TRAPEZOIDAL INTUITIONISTIC FUZZY PRIORITIZED AGGREGATION OPERATORS AND APPLICATION TO MULTI-ATTRIBUTE DECISION MAKING

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Abstract. In some multi-attribute decision making (MADM) problems, various relationships among the decision attributes should be considered. This paper investigates the prioritization relationship of attributes in MADM with trapezoidal intuitionistic fuzzy numbers (TrIFNs). TrIFNs are a special intuitionistic fuzzy set on a real number set and have better capability to model ill-known quantities. Firstly, the weighted possibility means of membership and non-membership functions for TrIFNs are defined. Hereby, a new lexicographic ranking method for TrIFNs is presented. Then, a series of trapezoidal intuitionistic fuzzy prioritized aggregation operators are developed, including trapezoidal intuitionistic fuzzy prioritized score (TrIFPS) operator, trapezoidal intuitionistic fuzzy prioritized weighted average (TrIFPWA) operator, trapezoidal intuitionistic fuzzy prioritized and (TrIFP-AND) operator and trapezoidal intuitionistic fuzzy prioritized or (TrIFP-OR) operator. Some desirable properties of these operators are also discussed. By utilizing the TrIFPWA operator, the attribute values of alternatives are integrated into the overall ones, which are used to rank the alternatives. Thus, a new method is proposed for solving the prioritized MADM problems with TrIFNs. Finally, the applicability of the proposed method is illustrated with a supply chain collaboration example.

1. Introduction

In real decision problems, the decision maker (DM) usually gives the information of assessment with some hesitancy degrees. Intuitionistic fuzzy set (IFS) introduced by Atanassov [1] is a suitable tool to represent this kind of information with hesitancy degrees. The notable characteristic of IFS is that IFS adds the degree of non-membership on the basis of the fuzzy set (FS) [43], i.e., the degrees of elements of the universal set belonging to the IFS are expressed with two characteristic functions: the membership degree and the non-membership degree. Therefore, IFS is a generalization of the FS and has stronger ability for representing imprecise and uncertain information than the FS. The IFSs have received considerable attention since its appearance. Atanassov and Gargov [2] further generalized the IFS and defined the interval-valued intuitionistic fuzzy set (IVIFS) using intervals to express the membership and non-membership degrees. There are plenty of researches on

Received: April 2014; Revised: July 2014; Accepted: May 2015

Key words and phrases: Multi-attribute decision making, Trapezoidal intuitionistic fuzzy number, Prioritized aggregation operators, Weighted possibility mean.

IFSs and IVIFSs in the applications to multi-attribute decision making (MADM) and multi-attribute group decision making (MAGDM). These researches focus on several aspects, such as critical infrastructure evaluation [18], information aggregation operators [19, 31], extension of classical decision making methods [16, 17], and preference relation [40].

Meanwhile, the research on the intuitionistic fuzzy numbers (IFNs) also has received considerable interest [5]-[33]. Analogous to the fuzzy number, IFN is a special IFS defined on the set of real numbers. As a generalization of fuzzy numbers, IFN seems to suitably describe an ill-known quantity [5]. Currently, there are three kinds of typical IFNs: triangular IFN (TIFN)[5]-[15], trapezoidal IFN (TrIFN) [25, 20, 32, 44, 21, 23] and interval-valued trapezoidal IFN (IVTrIFN) [23, 24, 33]. In a similar way to the fuzzy number, Shu et al. [11] introduced the concept of TIFN and applied to intuitionistic fuzzy fault tree analysis. Li [5] corrected some errors in the four arithmetic operations over the TIFNs in [11]. Li [6] discussed the concept of TIFN in depth and utilized ratio of value index to the ambiguity index to rank TIFNs. Nan et al. [10] defined ranking order relations of TIFNs, which are applied to matrix games with payoffs of TIFNs. Li et al. [7] defined values and ambiguities of membership degree and non-membership degree for TIFNs as well as the value-index and ambiguity-index. Hereby, a value and ambiguity based method is developed to rank TIFNs and applied to solve MADM problems with TIFNs. Wan et al.[12] introduced the possibility mean, variance and covariance of TIFNs. Wan and Li[13] developed the possibility mean and variance based method for MADM problems with TIFNs. Wan et al. [14] proposed the extended VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method for MAGDM with TIFNs. Wang et al. [27] proposed new arithmetic operations and logic operators for TIFNs, which can be employed in fault analysis of a printed circuit board assembly system. Wan and Dong [15] presented the possibility method for MAGDM with TIFNs and incomplete weight information.

Wang [25] defined TrIFN and IVTrIFN in the same way of TIFN. Wan and Dong [20] gave the expectation and expectant score of TrIFNs, and developed the ordered weighted aggregation operator and hybrid aggregation operator for TrIFNs, which can be used in MAGDM. Wei [29] proposed the ordered weighted averaging operator and hybrid aggregation operator for TrIFNs and applied to MAGDM with TrIFNs. Zhang et al.[44] proposed a grey relational projection method for MAGDM based on TrIFNs. Wang and Zhang [26] gave the TrIFN weighted arithmetic averaging and weighted geometric averaging operators, and discussed their applications to MADM problems. Wu and Cao [32] developed some families of geometric aggregation operators with TrIFNs and applied to MAGDM problems. Wan [21] investigated power average operators of TrIFNs and applied to MAGDM. In [22], Wan and Dong developed some power geometric operators of TrIFNs, which can be employed in MAGDM with TrIFNs. Wan [23] firstly defined some operational laws of IVTrIFNs and developed the weighted arithmetical average operator and weighted geometrical average operator for IVTrIFNs. An approach to ranking IVTrIFNs is presented based on the score and accurate functions. The MAGDM method using IVTrIFNs is then proposed. Wan [24] defined the Hamming and Euclidean distances for IVTrIFNs and developed a fractional programming method for the MADM problems using IVTrIFNs. Wu and Liu [33] investigated the attitudinal score and accuracy expected functions for IVTrIFNs, defined some geometric operators for IVTrIFNs, and thereby proposed a method for MAGDM with IVTrIFNs.

The above researches about IFNs mainly focus on the operation laws [5, 11, 6, 10, 44, 21], aggregation operators [27, 20, 32, 21, 33], ranking methods [6, 14, 20, 24, 33] and decision making methods [15, 44, 23, 24, 33]. The existing aggregation operators of TrIFNs did not consider the priority relationships between the TrIFNs being fused. However, the prioritized MADM problems are of great importance for scientific researches and real applications [35, 36, 37, 38, 42, 30]. For example, gasoline cost and safety are the most important two factors affecting airline. For an airline pilot, a benefit from cost of fuel should not be allowed to compensate a loss from safety. Namely, safety has a higher priority than cost [35]. Therefore, Yager [35] studied the prioritized MADM problems and provided some models using the ordered weighted averaging (OWA) operator method. Yager in [36], introduced a prioritized scoring operator, a closely related prioritized average operator, prioritized anding and oring operators for real numbers. Yager [37] proposed the prioritized OWA operator. Yan et al. [38] proposed prioritized weighted aggregation operator based on OWA operator and triangular norms. Wang et al. [28] investigated the prioritized aggregation for non-homogeneous group decision making in water resource management. These research results about prioritized aggregation operators [35, 36, 37, 38, 28] are only appropriate for the case in which the attribute values and weights are real numbers. To extend the prioritized aggregation operators, Yu and Xu [39] developed a prioritized intuitionistic fuzzy aggregation operator. Yu [41] proposed intuitionistic fuzzy prioritized weighted average (IFPWA) operator and intuitionistic fuzzy prioritized weighted geometric (IFPWG) operator. Xu et al. [34] defined intuitionistic fuzzy prioritized OWA operator and applied to MADM. Li and He [8] developed the intuitionistic fuzzy PRI-AND and PRI-OR aggregation operators. Yu et al. [42] proposed prioritized weighted average operator and prioritized weighted geometric operator for IVIFS. Wei [30] investigated the prioritized aggregation operators for hesitant fuzzy information and applied to hesitant fuzzy MADM problems in which the attributes are in different priority level.

It should be pointed out that TrIFNs use the trapezoidal fuzzy numbers to describe membership and non-membership functions, which can facilitate the decision makers (DMs) to express the uncertain and vague information in different dimensions [26]. Thus, TrIFNs have more flexibility to model ill-known quantities than IFSs. As far as we know, there is no investigation about the prioritized MADM problems in which the attribute values are in the form of TrIFNs up to date. To overcome this drawback, the purpose of this paper is to develop some trapezoidal intiutionistic prioritized aggregation operators and apply to MADM with TrIFNs. These operators sufficiently consider their priority relationships of TrIFNs being aggregated and can be used to effectively solve the prioritized MADM problems under trapezoidal intiutionistic fuzzy environment.

The remaining of this paper is structured as follows. In Section 2, the definition, operational rules, and weighted possibility means of TrIFN are introduced. Hereby, a ranking method of TrIFNs is presented in Section 2. In Section 3, some trapezoidal intiutionistic prioritized aggregation operators are developed and their desirable properties are also discussed. A decision method for the prioritized MADM problems with TrIFNs is proposed in Section 4. A supply chain collaboration problem and comparison analysis are given in Section 5. Short conclusions are made in Section 6.

2. Preliminaries for Trapezoidal Intuitionistic Fuzzy Numbers

In this section, the weighted lower and upper possibility means of membership and non-membership functions for TrIFNs are introduced as well as the weighted possibility means. Thereby, a ranking method of TrIFNs is presented.

2.1. Definition and Operation Laws of TrIFNs.

Definition 2.1. [25, 26, 21] A TrIFN $\tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$ is a special IFS on a real number set R, whose membership and non-membership functions are defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a}\omega_{\tilde{a}}, \text{if } \ a \leq x < b \\ \omega_{\tilde{a}}, \text{if } \ b \leq x \leq c \\ \frac{d-x}{d-c}\omega_{\tilde{a}}, \text{if } \ c < x \leq d \\ 0, \text{if } \ x < a \ \ or \ \ x > d \end{cases}$$

and

$$\nu_{\tilde{a}}(x) = \begin{cases} \frac{b-x+(x-a)u_{\tilde{a}}}{b-a}, \text{if } \ a \leq x < b \\ u_{\tilde{a}}, \text{if } \ b \leq x \leq c \\ \frac{x-c+(d-x)u_{\tilde{a}}}{d-c}, \text{if } \ c < x \leq d \\ 1, \text{if } \ x < a \ \ or \ \ x > d \end{cases}$$

respectively, where a, b, c and d are all real numbers. The values $\omega_{\tilde{a}}$ and $u_{\tilde{a}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy the conditions: $0 \leq \omega_{\tilde{a}} \leq 1, 0 \leq u_{\tilde{a}} \leq 1$ and $\omega_{\tilde{a}} + u_{\tilde{a}} \leq 1$. The function $\Pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - v_{\tilde{a}}(x)$ denotes the hesitation of an element x to $\tilde{a}$, which is called an intuitionistic fuzzy index. The smaller $\Pi_{\tilde{a}}(x)$, the more certain of TrIFN $\omega_{\tilde{a}}$. When $b = c$, a TrIFN reduces to a TIFN.

A TrIFN $\tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$ can be explained as an ill-known quantity approximate the interval $[b, c]$, which is expressed using any value between a and d with different membership and non-membership degrees. That is to say, the most likely value is between the interval $[b, c]$ with membership degree $\omega_{\tilde{a}}$ and non-membership degree $u_{\tilde{a}}$; the pessimistic value is a with membership degree 0 and non-membership degree 1; the optimistic value is d with membership degree 0 and non-membership degree 1; other values are any $x \in (a, d)$ with the membership degree $\mu_{\tilde{a}}(x)$ and non-membership degree $\nu_{\tilde{a}}(x)$.

If $\omega_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, the TrIFN $\tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$ is reduced to a trapezoidal fuzzy number $\tilde{a} = [a, b, c, d]$. The parameters $\omega_{\tilde{a}}$ and $u_{\tilde{a}}$ are used to reflect the confidence and non-confidence levels of the TrIFN, respectively. A TrIFN can represent more uncertainty than trapezoidal fuzzy number [21]. If $a \geq 0$ and one of the four values a, b, c and d is not equal to 0, then the TrIFN $\tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$ is called a positive TrIFN, denoted by $\tilde{a} > 0$[21]. The TrIFNs discussed in this paper are all positive TrIFNs.

Definition 2.2. [21] Let $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})(i = 1, 2)$ be two TrIFNs and $\lambda \geq 0$. Then the operation laws for TrIFNs are defined as follows:

(1) $\tilde{a}_1 + \tilde{a}_2 = ((a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); \omega_{\tilde{a}_1} \wedge \omega_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$, where the symbols $\wedge$ and $\vee$ mean min and max operators, respectively;
(2) $\lambda\tilde{a}_1 = ((\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); \omega_{\tilde{a}_1}, u_{\tilde{a}_1})$;
(3) $\tilde{a}_1\tilde{a}_2 = ((a_1a_2, b_1b_2, c_1c_2, d_1d_2); \omega_{\tilde{a}_1} \wedge \omega_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$;
(4) $\tilde{a}_1^{\lambda} = ((a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}, d_1^{\lambda}); \omega_{\tilde{a}_1}, u_{\tilde{a}_1})$.

