a^- \odot a) = (a^-)^2$;
$a^\sim = 1 \odot a^\sim = (a \vee a^\sim) \odot a^\sim = (a \odot a^\sim) \vee (a^\sim)^2 = (a^\sim)^2$.
(2) and (3) follow from (c_8), taking into consideration that $a \vee a^- = a \vee a^\sim = 1$ and applying (c_3). □

Corollary 3.4. *If $a \in B(A)$, then $a^{-\sim} = a^{\sim-} = a$.*

Proof. Since a and $a^{-\sim}$ are complements of a^-, hence by Proposition 3.1 it follows that $a^{-\sim} = a$. Similarly, $a^{\sim-} = a$. □

Proposition 3.5. *If $a \in B(A)$ and $x \in A$, then we have:*
(1) *$a \odot x = x \odot a = a \odot x \odot a$;*
(2) *$a^- \odot x = x \odot a^- = a^- \odot x \odot a^- = a^\sim \odot x \odot a^\sim = a^\sim \odot x = x \odot a^\sim$.*

Proof. (1) Applying the properties of a residuated lattice we get:

$$a \odot x = a \odot x \odot 1 = a \odot x \odot (a \vee a^\sim) =$$
$$(a \odot x \odot a) \vee (a \odot x \odot a^\sim) = a \odot x \odot a$$

(we have used the fact that $a \odot x \odot a^\sim \leq a \odot a^\sim = 0$, so $a \odot x \odot a^\sim = 0$).

(2) We have:

$$a^- \odot x = a^- \odot x \odot 1 = a^- \odot x \odot (a^- \vee a) =$$
$$(a^- \odot x \odot a^-) \vee (a^- \odot x \odot a) = a^- \odot x \odot a^-$$

(since $a^- \odot x \odot a \leq a^- \odot a = 0$, we have $a^- \odot x \odot a = 0$).

$$x \odot a^\sim = (a^\sim \vee a) \odot (x \odot a^\sim) =$$
$$(a^\sim \odot x \odot a^\sim) \vee (a \odot x \odot a^\sim) = a^\sim \odot x \odot a^\sim$$

(since $a \odot x \odot a^\sim \leq a \odot a^\sim = 0$, it follows that $a \odot x \odot a^\sim = 0$).
Now, since $a^- = a^\sim$, the assertion follows. □

Proposition 3.6. *If $a \in B(A)$, then the filters $< a >$, $< a^- >$ and $< a^\sim >$ are normal.*

Proof. Since $a^2 = a$, we have:
$< a >= \{x \in A \mid a^n \leq x \text{ for some } n \in \mathbb{N}, n \geq 1\} = \{x \in A \mid a \leq x\}$.
It follows that:

$$x \to y \in < a > \text{ iff } a \leq x \to y \text{ iff } a \odot x \leq y \text{ iff}$$
$$x \odot a \leq y \text{ iff } a \leq x \rightsquigarrow y \text{ iff } x \rightsquigarrow y \in < a >.$$

Thus, $< a >$ is a normal filter of A. Using the identities $(a^-)^2 = a^-$ and $(a^\sim)^2 = a^\sim$, it follows that $< a^- >= \{x \in A \mid a^- \leq x\}$ and $< a^\sim >= \{x \in A \mid a^\sim \leq x\}$.
The proofs for the normality of $< a^- >$ and $< a^\sim >$ are similar. □

In the next section we prove that $B(A)$ is the universe of a Boolean subalgebra of A.

4. Directly Indecomposable Residuated Lattices

Recall that [4], if A is an algebra and $\theta_1, \theta_2 \in Con(A)$, then $\theta_1 \circ \theta_2$ is the binary relation on A defined by $(x, y) \in \theta_1 \circ \theta_2$ iff there exists $z \in A$ such that $(x, z) \in \theta_1$ and $(z, y) \in \theta_2$. If $\theta_1, \theta_2 \in Con(A)$ such that $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ we say that θ_1 and θ_2 are *permutable*. An algebra A is called *congruence permutable* if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ for all $\theta_1, \theta_2 \in Con(A)$.
According to [17] every residuated lattice is congruence permutable.
For any algebra A, two permutable congruences θ_1, θ_2 of A are *complementary factor congruences* if $\theta_1 \vee \theta_2 = \mathbf{1}$ and $\theta_1 \wedge \theta_2 = \mathbf{0}$ ($\mathbf{1}$ and $\mathbf{0}$ are top and respectively bottom elements in the lattice $Con(A)$).
The mapping $p_i : A_1 \times A_2 \longrightarrow A_i$, $i \in \{1, 2\}$ defined by $p_i((a_1, a_2)) = a_i$ is called the *projection map on the i^{th} coordinate* of $A_1 \times A_2$.
An algebra A is *directly indecomposable* if A is not isomorphic to a direct product of two nontrivial algebras.
A *subdirect representation* of an algebra A with factors A_i is an embedding $f : A \longrightarrow \Pi_{i \in I} A_i$ such that each $f_i = p_i \circ f$ is onto A_i for all $i \in I$. A is also called a *subdirect product* of A_i. An algebra A is *subdirectly irreducible* iff it is nontrivial and for any subdirect representation $f : A \longrightarrow \Pi_{i \in I} A_i$, there exists $j \in I$ such that f_j is an isomorphism of A onto A_j.
An algebra A is said to be *simple* if it has a two-element congruence lattice.
In what follows we recall some results from [4].