From Definition 2.2, the following properties are proven [21]:

(1) Commutativity: $\tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1$, $\tilde{a}_1\tilde{a}_2 = \tilde{a}_2\tilde{a}_1$;
(2) Distributivity: $\lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda\tilde{a}_1 + \lambda\tilde{a}_2$,$\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_1 = (\lambda_1 + \lambda_2)\tilde{a}_1$, where $\lambda \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0$;
(3) Associativity: $\tilde{a}_1^{\lambda_1}\tilde{a}_1^{\lambda_2} = \tilde{a}_1^{\lambda_1+\lambda_2}$, $(\tilde{a}_1^{\lambda})^k = \tilde{a}_1^{\lambda k}$, where $\tilde{a}_1 > 0$ and $\lambda \geq 0, k \geq 0$.

2.2. **Weighted Possibility Means of TrIFNs.**

Analogous to cut sets of TIFNs in [6], the definitions of cut sets for TrIFNs are given as follows.

Definition 2.3. For a TrIFN $\tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$, the (α, β)-cut set, α-cut set and β-cut set are defined as follows:

$$\tilde{a}_{\alpha,\beta} = \{x|\mu_{\tilde{a}}(x) \geq \alpha, \nu_{\tilde{a}}(x) \leq \beta\},$$
$$\tilde{a}_{\alpha} = \{x|\mu_{\tilde{a}}(x) \geq \alpha\},$$

and

$$\tilde{a}_{\beta} = \{x|\nu_{\tilde{a}}(x) \leq \beta\},$$

respectively, where $0 \leq \alpha \leq \omega_{\tilde{a}}$,$u_{\tilde{a}} \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1$.

It directly follows from Figure 1, Definitions 2.1 and 2.3 that

$$\tilde{a}_{\alpha} = [a_{\alpha}^l, a_{\alpha}^u] = [a + \frac{(b-a)\alpha}{\omega_{\tilde{a}}}, d - \frac{(d-c)\alpha}{\omega_{\tilde{a}}}], \tag{1}$$

$$\tilde{a}_{\beta} = [a_{\beta}^l, a_{\beta}^u] = [\frac{b - au_{\tilde{a}} - \beta(b-a)}{1-u_{\tilde{a}}}, \frac{c - du_{\tilde{a}} + \beta(d-c)}{1-u_{\tilde{a}}}]. \tag{2}$$

Motivated by [4], we give the definitions of weighted possibility means of TrIFNs as follows.

Definition 2.4. A weighting function $f : [0, \omega_{\tilde{a}}] \to R$ is non-negative, monotone increasing satisfying the conditions: $\int_0^{\omega_{\tilde{a}}} f(\alpha)\mathrm{d}\alpha = \omega_{\tilde{a}}$ and $f(0) = 0$. The f weighted lower and upper possibility means of membership function for a TrIFN $\tilde{a}$ are defined as

$$\underline{m}_{\mu}(\tilde{a}) = \int_0^{\omega_{\tilde{a}}} f(\mathrm{Pos}[\tilde{a} \leq a_{\alpha}^l])a_{\alpha}^l\mathrm{d}\alpha \tag{3}$$

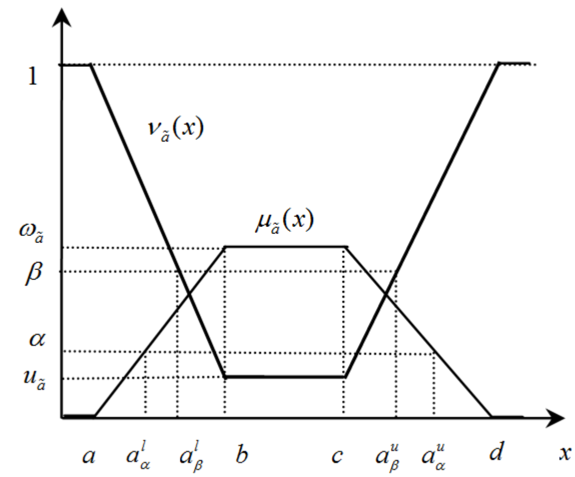

FIGURE 1. The α-cut Set of Membership Function and β-cut Set of Non-membership Function for a TrIFN $\tilde{a}$

and

$$\overline{m}_\mu(\tilde{a}) = \int_0^{\omega_{\tilde{a}}} f(\mathrm{Pos}[\tilde{a} \geq a^u_\alpha]) a^u_\alpha \mathrm{d}\alpha, \tag{4}$$

respectively, where Pos means possibility [4] and

$$\mathrm{Pos}[\tilde{a} \leq a^l_\alpha] = \sup_{x \leq a^l_\alpha} \mu_{\tilde{a}}(x) = \alpha, \tag{5}$$

$$\mathrm{Pos}[\tilde{a} \geq a^u_\alpha] = \sup_{x \geq a^u_\alpha} \mu_{\tilde{a}}(x) = \alpha. \tag{6}$$

It yields from Equations (5) and (6) that

$$\underline{m}_\mu(\tilde{a}) = \int_0^{\omega_{\tilde{a}}} f(\mathrm{Pos}[\tilde{a} \leq a^l_\alpha]) a^l_\alpha \mathrm{d}\alpha = \int_0^{\omega_{\tilde{a}}} f(\alpha) a^l_\alpha \mathrm{d}\alpha,$$

$$\overline{m}_\mu(\tilde{a}) = \int_0^{\omega_{\tilde{a}}} f(\mathrm{Pos}[\tilde{a} \geq a^u_\alpha]) a^u_\alpha \mathrm{d}\alpha = \int_0^{\omega_{\tilde{a}}} f(\alpha) a^u_\alpha \mathrm{d}\alpha.$$

Apparently, $\underline{m}_\mu(\tilde{a})$ is the f weighted lower possibility weighted average of the minimum of α-cut set and it is why we call it the f weighted lower possibility mean of membership function. $\overline{m}_\mu(\tilde{a})$ is the f weighted upper possibility weighted average of the maximum of α-cut set and it is why we call it the f weighted upper possibility mean of membership function.

Definition 2.5. A weighting function $g : [u_{\tilde{a}}, 1] \to R$ is non-negative, monotone decreasing satisfying the conditions: $\int_{u_{\tilde{a}}}^1 g(\beta)\mathrm{d}\beta = 1 - u_{\tilde{a}}$ and $g(1) = 1$. The g weighted lower and upper possibility means of non-membership function for a TrIFN $\tilde{a}$ are defined as

$$\underline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g(\mathrm{Pos}[\tilde{a} \leq a^l_\beta]) a^l_\beta \mathrm{d}\beta, \tag{7}$$

$$\overline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^{1} g(\text{Pos}[\tilde{a} \geq a_\beta^u]) a_\beta^u \mathrm{d}\beta, \tag{8}$$

respectively, where

$$\text{Pos}[\tilde{a} \leq a_\beta^l] = \sup_{x \leq a_\beta^l} \nu_{\tilde{a}}(x) = \beta, \tag{9}$$

$$\text{Pos}[\tilde{a} \geq a_\beta^u] = \sup_{x \geq a_\beta^u} \nu_{\tilde{a}}(x) = \beta. \tag{10}$$

It derives from Equations (9) and (10) that

$$\underline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^{1} g(\text{Pos}[\tilde{a} \leq a_\beta^l]) \tilde{a}_\beta^l \mathrm{d}\beta = \int_{u_{\tilde{a}}}^{1} g(\beta) a_\beta^l \mathrm{d}\beta,$$

$$\overline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^{1} g(\text{Pos}[\tilde{a} \geq a_\beta^u]) a_\beta^u \mathrm{d}\beta = \int_{u_{\tilde{a}}}^{1} g(\beta) a_\beta^u \mathrm{d}\beta.$$

We call $\underline{m}_\nu(\tilde{a})$ the g weighted lower possibility mean of non-membership function since $\underline{m}_\nu(\tilde{a})$ is just the g weighted lower possibility weighted average of the minimum of β-cut set. Likewise, we call $\overline{m}_\nu(\tilde{a})$ the g weighted upper possibility mean of non-membership function.

Definition 2.6. For a TrIFN $\tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$, the f weighted possibility mean of membership function is defined as follows:

$$m_\mu(\tilde{a}, \theta) = (1-\theta)\underline{m}_\mu(\tilde{a}) + \theta \overline{m}_\mu(\tilde{a}) \tag{11}$$

and the g weighted possibility mean of non-membership function is defined as follows:

$$m_\nu(\tilde{a}, \theta) = (1-\theta)\underline{m}_\nu(\tilde{a}) + \theta \overline{m}_\nu(\tilde{a}), \tag{12}$$

where $\theta \in [0, 1]$ is the preference parameter of DM and can reflect different importance to the weighted lower and upper possibility means. Different DMs have different preferences for the lower and upper possibility means. $\theta \in (0.5, 1]$ implies that DM prefers the weighted upper possibility mean, namely DM is pessimistic; $\theta \in [0, 0.5)$ shows that DM prefers the weighted lower possibility mean, namely DM is optimistic; $\theta = 0.5$ indicates that DM is indifferent to the weighted lower and upper possibility means, namely, the preference of DM is neutral.

If $\theta = 0$, then $m_\mu(\tilde{a}, 0) = \underline{m}_\mu(\tilde{a})$ and $m_\nu(\tilde{a}, 0) = \underline{m}_\nu(\tilde{a})$; If $\theta = 1$, then $m_\mu(\tilde{a}, 1) = \overline{m}_\mu(\tilde{a})$ and $m_\nu(\tilde{a}, 1) = \overline{m}_\nu(\tilde{a})$; If $\theta = 0.5$, then $m_\mu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\mu(\tilde{a}) + \overline{m}_\mu(\tilde{a})]$ and $m_\nu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\nu(\tilde{a}) + \overline{m}_\nu(\tilde{a})]$. Thus, if a TrIFN $\tilde{a} = ([a, b, c, d]; \omega_{\tilde{a}}, u_{\tilde{a}})$ degenerates to a trapezoidal fuzzy number $\tilde{a} = [a, b, c, d]$, i.e., $\omega_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, $m_\mu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\mu(\tilde{a}) + \overline{m}_\mu(\tilde{a})]$ (or $m_\nu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\nu(\tilde{a}) + \overline{m}_\nu(\tilde{a})]$) is just the f weighted possibility mean of fuzzy number defined in Definition 2.2 of [4] (see pp. 365).

Obviously, $m_\mu(\tilde{a}, \theta)$ synthetically reflects the information on every membership degree, and $m_\mu(\tilde{a}, 0.5)$ may be regarded as a central value that represents from the membership function point of view. Likewise, $m_\nu(\tilde{a}, \theta)$ synthetically reflects the information on every non-membership degree, and $m_\nu(\tilde{a}, 0.5)$ may be regarded as a central value that represents from the non-membership function point of view.

Example 2.7. If f and g are chosen as follows:

$$f(\alpha) = 2\alpha/\omega_{\tilde{a}}(\alpha \in [0, \omega_{\tilde{a}}]) \tag{13}$$

and

$$g(\beta) = 2(1-\beta)/(1-u_{\tilde{a}})(\beta \in [u_{\tilde{a}}, 1]), \tag{14}$$

respectively, then, according to Equations (3), (4), (7) and (8), we have

$$\underline{m}_\mu(\tilde{a}) = \tfrac{1}{3}(a+2b)\omega_{\tilde{a}}, \tag{15}$$

$$\overline{m}_\mu(\tilde{a}) = \tfrac{1}{3}(d+2c)\omega_{\tilde{a}}, \tag{16}$$

$$\underline{m}_v(\tilde{a}) = \tfrac{1}{3}(a+2b)(1-u_{\tilde{a}}), \tag{17}$$

$$\overline{m}_v(\tilde{a}) = \tfrac{1}{3}(d+2c)(1-u_{\tilde{a}}). \tag{18}$$

Further, from Equations (11) and (12), it yields that

$$m_\mu(\tilde{a},\theta) = \tfrac{1}{3}[(1-\theta)(a+2b)+\theta(2c+d)]\omega_{\tilde{a}}, \tag{19}$$

$$m_v(\tilde{a},\theta) = \tfrac{1}{3}[(1-\theta)(a+2b)+\theta(2c+d)](1-u_{\tilde{a}}). \tag{20}$$

Remark 2.8. If a TrIFN $\tilde{a} = ((a,b,c,d);\omega_{\tilde{a}},u_{\tilde{a}})$ degenerates to a triangular fuzzy number $\tilde{a} = (\underline{a}, a, \overline{a})$, i.e., $\omega_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, it follows from Equations (15), (16) (or (17), (18)) and (19) (or (20) with $\theta = 0.5$) that the weighted lower possibility mean, weighted upper possibility mean and weighted possibility mean of a triangular fuzzy number $\tilde{a} = (\underline{a}, a, \overline{a})$ are obtained as follows:$M_*(\tilde{a}) = (\underline{a}+2a)/3$, $M^*(\tilde{a}) = (2a+\overline{a})/3$, and $\overline{M}(\tilde{a}) = (\underline{a}+4a+\overline{a})/6$, respectively. These results of a triangular fuzzy number are the same as those of a triangular fuzzy number in Examples 2.1 of [3].

The weighted possibility means have linear properties listed in Theorem 2.9.