For $i \in \{1,2\}$, the mapping $p_i : A_1 \times A_2 \longrightarrow A_i$ is a surjective homomorphism from $A = A_1 \times A_2$ to A_i. Moreover, in $Con(A_1 \times A_2)$, $ker(p_1)$ and $ker(p_2)$ are permutable and $ker(p_1) \cap ker(p_2) = \mathbf{0}$, $ker(p_1) \vee ker(p_2) = \mathbf{1}$.
If θ_1, θ_2 are complementary factor congruences of an algebra A, then $A \cong A/\theta_1 \times A/\theta_2$.
As a consequence, an algebra A is directly indecomposable iff the only factor congruences on A are $\mathbf{0}$ and $\mathbf{1}$.
A subdirectly irreducible algebra is directly indecomposable.
Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras (Birkhoff's theorem).

Proposition 4.1. *Let A be a residuated lattice and $a \in B(A)$. Then the congruences $\theta_{<a>}$ and $\theta_{<a^->}$ form a pair of complementary factor congruences.*

Proof. We proved in Proposition 3.6 that $< a >$ and $< a^- >$ are normal filters of A, so by Theorem 2.10 they are congruences on A.
Since every residuated lattice is congruence permutable, it follows that the congruences $\theta_{<a>}$ and $\theta_{<a^->}$ are permutable. Hence, we must prove that $\theta_{<a>} \cap \theta_{<a^->} = \{1\}$ and $\theta_{<a>} \vee \theta_{<a^->} = A$ with the join defined in the lattice of filters of A. In the proof of Proposition 3.6 we showed that $< a >= \{x \in A \mid x \geq a\}$ and $< a^- >= \{x \in A \mid x \geq a^-\}$. We have $x \in < a > \cap < a^- >$ iff $x \geq a$ and $x \geq a^-$ iff $x \geq a \vee a^- = 1$, so $< a > \cap < a^- >= \{1\}$.
Since $< a > \vee < a^- >=< \{x \in A \mid x \geq a\} \cup \{x \in A \mid x \geq a^-\} >$, applying Remark 2.6(3) we get $< a > \vee < a^- >= \{x \in A \mid x \geq a \odot a^- = \{x \in A \mid x \geq 0\} = A$. □

Corollary 4.2. *If A is a residuated lattice and $a \in B(A)$, then the congruences $\theta_{<a>}$ and $\theta_{<a^\sim>}$ form a pair of complementary factor congruences.*

Proof. The corollary follows since $< a^\sim >=< a^- >$. □

Theorem 4.3. *A nontrivial residuated lattice A is directly indecomposable iff $B(A) = \{0,1\}$.*

Proof. As we have already mentioned, a residuated lattice A is directly indecomposable iff the only factor congruences on A are $\mathbf{0}$ and $\mathbf{1}$. By Proposition 4.1, the number of pairs of complementary factor congruences coincide with the number of elements of $B(A)$. Thus, A is directly indecomposable iff $B(A)$ has only two elements; i.e. $B(A) = \{0,1\}$. □

Corollary 4.4. *A simple residuated lattice is subdirectly irreducible and a subdirectly irreducible residuated lattice is directly indecomposable.*

Proof. The corollary follows from the definitions of simple and subdirectly irreducible algebras and Theorem 4.3. □

Example 4.5. If A is the residuated lattice of Example 2.2, we can see that $B(A) = \{0,1\}$, so it is directly indecomposable.