Theorem 2.9. *Let $\tilde{a}_i = ((a_i,b_i,c_i,d_i);\omega_{\tilde{a}_i},u_{\tilde{a}_i})\quad (i=1,2)$ be two TrIFNs with $\omega_{\tilde{a}_1} = \omega_{\tilde{a}_2}$ and $u_{\tilde{a}_1} = u_{\tilde{a}_2}$. Then for any $\gamma > 0$ and $\tau > 0$, the following equalities are valid:*

$$m_\mu(\gamma\tilde{a}_1+\tau\tilde{a}_2,\theta) = \gamma m_\mu(\tilde{a}_1,\theta)+\tau m_\mu(\tilde{a}_2,\theta), \tag{21}$$

$$m_\nu(\gamma\tilde{a}_1+\tau\tilde{a}_2,\theta) = \gamma m_\nu(\tilde{a}_1,\theta)+\tau m_\nu(\tilde{a}_2,\theta). \tag{22}$$

Proof. Since $\gamma > 0$ and $\tau > 0$, by Definition 2.3, we get the α-cut set of TrIFN $\gamma\tilde{a}+\tau\tilde{b}$ as follows:

$$(\gamma\tilde{a}_1+\tau\tilde{a}_2)_\alpha = [\gamma a^l_{1\alpha}+\tau a^l_{2\alpha}, \gamma a^u_{1\alpha}+\tau a^u_{2\alpha}].$$

By Equation (11), $\omega_{\tilde{a}_1} = \omega_{\tilde{a}_2}$ and $u_{\tilde{a}_1} = u_{\tilde{a}_2}$, we obtain

$m_\mu(\gamma\tilde{a}_1+\tau\tilde{a}_2,\theta) = (1-\theta)\underline{m}_\mu(\gamma\tilde{a}_1+\tau\tilde{a}_2)+\theta\overline{m}_\mu(\gamma\tilde{a}_1+\tau\tilde{a}_2)$
$= (1-\theta)\int_0^{\omega_{\tilde{a}_1}\wedge\omega_{\tilde{a}_2}} f(\alpha)(\gamma a^l_{1\alpha}+\tau a^l_{2\alpha})\mathrm{d}\alpha+\theta\int_0^{\omega_{\tilde{a}_1}\wedge\omega_{\tilde{a}_2}} f(\alpha)(\gamma a^u_{1\alpha}+\tau a^u_{2\alpha})\mathrm{d}\alpha$
$= \gamma[(1-\theta)\underline{m}_\mu(\tilde{a}_1)+\theta\overline{m}_\mu(\tilde{a}_1)]+\tau[(1-\theta)\underline{m}_\mu(\tilde{a}_2)+\theta\overline{m}_\mu(\tilde{a}_2)]$
$= \gamma m_\mu(\tilde{a}_1,\theta)+\tau m_\mu(\tilde{a}_2,\theta)$.

Thus, Equation (21) holds. In the same way, Equation (22) can be proven. Namely, Theorem 2.9 is proven. $\square$

It is noted that if $\gamma = \tau = 1$, then by Theorem 2.9 the following are valid:

$$m_\mu(\tilde{a}_1 + \tilde{a}_2, \theta) = m_\mu(\tilde{a}_1, \theta) + m_\mu(\tilde{a}_2, \theta)$$

and

$$m_\nu(\tilde{a}_1 + \tilde{a}_2, \theta) = m_\nu(\tilde{a}_1, \theta) + m_\nu(\tilde{a}_2, \theta).$$

2.3. Ranking Method of TrIFNs Based on Weighted Possibility Means.

The possibility means of fuzzy numbers are similar to the mean of random variables. They can be used to quantitatively characterize the values of fuzzy numbers. Obviously, the greater the possibility mean, the bigger the corresponding fuzzy number.

Let $m_\mu(\tilde{a}_i, \theta)$ and $m_\nu(\tilde{a}_i, \theta)$ be the weighted possibility means of the membership and non-membership functions for TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2)$, respectively. Thereby, a lexicographic ranking method between two TrIFNs $\tilde{a}_1$ and $\tilde{a}_2$ can be summarized as follows:

(1) If $m_\mu(\tilde{a}_1, \theta) < m_\mu(\tilde{a}_2, \theta)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
(2) If $m_\mu(\tilde{a}_1, \theta) > m_\mu(\tilde{a}_2, \theta)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;
(3) If $m_\mu(\tilde{a}_1, \theta) = m_\mu(\tilde{a}_2, \theta)$, then
  (a) If $m_\nu(\tilde{a}_1, \theta) < m_\nu(\tilde{a}_2, \theta)$, then $\tilde{a}_1 < \tilde{a}_2$;
  (b) If $m_\nu(\tilde{a}_1, \theta) > m_\nu(\tilde{a}_2, \theta)$, then $\tilde{a}_1 > \tilde{a}_2$;
  (c) If $m_\nu(\tilde{a}_1, \theta) = m_\nu(\tilde{a}_2, \theta)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$.

Remark 2.10. The weighting functions f and g can be chosen as several forms, for example,

$$f(\alpha) = (n+1)\alpha^n / (\omega_{\tilde{a}})^n (\alpha \in [0, \omega_{\tilde{a}}]),$$

$$g(\beta) = (n+1)(1-\beta)^n / (1-u_{\tilde{a}})^n (\beta \in [u_{\tilde{a}}, 1]),$$

where the power n is any positive integer, such as $n = 1$, $n = 2$, etc.. These power forms of weighting functions are motivated by Carlsson and Fullr [4] (see Examples 1-3 in [4]). Hence, by introducing different weighting functions f and g, we can give different (case-dependent) importance to α-cut set and β-cut set of a TrIFN $\tilde{a}$. For computation convenience, the weighting functions f and g are respectively chosen as Equations (13) and (14) in the following.

3. Trapezoidal Intuitionistic Fuzzy Prioritized Operators

In this section, some trapezoidal intuitionistic fuzzy prioritized operators are developed and their desirable properties are discussed in detail.

In the sequel, suppose that all TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$ are normalized, i.e., $0 \le a_i \le b_i \le c_i \le d_i \le 1 \quad (i = 1, 2, \cdots, n)$, then the maximum TrIFN is $\tilde{a}^{\max} = ([1, 1, 1, 1]; 1, 0)$ and the minimum TrIFN is $\tilde{a}^{\min} = ([0, 0, 0, 0]; 0, 1)$.

3.1. Trapezoidal Intuitionistic Fuzzy Prioritized Scoring Operator.

Yager [36] introduced prioritized scoring operator for real numbers as follows:

Definition 3.1. [36] Let $C = \bigcup\limits_{i=1}^{q} H_i$ be an attribute set, where $H_i = \{C_{i1}, C_{i2}, \cdots, C_{in_i}\}$. Suppose that there exists a linear ordering $H_1 \succ H_2 \succ \cdots \succ H_q$ between the attributes which shows that the priority of attribute C_{ij} is higher than that of C_{kl} if $i < k$, for any $j = 1, 2, \cdots n_i$ and $l = 1, 2 \cdots n_k$. Denote the satisfaction of any alternative x on attribute C_{ij} by $C_{ij}(x)$ which satisfies $C_{ij}(x) \in [0, 1]$. If

$$PS_w(C_{ij}(x)) = \sum_{i=1}^{q} (\sum_{j=1}^{n_i} w_{ij} C_{ij}(x)), \tag{23}$$

where $w_{ij} = T_i$, $T_i = \prod\limits_{k=1}^{i} S_{k-1}(x)$, $S_i = \min\limits_{j}\{C_{ij}(x)\}$ and $T_1 = S_0 = 1$, then PS is called a prioritized scoring operator.

Motivated by PS operator, we extend the prioritized scoring operator of real numbers to the case in which the arguments to be aggregated are TrIFNs. Suppose that the TrIFNs $\tilde{a}_{ij}$ are considered as the satisfactions of any alternative x on attributes C_{ij} $(i = 1, 2, \cdots, q;\ j = 1, 2, \cdots, n_i)$. Figure 2 shows the prioritization of attributes.

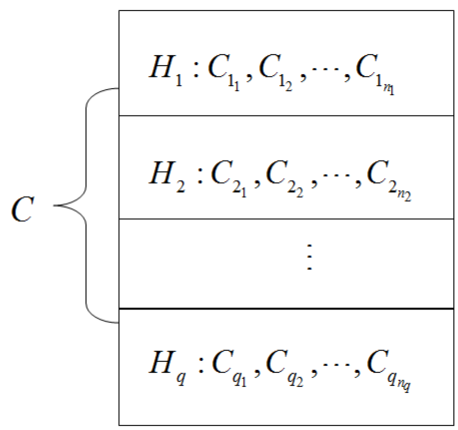

Figure 2. Prioritization of Attributes

Definition 3.2. For TrIFNs $\tilde{a}_{ij} = ((a_{ij}, b_{ij}, c_{ij}, d_{ij}); \omega_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$ $(i = 1, 2, \cdots, q; j = 1, 2, ..., n_i)$, if TrIFPS: $\Omega^n \to \Omega$, such that

$$\text{TrIFPS}_w(\tilde{a}_{ij}) = \sum_{i=1}^{q} (\sum_{j=1}^{n_i} w_{ij} \tilde{a}_{ij}), \tag{24}$$

where Ω is the set of all TrIFNs, $w = (w_{1j}, w_{2j}, \cdots, w_{qj})^{\mathrm{T}}$ is the priority based weight vector, then TrIFPS is called a trapezoidal intuitionistic fuzzy prioritized scoring operator.

The priority based weight vector $w = (w_{1j}, w_{2j}, \cdots, w_{qj})^{\mathrm{T}}$ can be determined involving the following steps:

Step 1: Calculate the weighted possibility means of TrIFNs $\tilde{a}_{ij}$, i.e., $m_\mu(\tilde{a}_{ij},\theta)$ $(i=1,2,\cdots,q;\ j=1,2,\cdots,n_i)$. Due to the fact that all TrIFNs $\tilde{a}_{ij}$ are already normalized, it yields by Equation (19) that $m_\mu(\tilde{a}_{ij},\theta)\in[0,1]$.

Step 2: For each priority category H_i, we calculate $S_i=\min\limits_j\{m_\mu(\tilde{a}_{ij},\theta)\}\quad(i=1,2,\cdots,q-1)$, $S_0=1$.

Step 3: The priority weight vector $w=(w_{1j},w_{2j},\cdots,w_{qj})^{\mathrm{T}}$ can be obtained as follows:

$$w_{ij}=T_i(j=1,2,\cdots,n_i),T_1=S_0=1,T_i=\prod_{k=1}^{i}S_{k-1}\quad(i=2,3,\cdots,q)\tag{25}$$

If letting $A_i=\sum\limits_{j=1}^{n_i}a_{ij}$, then according to Definition 3.2 and Equation (25), Equation (24) can be rewritten as follows:

$$\mathrm{TrIFPS}_w(\tilde{a}_{ij})=\sum_{i=1}^{q}T_i(\sum_{j=1}^{n_i}\tilde{a}_{ij})=\sum_{i=1}^{q}T_iA_i.$$

The above process of determining the priority based weight vector $w=(w_{1j},w_{2j},\cdots,w_{qj})^{\mathrm{T}}$ reflects a fundamental feature of prioritization relationship, i.e., the attributes in H_i contribute proportionally to the product of satisfaction of the higher order attributes since the weight associated with elements in the category H_i is $T_i=\prod\limits_{k=1}^{i}S_{k-1}$. The poor satisfaction to any higher attribute decreases the ability for compensation by lower priority attribute.

Theorem 3.3. *For TrIFNs $\tilde{a}_i=((a_i,b_i,c_i,d_i);\omega_{\tilde{a}_i},u_{\tilde{a}_i})$ ($i=1,2,\cdots,q$), their aggregated result by using the TrIFPS operator is still a TrIFN as follows:*

$$\mathrm{TrIFPS}_w(\tilde{a}_{ij})=((\sum_{i=1}^{q}\sum_{j=1}^{n_i}w_{ij}a_{ij},\sum_{i=1}^{q}\sum_{j=1}^{n_i}w_{ij}b_{ij},\sum_{i=1}^{q}\sum_{j=1}^{n_i}w_{ij}c_{ij},$$
$$\sum_{i=1}^{q}\sum_{j=1}^{n_i}w_{ij}d_{ij});\bigwedge_{i,j}\omega_{\tilde{a}_{ij}},\bigvee_{i,j}u_{\tilde{a}_{ij}})\tag{26}$$

Proof. The proof can be easily completed according to Definitions 2.2 and 3.2 by using mathematical induction. □

According to Theorem 3.3, the TrIFPS operator is monotonic, i.e. if $\tilde{a}_{ij}$ increases, then the aggregated value $\mathrm{TrIFPS}_w(\tilde{a}_{ij})$ cannot decrease. The property can be described as Theorem 3.4.

Theorem 3.4. *(Monotonicity). Let $\tilde{a}_{ij}=((a_{ij1},b_{ij1},c_{ij1},d_{ij1});\omega_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}})$ and $\tilde{b}_{ij}=((a_{ij2},b_{ij2},c_{ij2},d_{ij2});\omega_{\tilde{b}_{ij}},u_{\tilde{b}_{ij}})$ ($i=1,2,\cdots,q$; $j=1,2,\cdots,n_i$) be two series of TrIFNs. If $\tilde{a}_{ij}\le\tilde{b}_{ij}$ with $\omega_{\tilde{a}_{ij}}=\omega_{\tilde{b}_{ij}}$ and $u_{\tilde{a}_{ij}}=u_{\tilde{b}_{ij}}$ ($i=1,2,\cdots,q$; $j=1,2,\cdots,n_i$), the following inequality holds:*

$$\mathrm{TrIFPS}_w(\tilde{a}_{ij})\le\mathrm{TrIFPS}_{w'}(\tilde{b}_{ij}),$$

where the priority based weight vector $w' = (w'_{1j}, w'_{2j}, \cdots, w'_{qj})^{\mathrm{T}}$ *can be determined similarly by Equation (25).*

Proof. Since $\tilde{a}_{ij} \leq \tilde{b}_{ij}$, according to Equations (19) and (25), we have

$m_\mu(\tilde{a}_{ij}, \theta) = \frac{1}{3}[(1-\theta)(a_{ij1} + 2b_{ij1}) + \theta(2c_{ij1} + d_{ij1})]\omega_{\tilde{a}_{ij}} \leq m_\mu(\tilde{b}_{ij}, \theta) = \frac{1}{3}[(1-\theta)(a_{ij2} + 2b_{ij2}) + \theta(2c_{ij2} + d_{ij2})]\omega_{\tilde{b}_{ij}}$,

$w_{11} = w'_{11} = 1, w_{ij} = T_i = \prod_{k=1}^{i} \min_j \{m_\mu(\tilde{a}_{ij}, \theta)\} \leq \prod_{k=1}^{i} \min_j \{m_\mu(\tilde{b}_{ij}, \theta)\} = T'_i = w'_{ij}$ for $i = 2, 3, \cdots, q$.