Proposition 4.6. *Any linearly ordered residuated lattice is directly indecomposable.*

Proof. Let A be a linearly ordered residuated lattice and $a \in A$. By Corollary 3.2 we have $a \vee a^- = 1$. Since A is linearly ordered, it follows that $a \leq a^-$ or $a^- \leq a$, hence $a = 1$ or $a^- = 1$. If $a^- = 1$, we get $1 = a^- = a \to 0$, so $1 \odot a \leq 0$, that is $a = 0$. Thus, $a \in \{0, 1\}$. We conclude that $B(A) = \{0, 1\}$ and hence, by Theorem 4.3, A is directly indecomposable. □

Theorem 4.7. *Any local residuated lattice is directly indecomposable.*

Proof. Let A be a local residuated lattice and $a \in A$. By Proposition 2.8 we obtain $ord(a) < \infty$ or $ord(a^-) < \infty$. It follows that there exists $n \in \mathbb{N}, n \geq 1$ such that $a^n = 0$ or $(a^-)^n = 0$. However, by Proposition 3.1 and Corollary 3.3 we have $a^n = a$ and $(a^-)^n = a^-$. Since a^- is the complement of a, it follows that $a, a^- \in B(A)$. Hence, $a = 0$ or $a^- = 0$. If $a^- = 0$, then $a^{-\sim} = 1$ and by Corollary 3.4 we get $a = a^{-\sim} = 1$. Thus, $a \in \{0, 1\}$, hence $B(A) = \{0, 1\}$. Finally, by Theorem 4.3, it follows that A is directly indecomposable. □

Theorem 4.8. *Any locally finite residuated lattice is directly indecomposable.*

Proof. Let A be a locally finite residuated lattice; i.e. $D(A) = \{1\}$. By Proposition 2.7, it follows that A is local and hence by Proposition 4.7, we conclude that A is directly indecomposable. □

5. **Applications**

In this section, we give some applications of the previous results.
We first prove two results for the case of commutative residuated lattices following the method of [18].

Proposition 5.1. *If A is a residuated lattice and $a \in B(A)$, then $a \odot x = x \odot a = a \wedge x$ for all $x \in A$.*

Proof. By Birkhoff's theorem, A is isomorphic to a subdirect product of subdirectly irreducible algebras. Consider $f_i (i \in I)$, the homomorphisms which define the subdirect representation of A with the factors $A_i (i \in I)$. By Theorem 4.3, we obtain $f_i(a) \in \{0, 1\}$, so $f_i(a) \odot f_i(x) = f_i(a) \wedge f_i(x)$ for all $i \in I$. Hence $f_i(a \odot x) = f_i(a) \odot f_i(x) = f_i(a) \wedge f_i(x) = f_i(a \wedge x)$ for all $i \in I$. Thus, $a \odot x = a \wedge x$. Similarly, $x \odot a = x \wedge a = a \wedge x$. □

Proposition 5.2. *$B(A)$ is the universe of a Boolean subalgebra of A.*

Proof. We must prove that $(B(A), \wedge, \vee)$ is distributive and closed under the operations $\wedge, \vee, \odot, \to$ and $\rightsquigarrow$.
For distributivity, we must prove the identity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in B(A)$. By Proposition 5.1 we have:

$$a \wedge (b \vee c) = a \odot (b \vee c) = (a \odot b) \vee (a \odot c) = (a \wedge b) \vee (a \wedge c).$$

By hypothesis, there exist $a', b' \in A$ such that:

$$a \wedge a' = 0, a \vee a' = 1, b \wedge b' = 0, b \vee b' = 1.$$

We first prove that $a' \vee b'$ is the complement of $a \wedge b$:

$$(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b') = 0 \vee 0 = 0;$$

$$(a \wedge b) \vee (a^{'} \vee b^{'}) = (a^{'} \vee b^{'} \vee a) \wedge (a^{'} \vee b^{'} \vee b) = 1 \wedge 1 = 1.$$

Now we prove that $a^{'} \wedge b^{'}$ is the complement of $a \vee b$:

$$(a \vee b) \wedge (a^{'} \wedge b^{'}) = (a^{'} \wedge b^{'} \wedge a) \vee (a^{'} \wedge b^{'} \wedge b) = 0 \vee 0 = 0;$$
$$(a \vee b) \vee (a^{'} \wedge b^{'}) = (a \vee b \vee a^{'}) \wedge (a \vee b \vee b^{'}) = 1 \wedge 1 = 1.$$