Since $\omega_{\tilde{a}_{ij}} = \omega_{\tilde{b}_{ij}}$ and $u_{\tilde{a}_{ij}} = u_{\tilde{b}_{ij}}$ $(i = 1, 2, \cdots, q;\ j = 1, 2, \cdots, n_i\ )$, according to Theorem 2.9 and that $\theta \in [0, 1]$, we obtain

$$m_\mu(\sum_{i=1}^{q}\sum_{j=1}^{n_i} w_{ij}\tilde{a}_{ij}, \theta) = \sum_{i=1}^{q}\sum_{j=1}^{n_i} w_{ij} m_\mu(\tilde{a}_{ij}, \theta)$$
$$\leq \sum_{i=1}^{q}\sum_{j=1}^{n_i} w'_{ij} m_\mu(\tilde{b}_{ij}, \theta) = m_\mu(\sum_{i=1}^{q}\sum_{j=1}^{n_i} w'_{ij}\tilde{b}_{ij}, \theta).$$

It directly yields by Theorem 3.3 that

$$\mathrm{TrIFPS}_w(\tilde{a}_{ij}) \leq \mathrm{TrIFPS}_{w'}(\tilde{b}_{ij}).$$

Hence, the proof of Theorem 3.4 is completed. □

Corollary 3.5. *If there is some category such that* $\tilde{a}^{\min} = ([0, 0, 0, 0]; 0, 1)$ *for some criteria in* H_r*, then*

$$\mathrm{TrIFPS}_w(\tilde{a}_{ij}) = \sum_{i=1}^{r} T_i(\sum_{j=1}^{n_i} a_{ij}).$$

Proof. According to Definition 3.2 and Equation (25), we have $S_r = 0$,$T_i = 0$ for $i > r$. Thus, according to Theorem 3.3, Corollary 3.5 is proved. □

Example 3.6. Considering the following prioritized collection of attributes: $H_1 = \{C_{11}, C_{12}\}, H_2 = \{C_{21}, C_{22}, C_{23}\}, H_3 = \{C_{31}, C_{32}\}, H_4 = \{C_{41}\}$. Let TrIFNs $\tilde{a}_{ij}(x)$ be the satisfaction of any alternative x on attribute C_{ij}, where

$\tilde{a}_{11}(x)$=((0.1,0.2,0.3,0.5); 0.5,0.4), $\tilde{a}_{12}(x)$=((0.1,0.3,0.4,0.6); 0.7,0.2),
$\tilde{a}_{21}(x)$=((0.2,0.4,0.6,0.8); 0.3,0.5), $\tilde{a}_{22}(x)$= ((0.1,0.4,0.6,0.9); 0.6,0.4),
$\tilde{a}_{23}(x)$=((0.3,0.5,0.6,0.8); 0.2,0.5), $\tilde{a}_{31}(x)$= ((0.4,0.5,0.6,0.7); 0.3,0.6),
$\tilde{a}_{32}(x)$=((0.2,0.3,0.5,0.8); 0.8,0.1), $\tilde{a}_{41}(x)$= ((0.3,0.5,0.6,0.9); 0.6,0.3).

Assume that DM is indifferent to the lower and upper possibility means, namely, the preference of DM is neutral. and $\theta = 0.5$. Combining Definition 3.2 with Equation (25), we have

$$S_0 = 1, S_1 = 0.1333, S_2 = 0.1100, S_3 = 0.1650, S_4 = 0.3467.$$

Then, $T_1 = S_0 = 1, T_2 = S_1T_1 = 0.1333, T_3 = S_2T_2 = 0.0147, T_4 = S_3T_3 = 0.0024, w_{11} = w_{12} = T_1 = 1, w_{21} = w_{22} = w_{23} = T_2 = 0.1333, w_{31} = w_{32} = T_3 = 0.0147, w_{41} = T_4 = 0.0024$.

Thus, according to Theorem 3.3, the aggregated value is computed as:

$$\tilde{a}(x) = \text{TrIFPS}_w(\tilde{a}_{11}(x), \tilde{a}_{12}(x); \tilde{a}_{21}(x), \tilde{a}_{22}(x), \tilde{a}_{23}(x); \tilde{a}_{31}(x), \tilde{a}_{32}(x);\ \tilde{a}_{41}(x))$$
$$= ((0.2895,\ 0.6862, 0.9576, 1.4575); 0.2, 0.6).$$

Let $\theta = 0.5$, according to Equations (19) and (20), the weighted possibility means of membership and non-membership functions for the overall aggregated value $\tilde{a}(x)$ are respectively obtained as follows:

$$m_\mu(\tilde{a}(x), 0.5) = 0.1678, m_v(\tilde{a}(x), 0.5) = 0.3356.$$

3.2. **Trapezoidal Intuitionistic Fuzzy Prioritized Weighted Average Operator.**

Yager [36] introduced an aggregation operator called prioritized average operator for real numbers as follows:

Definition 3.7. [36] Let $C = \{C_1, C_2, \cdots, C_n\}$ be an attribute set with a linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$, i.e., the priority of attribute C_j is higher than that of C_k if $j < k$. Denote the satisfaction of any alternative x on attribute C_j by $C_j(x)$ which satisfies $C_j(x) \in [0, 1]$. If

$$PA_w(C_1(x), C_2(x), \cdots, C_n(x)) = \sum_{j=1}^{n} w_j C_j(x), \tag{27}$$

where $w_j = T_j / \sum_{i=1}^{n} T_i$, $T_j = \prod_{k=1}^{j} C_{k-1}(x)$, $T_1 = C_0(x) = 1$, then PA is called the prioritized average operator.

The PS operator allows ties for attributes in the same category, whereas the PA operator allows no ties. Motivated by the PA operator, we extend the PA operator of real numbers to the case in which the arguments to be aggregated are TrIFNs. Suppose that the TrIFN $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ is considered as the satisfaction of any alternative x on attribute C_i.

Definition 3.8. For TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$, if TrIFPWA: $\Omega^n \to \Omega$ such that

$$\text{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{a}_j, \tag{28}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the priority based weight vector, then TrIFPWA is called a trapezoidal intuitionistic fuzzy prioritized weighted average operator.

The priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ can be determined including the following steps:

Step 1. Calculate the weighted possibility means of membership function for TrIFNs $\tilde{a}_i$, i.e., $m_\mu(\tilde{a}_i, \theta) \quad (i = 1, 2, \cdots, n)$.

Step 2. Compute the un-normalized priority based weights: $T_1 = 1, T_j = \prod_{i=1}^{j} m_\mu(\tilde{a}_{i-1}, \theta) \ \ (j = 2, 3, \cdots, n)$.

Step 3. Normalize the $T_j \quad (j = 1, 2, \cdots, n)$ to obtain the normalized priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, where

$$w_j = T_j / \sum_{i=1}^{n} T_i \quad (j = 1, 2, \cdots, n). \tag{29}$$

Theorem 3.9. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$*, their aggregated result by using TrIFPWA operator is also a TrIFN, and*

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = ((\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j); \wedge_{j=1}^{n} \omega_{\tilde{a}_j}, \vee_{j=1}^{n} u_{\tilde{a}_j}) \tag{30}$$

Proof. It derives directly from Definitions 2.2 and 3.8 that the first result is valid. By induction on n, we can prove the second result according to Definition 2.2.

For $n = 2$, since

$$w_1 \tilde{a}_1 = ((w_1 a_1, w_1 b_1, w_1 c_1, w_1 d_1); \omega_{\tilde{a}_1}, u_{\tilde{a}_1}),$$

$$w_2 \tilde{a}_2 = ((w_2 a_2, w_2 b_2, w_2 c_2, w_2 d_2); \omega_{\tilde{a}_2}, u_{\tilde{a}_2}),$$

according to Equations (28) and (29) and Definition 2.2, we have

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2) = ((\sum_{j=1}^{2} w_j a_j, \sum_{j=1}^{2} w_j b_j, \sum_{j=1}^{2} w_j c_j, \sum_{j=1}^{2} w_j d_j); \omega_{\tilde{a}_1} \wedge \omega_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$$

Assume that Equation (30) holds for $n = k$, i.e.,

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = ((\sum_{j=1}^{k} w_j a_j, \sum_{j=1}^{k} w_j b_j, \sum_{j=1}^{k} w_j c_j, \sum_{j=1}^{k} w_j d_j); \wedge_{j=1}^{k} \omega_{\tilde{a}_j}, \vee_{j=1}^{k} u_{\tilde{a}_j}).$$

Then, as $n = k + 1$, it follows from Definition 2.2 that

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k, \tilde{a}_{k+1}) = \sum_{j=1}^{k} w_j \tilde{a}_j + w_{k+1} \tilde{a}_{k+1}$$

$$= ((\sum_{j=1}^{k} w_j a_j, \sum_{j=1}^{k} w_j b_j, \sum_{j=1}^{k} w_j c_j, \sum_{j=1}^{k} w_j d_j); \wedge_{j=1}^{k} \omega_{\tilde{a}_j}, \vee_{j=1}^{k} u_{\tilde{a}_j})$$

$$+((w_{k+1} a_{k+1}, w_{k+1} b_{k+1}, w_{k+1} c_{k+1}, w_{k+1} d_{k+1}); \omega_{\tilde{a}_{k+1}}, u_{\tilde{a}_{k+1}})$$

$$= ((\sum_{j=1}^{k+1} w_j a_j, \sum_{j=1}^{k+1} w_j b_j, \sum_{j=1}^{k+1} w_j c_j, \sum_{j=1}^{k+1} w_j d_j); \wedge_{j=1}^{k+1} \omega_{\tilde{a}_j}, \vee_{j=1}^{k+1} u_{\tilde{a}_j}),$$

i.e., Equation (24) validates for $n = k + 1$. Hence, Equation (30) validates for all n. The proof of Theorem 3.9 is finished. □

The TrIFPWA operator has some desirable properties, such as idempotency, boundedness, and so on.

Theorem 3.10. *(Idempotency). For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$*, if* $\tilde{a}_1 = \tilde{a}_2 = \cdots = \tilde{a}_n = \tilde{a}$*, then* $\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}$.

Proof. Since $w_j = T_j / \sum_{i=1}^{n} T_i$, we have $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} (T_j / \sum_{i=1}^{n} T_i) = 1$. Thus,

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{a}_j = \sum_{j=1}^{n} w_j \tilde{a} = \tilde{a} \sum_{j=1}^{n} w_j = \tilde{a}.$$

□

It directly follows from Theorem 3.10 that the following corollaries 2 and 3 hold.

Corollary 3.11. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$*, if all TrIFNs* $\tilde{a}_i = \tilde{a}^{\max} \quad (i = 1, 2, \cdots, n)$*, then* $\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}^{\max}$.

Corollary 3.12. *(Non-compensatory). For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$*, if TrIFN* $\tilde{a}_1 = \tilde{a}^{\min}$*, then* $\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}^{\min}$.

Proof. Using Equation (19), we get $m_\mu(\tilde{a}_1, \theta) = \frac{1}{3}[(1-\theta)(a+2b)+\theta(2c+d)]\omega_{\tilde{a}} = 0$. Since $T_1 = 1$,$T_j = \prod_{k=1}^{j} m_\mu(\tilde{a}_{k-1}, \theta) = 1 \times 0 \times m_\mu(\tilde{a}_2, \theta) \times \cdots \times m_\mu(\tilde{a}_{j-1}, \theta) = 0$ for $j = 2, 3, \cdots, n$, we have

$$w_1 = 1, w_j = T_j / \sum_{i=1}^{n} T_i = 0 \quad \text{for} \quad j = 2, 3, \cdots, n.$$

Thus,

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{a}_j = w_1 \tilde{a}_1 + \sum_{j=2}^{n} w_j \tilde{a}_j = \tilde{a}_1 = \tilde{a}^{\min}.$$

□

Corollary 3.12 indicates that for the highest order attribute C_1 with the highest priority, if the satisfaction $\tilde{a}_1$ is the smallest TrIFN, then any compensation cannot be got from other attributes though these attributes are satisfied. Thus, this verifies that a TrIFPWA operator has the fundamental feature of prioritization relationship similar to the PS operator: poor satisfaction to any higher attribute decreases the ability for compensation by lower priority attributes [36].