Thus, $a \wedge b, a \vee b \in B(A)$.
If $a, b \in B(A)$, then $a \odot b = a \wedge b \in B(A)$, so $(B(A)$ is closed under $\odot$.
If $b^{'}$ is the complement of b, then $b^{'} = b^{-} = b^{\sim}$ and we have:
$a \to b = a \to b^{-} = (a \odot b)^{-} \in B(A)$ and $a \rightsquigarrow b = a \rightsquigarrow b^{\sim} = (b \odot a)^{\sim} \in B(A)$.
Thus, $B(A)$ is closed under $\to$ and $\rightsquigarrow$. □

Corollary 5.3. *(De Morgan's laws)* $(a \vee b)^{'} = a^{'} \wedge b^{'}$ *and* $(a \wedge b)^{'} = a^{'} \vee b^{'}$.

Corollary 5.4. $(B(A), \wedge, \vee)$ *is a de Morgan lattice.*

Proposition 5.5. *If A is a residuated lattice, then, for all $a, b \in B(A)$ and $x, y \in A$, we have:*
(b_1) $(x \to a) \odot x = x \odot (x \rightsquigarrow a) = a \wedge x$;
(b_2) $(a \to x) \odot a = a \odot (a \rightsquigarrow x) = a \wedge x$;
(b_3) $(x \to y) \odot a = [(x \odot a) \to (y \odot a)] \odot a$;
(b_4) $a \odot (x \rightsquigarrow y) = a \odot [(a \odot x) \rightsquigarrow (a \odot y)]$.

Proof. (b_1) By (c_4) we have $(x \to a) \odot x$, $x \odot (x \rightsquigarrow a) \leq a \wedge x$.
Since $a \leq x \to a$, $x \rightsquigarrow a$, hence

$$a \odot x \leq (x \to a) \odot x \text{ and } x \odot a \leq x \odot (x \rightsquigarrow a).$$

Now by Proposition 5.1, it follows that $a \wedge x \leq (x \to a) \odot x$, $x \odot (x \rightsquigarrow a)$.
Therefore, $(x \to a) \odot x = x \odot (x \rightsquigarrow a) = a \wedge x$.
(b_2) Similar to (b_1).
(b_3) By (c_{10}) we have $(x \to y) \odot a \leq [(x \odot a) \to (y \odot a)] \odot a$.
Conversely, by (c_4) we have:

$$[(x \odot a) \to (y \odot a)] \odot (x \odot a) \leq y \odot a \leq y$$

and since $x \odot a = a \odot x$,

$$[(x \odot a) \to (y \odot a)] \odot (a \odot x) \leq y.$$

Hence, $[(x \odot a) \to (y \odot a)] \odot a \leq x \to y$.
By right multiplication with a and the fact that $a^2 = a$, we obtain:

$$[(x \odot a) \to (y \odot a)] \odot a \leq (x \to y) \odot a.$$

(b_4) Similar to (b_3). □

Proposition 5.6. *If A is a residuated lattice, then, for all $a, b \in B(A)$ and $x, y \in A$, we have:*
(b_5) $(a \to b) \odot x = [(x \odot a) \to (x \odot b)] \odot x$;
(b_6) $x \odot (a \rightsquigarrow b) = x \odot [(a \odot x) \rightsquigarrow (b \odot x)]$;
(b_7) $a \to (x \to y) = (a \to x) \to (a \to y)$;
(b_8) $a \rightsquigarrow (x \rightsquigarrow y) = (a \rightsquigarrow x) \rightsquigarrow (a \rightsquigarrow y)$;
(b_9) $x \odot (a \rightsquigarrow y) = (a \to x) \odot y$.

Proof. (b_5) By (c_9), (c_1), (b_1) and the fact that $a \to b \in B(A))$, we have:

$$[(x \odot a) \to (x \odot b)] \odot x = [(x \odot a) \to (x \wedge b)] \odot x =$$
$$[(x \odot a \to x) \wedge (x \odot a \to b)] \odot x = (x \odot a \to b) \odot x =$$