Theorem 3.13. *(Boundedness). For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$*, let*

$$\tilde{a}^+ = ((\max\{a_i\}, \max\{b_i\}, \max\{c_i\}, \max\{d_i\}); \max\{\omega_{\tilde{a}_i}\}, \min\{u_{\tilde{a}_i}\}), \tag{31}$$

$$\tilde{a}^- = ((\min\{a_i\}, \min\{b_i\}, \min\{c_i\}, \min\{d_i\}); \min\{\omega_{\tilde{a}_i}\}, \max\{u_{\tilde{a}_i}\}), \tag{32}$$

then,

$$\tilde{a}^- \le \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \le \tilde{a}^+.$$

Proof. Since $w_j = T_j / \sum_{i=1}^{n} T_i$, we have $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} (T_j / \sum_{i=1}^{n} T_i) = 1$ and

$$\min\{a_j\} = \sum_{j=1}^{n} w_j \min\{a_j\} \le \sum_{j=1}^{n} w_j a_j \le \sum_{j=1}^{n} w_j \max\{a_j\} = \max\{a_j\},$$

$$\min\{b_j\} = \sum_{j=1}^{n} w_j \min\{b_j\} \le \sum_{j=1}^{n} w_j b_j \le \sum_{j=1}^{n} w_j \max\{b_j\} = \max\{b_j\},$$

$$\min\{c_j\} = \sum_{j=1}^{n} w_j \min\{c_j\} \le \sum_{j=1}^{n} w_j c_j \le \sum_{j=1}^{n} w_j \max\{c_j\} = \max\{c_j\},$$

$$\min\{d_j\} = \sum_{j=1}^{n} w_j \min\{d_j\} \le \sum_{j=1}^{n} w_j d_j \le \sum_{j=1}^{n} w_j \max\{d_j\} = \max\{d_j\}.$$

Let $\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a} = ((a, b, c, d); \omega_{\tilde{a}}, u_{\tilde{a}})$. Then by Theorem 3.9, it yields that

$$a = \sum_{j=1}^{n} w_j a_j, b = \sum_{j=1}^{n} w_j b_j, c = \sum_{j=1}^{n} w_j c_j, d = \sum_{j=1}^{n} w_j d_j, \omega_{\tilde{a}} = \mathop{\wedge}_{j} \omega_{\tilde{a}_j}, u_{\tilde{a}} = \mathop{\vee}_{j} u_{\tilde{a}_j}.$$

Since $\omega_{\tilde{a}} = \mathop{\wedge}_{j} \omega_{\tilde{a}_j} \le \max\{\omega_{\tilde{a}_j}\}, u_{\tilde{a}} = \mathop{\vee}_{j} u_{\tilde{a}_j} \ge \min\{u_{\tilde{a}_j}\}$, for a given $\lambda \in [0, 1]$, it holds that

$m_\mu(\tilde{a}, \theta) = \frac{1}{3}[(1-\theta)(a+2b) + \theta(2c+d)]\omega_{\tilde{a}} \le \frac{1}{3}[(1-\theta)(\max\{a_j\} + 2\max\{b_j\}) + \theta(2\max\{c_j\} + \max\{d_j\})]\max\{\omega_{\tilde{a}_i}\} = m_\mu(\tilde{a}^+, \theta)$,

$m_\mu(\tilde{a}, \theta) = \frac{1}{3}[(1-\theta)(a+2b) + \theta(2c+d)]\omega_{\tilde{a}} \ge \frac{1}{3}[(1-\theta)(\min\{a_j\} + 2\min\{b_j\}) + \theta(2\min\{c_j\} + \min\{d_j\})]\min\{\omega_{\tilde{a}_i}\} = m_\mu(\tilde{a}^-, \theta)$.

Thus, $\tilde{a}^- \le \tilde{a} \le \tilde{a}^+$, which completes the proof of Theorem 3.13. □

Theorem 3.14. *(Monotonicity). Let* $\tilde{a}_i = ((a_{1i}, b_{1i}, c_{1i}, d_{1i}); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ *and* $\tilde{b}_i = ((a_{2i}, b_{2i}, c_{2i}, d_{2i}); \omega_{\tilde{b}_i}, u_{\tilde{b}_i}) \quad (i = 1, 2, \cdots, n)$ *be two collections of TrIFNs. If* $\tilde{a}_i \le \tilde{b}_i$, $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ *and* $u_{\tilde{a}_i} = u_{\tilde{b}_i} \quad (i = 1, 2, \cdots, n)$, *then*

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \le \mathrm{TrIFPWA}_{w'}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n),$$

where the priority based weight vector $w' = (w'_1, w'_2, \cdots, w'_n)^{\mathrm{T}}$ *can be determined similarly by Equation (29).*

Proof. Since $\tilde{a}_i \le \tilde{b}_i$, $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ and $u_{\tilde{a}_i} = u_{\tilde{b}_i} \quad (i = 1, 2, \cdots, n)$, it follows that

$m_\mu(\tilde{a}_i, \theta) = \frac{1}{3}[(1-\theta)(a_{1i} + 2b_{1i}) + \theta(2c_{1i} + d_{1i})]\omega_{\tilde{a}_i} \le m_\mu(\tilde{b}_i, \theta) = \frac{1}{3}[(1-\theta)(a_{2i} + 2b_{2i}) + \theta(2c_{2i} + d_{2i})]\omega_{\tilde{b}_i}$,

$T_1 = T'_1 = 1, T_j = \prod_{k=1}^{j} m_\mu(\tilde{a}_i, \theta) \le \prod_{k=1}^{j} m_\mu(\tilde{b}_i, \theta) = T'_j$ for $j = 2, 3, \cdots, n$.

$w_1 = w'_1 = 1, w_j \le w'_j$, for $j = 2, 3, \cdots, n$.

For a given $\lambda \in [0, 1]$, according to Theorem 2.9, $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ and $u_{\tilde{a}_i} = u_{\tilde{b}_i} \quad (i = 1, 2, \cdots, n)$, we obtain

$$m_\mu(\sum_{j=1}^{n} w_j \tilde{a}_i, \theta) = \sum_{j=1}^{n} w_j m_\mu(\tilde{a}_i, \theta) \le \sum_{j=1}^{n} w'_j m_\mu(\tilde{b}_i, \theta) = m_\mu(\sum_{j=1}^{n} w'_j \tilde{b}_i, \theta).$$

It directly yields by Theorem 3.3 that

$$\mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \le \mathrm{TrIFPWA}_{w'}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n).$$

Hence, we end the proof of Theorem 3.14. □

Theorem 3.15. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$ *and* $\tilde{b} = ((a_{\tilde{b}}, b_{\tilde{b}}, c_{\tilde{b}}, d_{\tilde{b}}); \omega_{\tilde{b}}, u_{\tilde{b}})$*, we have*

$$\mathrm{TrIFPWA}_w(\tilde{a}_1 + \tilde{b}, \tilde{a}_2 + \tilde{b}, \cdots, \tilde{a}_n + \tilde{b}) = \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) + \tilde{b},$$

$$\mathrm{TrIFPWA}_w(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \cdots, \tilde{a}_n\tilde{b}) = \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \cdot \tilde{b},$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *is determined by Equation (29).*

Proof. By Theorem 3.9 and Definitions 2.2 and 3.8, we have

$$\mathrm{TrIFPWA}_w(\tilde{a}_1 + \tilde{b}, \tilde{a}_2 + \tilde{b}, \cdots, \tilde{a}_n + \tilde{b}) = \sum_{j=1}^{n} w_j(\tilde{a}_j + \tilde{b})$$

$$= ((\sum_{j=1}^{n} w_j(a_j + a_{\tilde{b}}), \sum_{j=1}^{n} w_j(b_j + b_{\tilde{b}}), \sum_{j=1}^{n} w_j(c_j + c_{\tilde{b}}), \sum_{j=1}^{n} w_j(d_j + d_{\tilde{b}})); \\ (\bigwedge_j \omega_{\tilde{a}_j}) \wedge \omega_{\tilde{b}}, (\bigvee_j u_{\tilde{a}_j}) \vee u_{\tilde{b}})$$

$$= ((\sum_{j=1}^{n} w_j a_j + a_{\tilde{b}}, \sum_{j=1}^{n} w_j b_j + b_{\tilde{b}}, \sum_{j=1}^{n} w_j c_j + c_{\tilde{b}}, \sum_{j=1}^{n} w_j d_j + d_{\tilde{b}}); (\bigwedge_j \omega_{\tilde{a}_j}) \wedge \omega_{\tilde{b}}, (\bigvee_j u_{\tilde{a}_j}) \vee u_{\tilde{b}})$$

$$= ((\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j); \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j}) + ((a_{\tilde{b}}, b_{\tilde{b}}, c_{\tilde{b}}, d_{\tilde{b}}); \omega_{\tilde{b}}, u_{\tilde{b}})$$

$$= \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) + \tilde{b}.$$

Likewise, we have

$$\mathrm{TrIFPWA}_w(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \cdots, \tilde{a}_n\tilde{b}) = \sum_{j=1}^{n} w_j \tilde{a}_j \tilde{b}$$

$$= ((\sum_{j=1}^{n} w_j a_j a_{\tilde{b}}, \sum_{j=1}^{n} w_j b_j b_{\tilde{b}}, \sum_{j=1}^{n} w_j c_j c_{\tilde{b}}, \sum_{j=1}^{n} w_j d_j d_{\tilde{b}}); (\bigwedge_j \omega_{\tilde{a}_j}) \wedge \omega_{\tilde{b}}, (\bigvee_j u_{\tilde{a}_j}) \vee u_{\tilde{b}})$$

$$= (((\sum_{j=1}^{n} w_j a_j) a_{\tilde{b}}, (\sum_{j=1}^{n} w_j b_j) b_{\tilde{b}}, (\sum_{j=1}^{n} w_j c_j) c_{\tilde{b}}, (\sum_{j=1}^{n} w_j d_j) d_{\tilde{b}}); (\bigwedge_j \omega_{\tilde{a}_j}) \wedge \omega_{\tilde{b}}, (\bigvee_j u_{\tilde{a}_j}) \vee u_{\tilde{b}})$$

$$= ((\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j); \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j}) \cdot ((a_{\tilde{b}}, b_{\tilde{b}}, c_{\tilde{b}}, d_{\tilde{b}}); \omega_{\tilde{b}}, u_{\tilde{b}})$$

$$= \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \cdot \tilde{b}.$$

Hence, the proof of Theorem 3.15 is completed. □

Theorem 3.16. *Let* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$ *be a collection of TrIFNs and* $\lambda \geq 0$*. Then,*

$$\mathrm{TrIFPWA}_w(\lambda\tilde{a}_1, \lambda\tilde{a}_2, \cdots, \lambda\tilde{a}_n) = \lambda \cdot \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n),$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *is determined by Equation (29).*

Proof. By Definition 2.2 and Theorem 3.9, we have

$$\mathrm{TrIFPWA}_w(\lambda\tilde{a}_1, \lambda\tilde{a}_2, \cdots, \lambda\tilde{a}_n) = \sum_{j=1}^{n} w_j \lambda\tilde{a}_j$$

$$= ((\sum_{j=1}^{n} w_j\lambda a_j, \sum_{j=1}^{n} w_j\lambda b_j, \sum_{j=1}^{n} w_j\lambda c_j, \sum_{j=1}^{n} w_j\lambda d_j); \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j})$$

$$= \lambda((\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j); \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j})$$

$$= \lambda \cdot \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

Hence, the proof of Theorem 3.16 is completed. □

Combined with Theorems 3.15 and 3.16, the following Theorem 3.17 can be easily proven.

Theorem 3.17. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ *and* $\tilde{b} = ((a_{\tilde{b}}, b_{\tilde{b}}, c_{\tilde{b}}, d_{\tilde{b}}); \omega_{\tilde{b}}, u_{\tilde{b}})$*, if* $\lambda \geq 0$*, then*

$$\mathrm{TrIFPWA}_w(\lambda\tilde{a}_1 + \tilde{b}, \lambda\tilde{a}_2 + \tilde{b}, \cdots, \lambda\tilde{a}_n + \tilde{b}) = \lambda \cdot \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) + \tilde{b},$$

$$\mathrm{TrIFPWA}_w(\lambda(\tilde{a}_1 + \tilde{b}), \lambda(\tilde{a}_2 + \tilde{b}), \cdots, \lambda(\tilde{a}_n + \tilde{b})) = \lambda \cdot \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) + \tilde{b},$$

$$\mathrm{TrIFPWA}_w(\lambda\tilde{a}_1\tilde{b}, \lambda\tilde{a}_2\tilde{b}, \cdots, \lambda\tilde{a}_n\tilde{b}) = \lambda \cdot \mathrm{TrIFPWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \cdot \tilde{b},$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *is determined by Equation (29).*

3.3. Trapezoidal Intuitionistic Fuzzy Prioritized and Operator.

Using the PA operator, Yager [36] further introduced an aggregation operator called prioritized and (PRI-AND) operator for real numbers as follows:

Definition 3.18. [36] Let $C = \{C_1, C_2, \cdots, C_n\}$ be an attribute set with a linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$, i.e., the priority of attribute C_j is higher than that of C_k if $j < k$. Denote the satisfaction of any alternative x on attribute C_j by $C_j(x)$ which satisfies $C_j(x) \in [0, 1]$. If

$$PRI-AND_w(C_1(x), C_2(x), \cdots, C_n(x)) = R(C_1(x)^{w_1}, C_2(x)^{w_2}, \cdots, C_n(x)^{w_n}), \tag{33}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ and $w_j \in [0, 1]$ is the importance weight associated with the argument $C_j(x)$, R is a t-norm, then, PRI-AND is called a prioritized and operator.

There are two most important t-norms:

(1) $R_M(x, y) = \min(x, y)$,
(2) $R_P(x, y) = xy$.

For different t-norms, the aggregation results are given as follows [36]:
(i) If R is the min t-norm $R = R_M(x, y) = \min\{x, y\}$, we have

$$PRI-AND_w(C_1(x), C_2(x), \cdots, C_n(x)) = \min_j\{C_j(x)^{w_j}\};$$

(ii) If R is the product t-norm $R = R_P(x, y) = xy$, we have

$$PRI-AND_w(C_1(x), C_2(x), \cdots, C_n(x)) = \prod_{j=1}^{n} C_j(x)^{w_j}.$$

In what follows, we also extend the PRI-AND operator of real numbers to the case in which the arguments to be aggregated are TrIFNs.

Definition 3.19. For TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$, if TrIFP-AND: $\Omega^n \to \Omega$ such that

$$\mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = R(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}), \tag{34}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the priority based weight vector associated with the argument $\tilde{a}_i$, which can be determined by Equation (29), R is a t-norm, then TrIFP-AND is called a trapezoidal intuitionistic fuzzy prioritized and operator.

Especially, if $R = R_M(x, y) = \min\{x, y\}$, we have

$$\mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = R_M(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}) = \min_j\{\tilde{a}_j^{w_j}\};$$

If $R = R_P(x, y) = xy$, we have

$$\mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = R_P(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}) = \prod_{j=1}^{n} \tilde{a}_j^{w_j}.$$

If $w_j = 0$, then $\tilde{a}_j^{w_j} = 1$. This observation shows that arguments with zero importance have no effect in the calculation of TrIFP-AND operator.

Theorem 3.20. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i}) \quad (i = 1, 2, \cdots, n)$, *their aggregated value by using TrIFP-AND operator is also a TrIFN, and*

$$\begin{aligned} R_M(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}) = \min_j\{((a_j^{w_j}, b_j^{w_j}, c_j^{w_j}, d_j^{w_j}); \\ \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j})\} \quad \text{for} \quad R = R_M(x, y); \end{aligned} \tag{35}$$

$$\begin{aligned} R_P(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}) = ((\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j}, \prod_{j=1}^{n} d_j^{w_j}); \\ \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j}) \quad \text{for} \quad R = R_P(x, y). \end{aligned} \tag{36}$$

Proof. The first result follows directly from Definitions 2.2 and 3.19. The second result can be easily proven by using mathematical induction on n according to Definition 3.19.

Clearly, for any TrIFNs $\tilde{a}_i (i = 1, 2, \cdots, n)$, we have $R_M(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}) \geq R_P(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n})$; For any t-norm R, we have

$$R_M(\tilde{a}_1^{w_1}, \tilde{a}_2^{w_2}, \cdots, \tilde{a}_n^{w_n}) \geq \mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

According to Theorem 3.20, there are some desirable properties for the TrIFP-AND operator, such as commutativity, monotonicity, associativity, and so on. □

Theorem 3.21. *(Commutativity). Let* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ *be a collection of TrIFNs. If* $(\dot{\tilde{a}}_1, \dot{\tilde{a}}_2, \cdots, \dot{\tilde{a}}_n)$*, is any permutation of* $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ *then*

$$\mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \mathrm{TrIFP-AND}_w(\dot{\tilde{a}}_1, \dot{\tilde{a}}_2, \cdots, \dot{\tilde{a}}_n),$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Proof. The result can be easily proved by Theorem 3.20. □

Theorem 3.22. *(Monotonicity). Let* $\tilde{a}_i = ((a_{1i}, b_{1i}, c_{1i}, d_{1i}); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ *and* $\tilde{b}_i = ((a_{2i}, b_{2i}, c_{2i}, d_{2i}); \omega_{\tilde{b}_i}, u_{\tilde{b}_i})$ $(i = 1, 2, \cdots, n)$ *be two collections of TrIFNs. If* $\tilde{a}_i \leq \tilde{b}_i$ $(i = 1, 2, \cdots, n)$*,* $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ *and* $u_{\tilde{a}_i} = u_{\tilde{b}_i} (i = 1, 2, \cdots, n)$*, then,*

$$\mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \mathrm{TrIFP-AND}_{w'}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n),$$

where the priority based weight vector $w' = (w'_1, w'_2, \cdots, w'_n)^{\mathrm{T}}$ *can be determined similarly by Equation (29).*

Proof. Since $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ and $u_{\tilde{a}_i} = u_{\tilde{b}_i} (i = 1, 2, \cdots, n)$, we have

$$\begin{aligned} m_\mu(\tilde{a}_i, \theta) &= \tfrac{1}{3}[(1-\theta)(a_{1i} + 2b_{1i}) + \theta(2c_{1i} + d_{1i})]\omega_{\tilde{a}_i} \\ &\leq m_\mu(\tilde{b}_i) = \tfrac{1}{3}[(1-\theta)(a_{2i} + 2b_{2i}) + \theta(2c_{2i} + d_{2i})]\omega_{\tilde{b}_i}, \end{aligned}$$

$$T_1 = T'_1 = 1, T_j = \prod_{k=1}^{j} m_\mu(\tilde{a}_i, \theta) \leq \prod_{k=1}^{j} m_\mu(\tilde{b}_i, \theta) = T'_j, \quad \text{for} \quad j = 2, 3, \cdots, n,$$

$$w_1 = w'_1 = 1, w_j \leq w'_j, \quad \text{for} \quad j = 2, 3, \cdots, n.$$

For a given $\lambda \in [0, 1]$, according to Theorem 2.9, $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ and $u_{\tilde{a}_i} = u_{\tilde{b}_i}$ $(i = 1, 2, \cdots, n)$, we obtain

$$m_\mu(\min_j\{\tilde{a}_j^{w_j}\}, \theta) = \min_j\{m_\mu(\tilde{a}_j^{w_j}, \theta)\} \leq \min_j\{m_\mu(\tilde{b}_j^{w'_j}, \theta)\} = m_\mu(\min_j\{\tilde{b}_j^{w'_j}\}, \theta)$$

$$\begin{aligned} m_\mu(\prod_{j=1}^{n} \tilde{a}_j^{w_j}, \theta) &= m_\mu((\prod_{j=1}^{n} a_{1j}^{w_j}, \prod_{j=1}^{n} b_{1j}^{w_j}, \prod_{j=1}^{n} c_{1j}^{w_j}, \prod_{j=1}^{n} d_{1j}^{w_j}); \bigwedge_j \omega_{\tilde{a}_j}, \bigvee_j u_{\tilde{a}_j}) \\ &\leq m_\mu(((\prod_{j=1}^{n} a_{2j}^{w'_j}, \prod_{j=1}^{n} b_{2j}^{w'_j}, \prod_{j=1}^{n} c_{2j}^{w'_j}, \prod_{j=1}^{n} d_{2j}^{w'_j}); \bigwedge_j \omega_{\tilde{b}_j}, \bigvee_j u_{\tilde{b}_j}), \theta) \\ &= m_\mu(\prod_{j=1}^{n} \tilde{b}_j^{w'_j}, \theta). \end{aligned}$$

It directly yields by Theorem 3.20 that

$$\mathrm{TrIFP-AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \mathrm{TrIFP-AND}_{w'}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n).$$

Hence, the proof of Theorem 3.22 is completed. □

Theorem 3.23. *(Associativity). For three TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, 3)$*, we have*

$$\mathrm{TrIFP} - \mathrm{AND}_w(\tilde{a}_1, \mathrm{TrIFP} - \mathrm{AND}_w(\tilde{a}_2, \tilde{a}_3))$$

$$= \mathrm{TrIFP} - \mathrm{AND}_w(\mathrm{TrIFP} - \mathrm{AND}_w(\tilde{a}_1, \tilde{a}_2), \tilde{a}_3),$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Proof. This Theorem follows directly from Definitions 2.2, 3.19 and Theorem 3.20. □

Theorem 3.24. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n+1)$*, we have*

$$\mathrm{TrIFP} - \mathrm{AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \geq \mathrm{TrIFP} - \mathrm{AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n, \tilde{a}_{n+1}),$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Proof. This Theorem can be directly proved from Definitions 2.2, 3.19 and Theorem 3.20. □

Theorem 3.25. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$*, if* $\tilde{a}_k = ((a_k, b_k, c_k, d_k); \omega_{\tilde{a}_k}, u_{\tilde{a}_k})$ *is not equal to* $\tilde{a}^{\max} = ([1, 1, 1, 1]; 1, 0)$*, and other TrIFNs* $\tilde{a}_i (i \neq k)$ *are* $\tilde{a}^{\max}$*, i.e.,* $\tilde{a}_i = \tilde{a}^{\max}$ *for* $i = 1, 2, \cdots, n$ *and* $i \neq k$*, then*

$$\mathrm{TrIFP} - \mathrm{AND}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}_k,$$

where the priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Proof. Theorem 3.25 can be proved from Definition 3.8 and Theorem 3.22. □

Remark 3.26. If the collection of attributes is partitioned into q categories $C = \bigcup_{i=1}^{q} H_i$ such that $H_i = \{C_{i1}, C_{i2}, \cdots, C_{in_i}\}$ with a linear ordering $H_1 \succ H_2 \succ \cdots \succ H_q$. Then, the importance weight w_{ij} associated with the TrIFN $\tilde{a}_{ij}$ can be determined as Equation (25). Thus, the TrIFP-AND operator developed in this paper can also suit this situation.

3.4. **Trapezoidal Intuitionistic Fuzzy Prioritized or Operator.**

Yager [36] also introduced an aggregation operator called prioritized or (PRI-OR) operator for real numbers as follows:

Definition 3.27. [36]. Let $C = \{C_1, C_2, \cdots, C_n\}$ be an attribute set with a linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$, i.e., the priority of attribute C_j is higher than that of C_k if $j < k$. Denote the satisfaction of any alternative x on attribute C_j by $C_j(x)$ which satisfies $C_j(x) \in [0, 1]$, if

$$PRI - OR_w(C_1(x), C_2(x), \cdots, C_n(x)) = P(w_1 C_1(x), w_2 C_2(x), \cdots, w_n C_n(x)), \quad (37)$$

where $w_j \in [0, 1]$ is the importance weight associated with the argument $C_j(x)$, P is a t-conorm, then, PRI- OR is called a prioritized or operator.

There are two most important t-conorms:
(a) $P_M(x,y) = \max(x,y)$;
(b) $P_S(x,y) = x + y - xy$.
Yager [36] obtained the following aggregation results:
(a) If P is the max t-conorm $P = P_M(x,y) = \max(x,y)$, we have

$$\begin{aligned} PIR - OR_w(C_1(x), C_2(x), \cdots, C_n(x)) &= P_M(w_1C_1(x), w_2C_2(x), \cdots, w_nC_n(x)) \\ &= \max_j\{w_jC_j(x)\}; \end{aligned}$$

(b) If P is the Lukasiewicz t-conorm $P = P_S(x,y) = x + y - xy$, we have

$$\begin{aligned} PIR - OR_w(C_1(x), C_2(x), \cdots, C_n(x)) &= P_S(w_1C_1(x), w_2C_2(x), \cdots, w_nC_n(x)) \\ &= 1 - \prod_{j=1}^{n}(1 - w_jC_j(x)). \end{aligned}$$

Analogously, we extend the PIR-OR operator of real numbers to the case where the arguments to be aggregated are TrIFNs.

Definition 3.28. For TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$, if TrIFP-OR: $\Omega^n \to \Omega$ such that

$$\mathrm{TrIFP} - \mathrm{OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = P(w_1\tilde{a}_1, w_2\tilde{a}_2, \cdots, w_n\tilde{a}_n), \tag{38}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ and $w_j \in [0,1]$ is the importance weight associated with the argument $C_j(x)$, which can be determined by Equation (29), P is a t-conorm, then TrIFP-OR is called a trapezoidal intuitionistic fuzzy prioritized or operator.

Since it is very difficult to define the substraction operation of TrIFNs, we only look at the issue of performing the max t-conorm aggregation as follows:
If P is the max t-conorm $P = P_M(x,y) = \max(x,y)$, then

$$\mathrm{TrIFP} - \mathrm{OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = P_M(w_1\tilde{a}_1, w_2\tilde{a}_2, \cdots, w_n\tilde{a}_n) = \max_j\{w_j\tilde{a}_j\}.$$

If $w_j = 0$, then $w_j\tilde{a}_j = 0$. Therefore, the attributes with zero importance have no effect in the calculation of TrIFP-OR operator.

Theorem 3.29. *For TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$, their aggregated value by using TrIFP-OR operator is also a TrIFN, and*

$$P_M(w_1\tilde{a}_1, w_2\tilde{a}_2, \cdots, w_n\tilde{a}_n) = \max_j\{((w_ja_j, w_jb_j, w_jc_j, w_jd_j); 1 - (1 - \omega_{\tilde{a}_j})^{w_j}, u_{\tilde{a}_j}^{w_j})\} \tag{39}$$

for $P = P_M(x,y)$, where priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ can be determined by Equation (29).

Proof. The first result follows directly from Definitions 2.2 and 3.28. □

It is apparent that for any t-conorm P, we have

$$P_M(w_1\tilde{a}_1, w_2\tilde{a}_2, \cdots, w_n\tilde{a}_n) \le \mathrm{TrIFP} - \mathrm{OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

The TrIFP-OR operator has also some desirable properties, such as commutativity, monotonicity, associativity, and so on.