$$[x \to (a \to b)] \odot x = (a \to b) \wedge x = (a \to b) \odot x$$

(b_6) Similarly, by (c_9), (c_1) and (b_2) we get:
$x \odot [(a \odot x) \rightsquigarrow (b \odot x)] = x \odot [(a \odot x) \rightsquigarrow (b \wedge x)] =$
$x \odot [(a \odot x \rightsquigarrow b) \wedge (a \odot x \rightsquigarrow x)] = x \odot (a \odot x \rightsquigarrow b) =$
$x \odot [x \rightsquigarrow (a \rightsquigarrow b)] = x \wedge (a \rightsquigarrow b) = x \odot (a \rightsquigarrow b)$;
(b_7) By (c_1) and (b_2) we have:
$a \to (x \to y) = (a \odot x) \to y = a \wedge x \to y =$
$(a \to x) \odot a \to y = (a \to x) \to (a \to y)$.
(b_8) Similar to (9);
(b_9) By (b_1) we obtain:
$x \odot (a \rightsquigarrow y) = (a \wedge x) \odot (a \rightsquigarrow y) = a \odot (a \rightsquigarrow x) \odot (a \rightsquigarrow y) =$
$(a \to x) \odot a \odot (a \rightsquigarrow y) = (a \to x) \odot (a \to y) \odot a =$
$(a \to x) \odot (a \wedge y) = (a \to x) \odot y$. □

Proposition 5.7. *If A is a residuated lattice, then, for all $a, b \in B(A)$ and $x, y \in A$, we have:*
(b_{10}) $a \vee (x \odot y) = (a \vee x) \odot (a \vee y)$;
(b_{11}) $a \wedge (x \odot y) = (a \wedge x) \odot (a \wedge y)$;
(b_{12}) $(a \boxdot_1 x) \boxdot_2 a = a$ *for all* $\boxdot_1, \boxdot_2 \in \{\to, \rightsquigarrow\}$.

Proof. (b_{10}) By (c_7) we get:
$(a \vee x) \odot (a \vee y) = [(a \vee x) \odot a] \vee [(a \vee x) \odot y] =$
$[(a \vee x) \wedge a] \vee [(a \odot y) \vee (x \odot y)] = a \vee (a \odot y) \vee (x \odot y) = a \vee (x \odot y)$.
(since $a \odot y \leq a$, we have $a \vee (a \odot y) = a$).
(b_{11}) By the properties of a residuated lattice we have:
$(a \wedge x) \odot (a \wedge y) = (a \odot x) \odot (a \odot y) = a \odot (x \odot a) \odot y =$
$a \odot (x \wedge a) \odot y = a \odot (a \wedge x) \odot y = a \odot (a \odot x) \odot y =$
$a^2 \odot x \odot y = a \odot (x \odot y) = a \wedge (x \odot y)$.
(b_{12}) Applying (c_5) and (c_6) we have:
$a \to 0 \leq a \to x$ and $(a \to x) \to a \leq (a \to 0) \to a = a^- \to a = a$.
Since by (c_3) $a \leq (a \to x) \to a$, we get $(a \to x) \to a = a$.
Thus, we proved (b_{12}) for $\boxdot_1 = \boxdot_2 = \to$.
The other cases can be proved similarly. □

In [9], a bounded Rℓ-monoid structure was defined on the subintervals $[a, b]$ of the interval $[0, 1]$ for all $a, b \in [0, 1], (a \leq b)$. Recall that a *bounded Rℓ-monoid* is a residuated lattice $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ which satisfies the *pseudo-divisibility* condition: $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ for all $x, y \in A$.
For the case of a residuated lattice we can endow the subinterval $[a, 1]$ with the structure of a residuated lattice for all $a \in A$. We will prove that, if $a, b \in B(A)$, then the subintervals of the forms $[0, a]$ and $[a, b]$ can also be endowed with a residuated lattice structure.

Theorem 5.8. [5] *Let $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ be a residuated latice and $a \in A$. Then, the algebra $A_a^1 = ([a, 1], \odot_a^1, \wedge, \vee, \to_a^1, \rightsquigarrow_a^1, a, 1)$ is a residuated lattice, where $x \odot_a^1 y = (x \odot y) \vee a$, $x \to_a^1 y = x \to y$ and $x \rightsquigarrow_a^1 y = x \rightsquigarrow y$.*

Theorem 5.9. *Let $(A, \odot, \wedge, \vee, \to, \rightsquigarrow, 0, 1)$ be a residuated lattice, $a \in B(A)$ and $A_0^a = ([0,a], \wedge, \vee, \odot_0^a, \to_0^a, \rightsquigarrow_0^a, 0, a)$, where $x \odot_0^a y = x \odot (a \rightsquigarrow y)$, $x \to_0^a y = (x \to y) \odot a$ and $x \rightsquigarrow_0^a y = a \odot (x \rightsquigarrow y)$. Then, A_0^a is a residuated lattice.*