Theorem 3.30. *(Commutativity). Let* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ *be a collection of TrIFNs. If* $(\dot{\tilde{a}}_1, \dot{\tilde{a}}_2, \cdots, \dot{\tilde{a}}_n)$ *is any permutation of* $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$*, then*

$$\mathrm{TrIFP-OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \mathrm{TrIFP-OR}_w(\dot{\tilde{a}}_1, \dot{\tilde{a}}_2, \cdots, \dot{\tilde{a}}_n),$$

where priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Theorem 3.31. *(Monotonicity). Let* $\tilde{a}_i = ((a_{1i}, b_{1i}, c_{1i}, d_{1i}); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ *and* $\tilde{b}_i = ((a_{2i}, b_{2i}, c_{2i}, d_{2i}); \omega_{\tilde{b}_i}, u_{\tilde{b}_i}), (i = 1, 2, \cdots, n)$ *be two collections of TrIFNs. If* $\tilde{a}_i \leq \tilde{b}_i, (i = 1, 2, \cdots, n)$*,* $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ *and* $u_{\tilde{a}_i} = u_{\tilde{b}_i}, (i = 1, 2, \cdots, n)$*, then*

$$\mathrm{TrIFP-OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \mathrm{TrIFP-OR}_{w'}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n),$$

where the priority based weight vector $w' = (w'_1, w'_2, \cdots, w'_n)^{\mathrm{T}}$ *can be determined similarly by Equation (29).*

Proof. Since $\tilde{a}_i \leq \tilde{b}_i \ \ (i = 1, 2, \cdots, n)$, $\omega_{\tilde{a}_i} = \omega_{\tilde{b}_i}$ and $u_{\tilde{a}_i} = u_{\tilde{b}_i}$, we have

$$\begin{aligned} m_\mu(\tilde{a}_i, \theta) &= \tfrac{1}{3}[(1-\theta)(a_{1i} + 2b_{1i}) + \theta(2c_{1i} + d_{1i})]\omega_{\tilde{a}_i} \\ &\leq m_\mu(\tilde{b}_i, \theta) = \tfrac{1}{3}[(1-\theta)(a_{2i} + 2b_{2i}) + \theta(2c_{2i} + d_{2i})]\omega_{\tilde{b}_i}, \end{aligned}$$

$$T_1 = T'_1 = 1, T_i = \prod_{k=1}^{i} m_\mu(\tilde{a}_k, \theta) \leq \prod_{k=1}^{i} m_\mu(\tilde{b}_k, \theta) = T'_i, \quad \text{for} \quad i = 2, 3, \cdots, n,$$

$$w_1 = w'_1 = 1, w_i \leq w'_i, \quad \text{for} \quad i = 2, 3, \cdots, n.$$

For a given $\lambda \in [0, 1]$, according to Theorem 2.9, we obtain

$$m_\mu(\max_j\{w_j\tilde{a}_j\}, \theta) = \max_j\{m_\mu(w_j\tilde{a}_j, \theta)\} \leq \max_j\{m_\mu(w'_j\tilde{b}_j, \theta)\} = m_\mu(\max_j\{w'_j\tilde{b}_j\}, \theta);$$

It directly yields by Theorem 3.29 that

$$\mathrm{TrIFP-OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \mathrm{TrIFP-OR}_{w'}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n).$$

Hence, the proof of Theorem 3.31 is completed. □

Theorem 3.32. *(Associativity). For three TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, 3)$*, we have*

$$\mathrm{TrIFP-OR}_w(\tilde{a}_1, \mathrm{TrIFP-OR}_w(\tilde{a}_2, \tilde{a}_3)) = \mathrm{TrIFP-OR}_w(\mathrm{TrIFP-OR}_w(\tilde{a}_1, \tilde{a}_2), \tilde{a}_3),$$

where priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Proof. This Theorem follows directly from Definitions 2.2 and 3.28 and Theorem 3.29. □

Theorem 3.33. *For TrIFNs* $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n+1)$*, we have*

$$\mathrm{TrIFP-OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \mathrm{TrIFP-OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n, \tilde{a}_{n+1}),$$

where priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ *can be determined by Equation (29).*

Proof. This Theorem can be proved from Definitions 2.2 and 3.28 and Theorem 3.29. □

Theorem 3.34. *For TrIFNs $\tilde{a}_i = ((a_i, b_i, c_i, d_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$, if $\tilde{a}_k = ((a_k, b_k, c_k, d_k); \omega_{\tilde{a}_k}, u_{\tilde{a}_k})$ is not equal to $\tilde{a}^{\min}$ and all other TrIFNs $\tilde{a}_i, (i \neq k)$ are $\tilde{a}^{\min}$, i.e., $\tilde{a}_i = \tilde{a}^{\min}(i \neq k)$, then*

$$\mathrm{TrIFP} - \mathrm{OR}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}_k,$$

where priority based weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ can be determined by Equation (29).

Proof. This Theorem can be proved from Definition 3.28 and Theorem 3.29. □

Remark 3.35. If the collection of attributes are partitioned into q categories $C = \bigcup_{i=1}^{q} H_i$ such that $H_i = \{C_{i1}, C_{i2}, \cdots, C_{in_i}\}$ with a linear ordering $H_1 \succ H_2 \succ \cdots \succ H_q$. Then, the importance weight w_{ij} associated with the TrIFN $\tilde{a}_{ij}$ can be determined including the following steps:

Step 1: Compute the weighted means of TrIFNs $\tilde{a}_{ij}$, i.e., $V_\lambda(\tilde{a}_{ij})$ $(i = 1, 2, \cdots, q;$ $j = 1, 2, \cdots, n_i)$;

Step 2: For each priority category H_i, we compute $S_i = \max_j\{V_\lambda(\tilde{a}_{ij})\}$ $(i = 1, 2, \cdots, q-1)$, $S_0 = 1$;

Step 3: The priority weight vector $w = (w_{1j}, w_{2j}, \cdots, w_{qj})^{\mathrm{T}}$ can be obtained, where

$$w_{ij} = T_i, T_i = \prod_{k=1}^{i} S_{k-1}, (i = 1, 2, \cdots, q). \tag{40}$$

Thus, the TrIFP-OR operator developed in this paper can also suit this situation.

4. Application to Prioritized MADM Problems with TrIFN Information

In this section, utilizing the proposed TrIFPWA operator, we develop a new decision making method to solve the prioritized MADM problems with TrIFNs.

4.1. Description of Prioritized MADM Problems Using TrIFNs.

For some prioritized MADM problems, there is an alternative set by $A = \{A_1, A_2, \cdots, A_m\}$ and an attribute set by $C = \{C_1, C_2, \cdots, C_n\}$. Assume that the priority of attribute C_i is higher than that of C_j if $i < j$ and there exists a linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$ between attributes. The rating of an alternative A_i on an attribute C_j may be represented as a TrIFN $\tilde{a}_{ij} = ((a_{ij}, b_{ij}, c_{ij}, d_{ij}); \omega_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$, where $\omega_{\tilde{a}_{ij}}$ denotes the maximum membership degree of alternative A_i belongs to trapezoidal fuzzy number $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ on attribute C_j, and $u_{\tilde{a}_{ij}}$ denotes the minimum non-membership degree of alternative A_i does not belong to the trapezoidal fuzzy number $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ on attribute C_j, such that they satisfy that $0 \leq \omega_{\tilde{a}_{ij}} \leq 1$, $0 \leq u_{\tilde{a}_{ij}} \leq 1$ and $0 \leq \omega_{\tilde{a}_{ij}} + u_{\tilde{a}_{ij}} \leq 1$. Hence, we can elicit a fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$, which is referred to as a TrIFN decision matrix usually used to represent the MADM problem.

To eliminate the impact of different dimensions on decision results, the matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ needs to be normalized into $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, where $\tilde{r}_{ij} = ((a_{\tilde{r}_{ij}}, b_{\tilde{r}_{ij}}, c_{\tilde{r}_{ij}}, d_{\tilde{r}_{ij}}); \omega_{\tilde{r}_{ij}}, u_{\tilde{r}_{ij}})$, $\omega_{\tilde{r}_{ij}} = \omega_{\tilde{a}_{ij}}$ and $u_{\tilde{r}_{ij}} = u_{\tilde{a}_{ij}}$. In this paper, the normalization method is chosen as follows:

For benefit attributes,

$$\tilde{r}_{ij} = ((a_{ij}/d_j^+, b_{ij}/d_j^+, c_{ij}/d_j^+, d_{ij}/d_j^+); \omega_{\tilde{r}_{ij}}, u_{\tilde{r}_{ij}}); \tag{41}$$

For cost attributes,

$$\tilde{r}_{ij} = ((a_j^-/d_{ij}, a_j^-/c_{ij}, a_j^-/b_{ij}, a_j^-/a_{ij}); \omega_{\tilde{r}_{ij}}, u_{\tilde{r}_{ij}}), \tag{42}$$

where $d_j^+ = \max\{d_{ij}|i = 1, 2, \cdots, m\}$ and $a_j^- = \min\{a_{ij}|i = 1, 2, \cdots, m\}, (j = 1, 2, \cdots, n)$.

4.2. Decision Method Based on Trapezoidal Intuitionistic Fuzzy Prioritized Weighted Average Operators.

A new method for solving the prioritized MADM problems with TrIFNs may be summarized as follows.

Step 1: Normalize the decision matrix $\tilde{A}$ into $\tilde{R}$ according to Equations (41) and (42).

Step 2: Utilized the TrIFPWA operator, the comprehensive attribute value of alternative A_i can be obtained as follows:

$$\tilde{r}_i = ((a_{\tilde{r}_i}, b_{\tilde{r}_i}, c_{\tilde{r}_i}, d_{\tilde{r}_i}); \omega_{\tilde{r}_i}, u_{\tilde{r}_i}) = \text{TrIFPWA}_{w_i}(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) = \sum_{j=1}^{n} w_{ij}\tilde{r}_{ij}, \tag{43}$$

for each $i = 1, 2, \cdots, m$, where the normalized priority based importance weight vector $w_i = (w_{i1}, w_{i2}, \cdots, w_{in})^{\mathrm{T}}$ can be obtained by Equation (29) as follows:

$$T_{i1} = 1, T_{ij} = \prod_{t=1}^{j} m_\mu(\tilde{r}_{i,t-1}, \theta) \quad (j = 2, 3, \cdots, n),$$
$$w_{ij} = T_{ij} / \sum_{j=1}^{n} T_{ij} \quad (j = 1, 2, \cdots, n).$$

Step 3: By using Equations (19) and (20), the weighted possibility means of membership and non-membership functions for the TrIFNs $\tilde{r}_i = ((a_{\tilde{r}_i}, b_{\tilde{r}_i}, c_{\tilde{r}_i}, d_{\tilde{r}_i}); \omega_{\tilde{r}_i}, u_{\tilde{r}_i})$ are respectively calculated as follows:

$$m_\mu(\tilde{r}_i, \theta) = \tfrac{1}{3}[(1-\theta)(a_{\tilde{r}_i} + 2b_{\tilde{r}_i}) + \theta(2c_{\tilde{r}_i} + d_{\tilde{r}_i})]\omega_{\tilde{r}_i}, \tag{44}$$

$$m_v(\tilde{r}_i, \theta) = \tfrac{1}{3}[(1-\theta)(a_{\tilde{r}_i} + 2b_{\tilde{r}_i}) + \theta(2c_{\tilde{r}_i} + d_{\tilde{r}_i})](1 - u_{\tilde{r}_i}). \tag{45}$$

Step 4: The ranking order of alternatives is generated by $m_\mu(\tilde{r}_i, \theta)$ and $m_v(\tilde{r}_i, \theta)$ $(i = 1, 2, \cdots, m)$ according to the ranking method developed in Subsection 2.3.

5. Supply Chain Collaboration Example and Comparison Analysis of Computational Results

In this section, a supply chain collaboration problem is illustrated to demonstrate the applicability and implementation process of the prioritized MADM method proposed in this paper. The comparison analyses of computational results are also conducted to show the superiority of the proposed method.