Proof. We prove that the axioms of a residuated lattice hold.
(A_1) It is clear that $([0,a], \wedge, \vee, 0, a)$ is a bounded lattice with the smallest element 0 and the greatest element a.
(A_2) Since for all $x \in [0,a]$ we have

$$x \odot_0^a a = x \odot (a \rightsquigarrow a) = x \odot 1 = x \text{ and}$$
$$a \odot_0^a x = a \odot (a \rightsquigarrow x) = a \wedge x = x,$$

it follows that a is a unit with respect to $\odot_0^a$.
The associativity follows from (b_9) and (b_2):

$$x \odot_0^a (y \odot_0^a z) = (a \to x) \odot (y \odot_0^a z) = (a \to x) \odot y \odot (a \rightsquigarrow z) =$$
$$(x \odot_0^a y) \odot (a \rightsquigarrow z) = (x \odot_0^a y) \odot_0^a z.$$

Thus, $([0,a], \odot_0^a, a)$ is a monoid with a unit a.
(A_3) Consider $x, y, z \in [0,a]$ such that $x \odot_0^a y \le z$, that is

$$(a \to x) \odot y \le z \text{ and } x \odot (a \rightsquigarrow y) \le z.$$

It follows that $a \to x \le y \to z$ and $a \rightsquigarrow y \le x \rightsquigarrow z$. Hence

$$(a \to x) \odot a \le (y \to z) \odot a = y \to_0^a z \text{ and}$$
$$a \odot (a \rightsquigarrow y) \le a \odot (x \rightsquigarrow z) = x \rightsquigarrow_0^a z.$$

Thus, $x = a \wedge x \le y \to_0^a z$ and $y = a \wedge y \le x \rightsquigarrow_0^a z$.
Conversely, suppose $x \le y \to_0^a z = (y \to z) \odot a$. It follows that

$$x \odot_0^a y = x \odot (a \rightsquigarrow y) \le (y \to z) \odot a \odot (a \rightsquigarrow y) =$$
$$(y \to z) \odot (a \wedge y) = (y \to z) \odot y \le y \wedge z \le z.$$

Similarly, since $y \le x \rightsquigarrow_0^a z$, hence $x \odot_0^a y \le z$.
Thus, A_0^a is a residuated lattice. □

Theorem 5.10. *Let $(A, \odot, \wedge, \vee, \to, \rightsquigarrow, 0, 1)$ be a residuated lattice, $a, b \in B(A), a \le b$ and $A_a^b = ([a,b], \wedge, \vee, \odot_a^b, \to_a^b, \rightsquigarrow_a^b, a, b)$, where $x \odot_a^b y = (x \odot (b \rightsquigarrow y)) \vee a$, $x \to_a^b y = (x \to y) \odot b$ and $x \rightsquigarrow_a^b y = b \odot (x \rightsquigarrow y)$. Then, A_a^b is a residuated lattice.*

Proof. By Theorem 5.8, the algebra $([a,1], \odot_a^1, \wedge, \vee, \to_a^1, \rightsquigarrow_a^1, a, 1)$ with the operations $x \odot_a^1 y = (x \odot y) \vee a$, $x \to_a^1 y = x \to y$ and $x \rightsquigarrow_a^1 y = x \rightsquigarrow y$ is a residuated lattice. Let $x, y \in [a,b]$. Since $x \le b$, by (c_6) we have $b \to y \le x \to y$. Hence

$$(x \to y) \odot b \ge (b \to y) \odot b = b \wedge y \ge a.$$

Similarly, $b \rightsquigarrow y \le x \rightsquigarrow y$, so:

$$b \odot (x \rightsquigarrow y) \ge b \odot (b \rightsquigarrow y) = b \wedge y \ge a.$$

By Theorem 5.9, the algebra $([a,b], \wedge, \vee, \odot_a^b, \to_a^b, \rightsquigarrow_a^b, a, b)$ is a residuated lattice with the operations:
$x \odot_a^b y = x \odot_a^1 (b \rightsquigarrow_a^1 y) = x \odot_a^1 (b \rightsquigarrow y) = (x \odot (b \rightsquigarrow y)) \vee a,$
$x \to_a^b y = (x \to_a^1 y) \odot_a^1 b = (x \to y) \odot_a^1 b = ((x \to y) \odot b) \vee a = (x \to y) \odot b,$
$x \rightsquigarrow_a^b y = b \odot_a^1 (x \rightsquigarrow_a^1 y) = b \odot_a^1 (x \rightsquigarrow y) = (b \odot (x \rightsquigarrow y)) \vee a = b \odot (x \rightsquigarrow y).$ □

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