	A_1	A_2	A_3	A_4	A_5
C_1	((2,4,5,7);0.6,0.2)	((1,2,3,6);0.5,0.3)	((2,3,4,5);0.7,0.1)	((2,3,4,6);0.5,0.1)	((3,5,6,7);0.3,0.5)
C_2	((2,3,4,6);0.5,0.4)	((2,3,4,6);0.6,0.3)	((1,3,4,6);0.5,0.3)	((1,4,5,6);0.7,0.2)	((1,2,3,6);0.6,0.3)
C_3	((1,2,3,4);0.3,0.5)	((3,4,6,7);0.6,0.2)	((1,2,4,5);0.4,0.3)	((2,4,5,6);0.6,0.2)	((2,3,4,6);0.5,0.5)
C_4	((4,5,7,9);0.7,0.1)	((1,3,4,5);0.6,0.2)	((2,3,4,6);0.6,0.3)	((2,3,4,6);0.4,0.2)	((1,3,4,5);0.6,0.1)
C_5	((1,2,3,4);0.2,0.5)	((2,3,5,6);0.4,0.6)	((1,2,4,6);0.4,0.4)	((3,4,5,6);0.5,0.1)	((2,3,5,6);0.6,0.2)
C_6	((3,5,6,8);0.5,0.3)	((1,2,5,6);0.4,0.2)	((3,4,6,8);0.3,0.3)	((2,3,5,6);0.7,0.2)	((1,4,5,6);0.5,0.2)

Table 1. The TrIFN Decision Matrix

	A_1	A_2	A_3
C_1	((0.29,0.57,0.71,1);0.6,0.2)	((0.14,0.29,0.43,0.88);0.5,0.3)	((0.29,0.43,0.58,0.71);0.7,0.1)
C_2	((0.17,0.25,0.33,0.5);0.5,0.4)	((0.17,0.25,0.33,0.5);0.6,0.3)	((0.17,0.25,0.33,1);0.5,0.3)
C_3	((0.25,0.33,0.5,1);0.3,0.5)	((0.14,0.17,0.25,0.33);0.6,0.2)	((0.2,0.25,0.5,1);0.4,0.3)
C_4	((0.44,0.55,0.77,1);0.7,0.1)	((0.11,0.33,0.44,0.56);0.6,0.2)	((0.22,0.33,0.44,0.67);0.6,0.3)
C_5	((0.17,0.33,0.5,0.67);0.2,0.5)	((0.33,0.5,0.83,1);0.4,0.6)	((0.17,0.33,0.67,1);0.4,0.4)
C_6	((0.38,0.62,0.75,1);0.5,0.3)	((0.13,0.25,0.67,0.75);0.4,0.2)	((0.38,0.5,0.75,1);0.3,0.3)

	A_4	A_5
C_1	((0.29,0.43,0.58,0.71);0.5,0.1)	((0.43,0.71,0.88,1);0.3,0.5)
C_2	((0.17,0.20,0.25,1);0.7,0.2)	((0.17,0.33,0.5,1);0.6,0.3)
C_3	((0.17,0.20,0.25,0.5);0.6,0.2)	((0.17,0.25,0.33,0.5);0.5,0.5)
C_4	((0.22,0.33,0.44,0.67);0.4,0.2)	((0.11,0.33,0.44,0.56);0.6,0.1)
C_5	((0.5,0.67,0.83,1);0.5,0.1)	((0.33,0.5,0.83,1);0.6,0.2)
C_6	((0.25,0.38,0.63,0.75);0.7,0.2)	((0.13,0.5,0.63,0.75);0.5,0.2)

Table 2. The Normalized TrIFN Decision Matrix

5.1. **Supply Chain Collaboration Problem and the Analysis Process.**

The proposed method is illustrated with a cooperative partner selection problem adapted from [9]. Suppose that a supply chain enterprise desires to select a cooperative partner. There are five potential candidates (i.e., alternatives)A_1, A_2, A_3, A_4 and A_5. The DM assesses the five candidates on the basis of six attributes, including quality and technology C_1, price and cost C_2, supply time C_3, service level C_4, innovation and agility C_5, logistics and information C_6. The DM agrees that there is a linear ordering $C_1 \succ C_2 \succ C_3 \succ C_4 \succ C_5 \succ C_6$ between the attributes. After the data acquisition and statistical treatment, the ratings of candidates with respect to attributes can be represented by TrIFNs as in Table 1. For example, ((2,4,5,7);0.6,0.2) is a TrIFN which indicates that the mark of the candidate A_1 with respect to the attribute C_1 is about interval [18, 19] with the maximum satisfaction degree is 0.6, while the minimum dissatisfaction degree is 0.2. In other words, the hesitation degree is 0.2. Other TrIFNs in Tables 1 are explained similarly. This is a MADM problem which has prioritization relationships over the attributes.

Step 1: The attributes C_2 and C_3 are cost attributes, and the attributes C_1, C_4, C_5 and C_6 are benefit attributes. According to Equations (41) and (42), the normalized TrIFN decision matrix is obtained as in Table 2.

Step 2: Assume that the preference of DM is neutral, i.e., the preference parameter of DM is $\theta = 0.5$. For alternative A_1, the comprehensive attribute value of TrIFNs $\tilde{r}_{1j} \quad (j = 1, 2, \cdots, 6)$ is calculated by Equation (29) as follows:

$$m_\mu(\tilde{r}_{11}, 0.5) = 0.3850, m_\mu(\tilde{r}_{12}, 0.5) = 0.1525, m_\mu(\tilde{r}_{13}, 0.5) = 0.3395,$$

$$m_\mu(\tilde{r}_{14}, 0.5) = 0.3400, m_\mu(\tilde{r}_{15}, 0.5) = 0.1250, m_\mu(\tilde{r}_{16}, 0.5) = 0.2060.$$

Then, the un-normalized priority based importance weights are calculated as follows:

$$T_{11} = 1, T_{12} = 0.3850, T_{13} = T_{12} \times 0.1525 = 0.0587, T_{14} = T_{13} \times 0.3395 = 0.0199,$$

$$T_{15} = T_{14} \times 0.3400 = 0.0068, T_{16} = T_{15} \times 0.1250 = 0.000847.$$

Thus, the normalized priority based importance weights are obtained as follows:

$$w_{11} = 0.6797, w_{12} = 0.2617, w_{13} = 0.0399,$$

$$w_{14} = 0.0135, w_{15} = 0.0046, w_{16} = 0.000575.$$

Utilized the TrIFPWA operator, the overall attribute value of alternative A_1 can be computed by Equation (43) as follows:

$$\tilde{r}_1 = \text{TrIFPWA}_{w_1}(\tilde{r}_{11}, \tilde{r}_{12}, \cdots, \tilde{r}_{16})$$

$$= \sum_{j=1}^{6} w_{1j}\tilde{r}_{1j} = ([0.2585, 0.4753, 0.6020, 0.8676]; 0.2, 0.5).$$

In the similar way, the overall attribute values of the other alternatives are obtained as follows:

$$\begin{aligned}
\tilde{r}_2 &= ((0.1456, 0.2788, 0.4058, 0.7890); 0.4, 0.6),\\
\tilde{r}_3 &= ((0.2535, 0.3730, 0.5113, 0.8000); 0.3, 0.4),\\
\tilde{r}_4 &= ((0.2565, 0.3658, 0.4878, 0.7629); 0.4, 0.2),\\
\tilde{r}_5 &= ((0.3504, 0.5893, 0.7530, 0.9617); 0.3, 0.5).
\end{aligned}$$

Step 3: Assume that the preference parameter of DM is $\theta = 0.5$, the weighted means and weighted ambiguities of membership and non-membership functions for the TrIFNs $\tilde{r}_i (i = 1, 2, \cdots, 5)$ are calculated by Equations (44) and (45) as follows:

$$\begin{aligned}
m_\mu(\tilde{r}_1, 0.5) &= 0.1094, m_\nu(\tilde{r}_1, 0.5) = 0.2734,\\
m_\mu(\tilde{r}_2, 0.5) &= 0.1560, m_\nu(\tilde{r}_2, 0.5) = 0.1560,\\
m_\mu(\tilde{r}_3, 0.5) &= 0.1416, m_\nu(\tilde{r}_3, 0.5) = 0.2831,\\
m_\mu(\tilde{r}_4, 0.5) &= 0.1847, m_\nu(\tilde{r}_4, 0.5) = 0.3694,\\
m_\mu(\tilde{r}_5, 0.5) &= 0.2069, m_\nu(\tilde{r}_5, 0.5) = 0.3448.
\end{aligned}$$

Step 4: Since $m_\mu(\tilde{r}_5, 0.5) > m_\mu(\tilde{r}_4, 0.5) > m_\mu(\tilde{r}_2, 0.5) > m_\mu(\tilde{r}_3, 0.5) > m_\mu(\tilde{r}_1, 0.5)$, according to the ranking method developed in Subsection 2.3, the ranking order of alternatives is generated as $A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$ and the best candidate is A_5.

For any other preference parameter values θ, in the same way, we can obtain the collective overall attribute values of candidates. The computation results and ranking orders of candidates are listed in Table 3.

It can be seen from Table 3 that, for different preference parameter values, the ranking orders of candidates are also not completely the same. For instance, if θ =0, namely DM is extremely optimistic, then the ranking order of candidates is $A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$, the best candidate is A_4; if θ=0.7, then the ranking order of candidates is $A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$, the best candidate is A_5; and if

θ	Ranking orders	Best candidates
0.0	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$	A_4
0.1	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$	A_4
0.2	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$	A_4
0.3	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.4	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.5	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.6	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.7	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.8	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.9	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	A_4
1.0	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	A_4

TABLE 3. Computation Results for Different Preference Parameter Values and Ranking Orders of Candidates

θ =1, namely DM is extremely pessimistic, then the ranking order of candidates is $A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$, the best candidate is A_4.

The above analysis suggests that the DM's preferences for the weighted lower and upper possibility means indeed play an important role in the decision making. Since TrIFN is a special kind of IFS, involving DM's preference to rank the TrIFNs is very reasonable and necessary. When the preference parameter values are different, the corresponding decision results may be different.

5.2. **Comparison Analysis with the TIFN MADM.**

In this subsection, we compare the differences obtained by the method [6] and the proposed method in this paper.

(1) The method [6] cannot be used to solve MADM problems with TrIFNs, whereas the method proposed in this paper is very suitable for MADM problems with TIFNs or TrIFNs.

For the TrIFNs $\tilde{a}_{ij} = ((a_{ij}, b_{ij}, c_{ij}, d_{ij}); \omega_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$ in Table 1, let $e_{ij} = \frac{1}{2}(b_{ij}+c_{ij})$, then all the TrIFNs $\tilde{a}_{ij} = ((a_{ij}, b_{ij}, c_{ij}, d_{ij}); \omega_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$ in Table 1 are reduced to the TIFNs $\tilde{a}_{ij} = ((a_{ij}, e_{ij}, d_{ij}); \omega_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$. Thus, the above supplier selection problem is reduced to MADM problems with TIFNs.

In the sequel, we use the method [6] to solve this MADM problem with TIFNs. Assume that the preference parameter of DM is also $\lambda = 0.5$ and the weights of the attributes are $\omega_1 = \omega_2 = \cdots = \omega_6 = 1/6$. Then the ranking order of the five candidates by the method[6] is generated as follows: $A_4 \succ A_1 \succ A_3 \succ A_5 \succ A_2$.

It is noted that the ranking orders obtained by this paper and by [6] are very different. This is because that all TrIFNs are transformed to TIFNs, which weakens the ability of information representation for TrIFNs. Hence, TrIFNs may better reflect the assessment information of decision problems than TIFNs.

(2) The method [6] is not applicable to prioritized MADM problems since it did not consider the prioritization between attributes, while this paper sufficiently considers the linear order between attributes, which makes the decision results more consistent with the actual situation.

5.3. **Comparison Analysis with the Existing TrIFNs Decision Method.**

To further illustrate the superiority of the method proposed in this paper, we apply the method [26] to the supply chain collaboration example in subsection 5.1.

Assume that the weights of the attributes are $\omega_1 = \omega_2 = \cdots = \omega_6 = 1/6$. Using the TrIFN weighted average (ITWAA) operator defined in [26], the score of the five alternatives is as follows: $S(A_1) = 0.0749, S(A_2) = 0.0626, S(A_3) = 0.0813, S(A_4) = 0.1673, S(A_5) = 0.1013$. Thus the ranking order of the five candidates by the method [26] is generated as follows: $A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$.

Obviously, the ranking order obtained by the proposed method in this paper is remarkably different from that obtained by the method[26]. The main reasons are as follows:

(1) The TrIFPWA operator defined in this paper takes the priority relationships into consideration, while the ITWAA operator defined in [26] did not consider the priority among the attributes.

(2) The method [26] calculated the score of TrIFNs to rank TrIFNs. Such a ranking method of TrIFNs is a single-index approach, which is not always feasible and effective. This paper, however, introduces the concepts of weighted lower and upper possibility means for membership and non-membership functions of TrIFNs as well as weighted possibility means. Since this paper takes into consideration not only the weighted possibility mean of membership function, but also the weighted possibility mean of non-membership functions for TrIFNs, the ranking method of TrIFNs proposed in this paper is a two-indices approach and more reasonable than that of [26].

(3) This paper sufficiently considers the different preferences between the lower and upper weighted possibility means for the DM, which makes the decision results be more consistent with the actual situation, while [26] did not consider the DM's preference (namely it assumes that the DM is risk-neutral).

6. Conclusions

TrIFNs are very useful for expressing ill-known quantities. This paper investigates the prioritized MADM problems in which the attribute values are in the form of TrIFNs. The weighted lower and upper possibility means for membership and non-membership functions of TrIFNs are firstly defined as well as weighted possibility means. Thereby, a new lexicographic ranking method for TrIFNs is developed. Four kinds of prioritized operators of TrIFNs are defined, including the TrIFPS operator, TrIFPWA operator, TrIFP-AND operator, and TrIFP-OR operator. The related properties for these operators are studied in depth. The notable characteristic of these operators is that they take the prioritization among attributes into consideration. The proposed MADM method is more reasonable and objective since it considers prioritization relationships among attributes. It can be applied to many areas, such as supplier management, risk investment and performance management. Meanwhile, the prioritized operators of TrIFNs provide a new tool of information fusion for solving decision problems under IF environments. The prioritized geometric operators of TrIFNs are also the challenging and interesting problems, which will be researched in the near future.

Acknowledgements. This work was partially supported by the National Natural Science Foundation of China (Nos. 71061006, 61263018 and 11461030), the Humanities Social Science Programming Project of Ministry of Education of China

(No. 09YGC630107), the Natural Science Foundation of Jiangxi Province of China (No. 20114BAB201012 and 20142BAB201011), Twelve five Programming Project of Jiangxi province Social Science (2013) (No. 13GL17) and the Excellent Young Academic Talent Support Program of Jiangxi University of Finance and Economics.

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