QUASI-CONTRACTIVE MAPPINGS IN FUZZY METRIC SPACES

A. AMINI-HARANDI AND D. MIHET

Abstract. We consider the concept of fuzzy quasi-contractions initiated by Ćirić in the setting of fuzzy metric spaces and establish fixed point theorems for quasi-contractive mappings and for fuzzy $\mathcal{H}$-contractive mappings on M-complete fuzzy metric spaces in the sense of George and Veeramani.The results are illustrated by a representative example.

1. Introduction

The notion of fuzzy metric space was introduced by Kramosil and Michalek [9] and later modified by George and Veeramani ([3]). In this paper we work in fuzzy metric spaces in the sense of George and Veeramani, defined as follows.

Definition 1.1. [3] A triple $(X, M, *)$ is called a fuzzy metric space (in the sense of George and Veeramani) if X is a nonempty set, $*$ is a continuous t-norm and $M : X^2 \times (0, \infty) \to [0, 1]$ is a fuzzy set satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1 \Leftrightarrow x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous for all $x, y \in X$.

If (GV4) is replaced by $M(x, z, max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$, then the space $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space. It should be noted that any non-Archimedean fuzzy metric space is a fuzzy metric space.
If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a fuzzy metric on X. George and Veeramani [3] proved that every fuzzy metric $(M, *)$ on X generates a topology τ_M on X which has as a base the family of sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $\epsilon \in (0, 1)$ and $t > 0$. It is well known that a sequence $(x_n)_{n \in \mathbb{N}}$ in X is convergent to $x \in X$ with respect to τ_M if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1, \ \forall \ t > 0$.

Definition 1.2. [3] Let $(X, M, *)$ be a fuzzy metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called Cauchy if $\lim_{m,n\to\infty} M(x_m, x_n, t) = 1$ for each $t > 0$. An M-complete fuzzy metric space is a fuzzy metric space in which every Cauchy sequence is convergent.

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Lemma 1.3. (see [14]) *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.*

The fixed point theory in fuzzy metric spaces started with the paper of Grabiec [4]. Later on, the concept of fuzzy contractive mappings, initiated by Gregori and Sapena in [5], have become of interest for many authors, see, e.g., the papers [5, 7, 10, 11, 12, 16].

The following class of fuzzy $\mathcal{H}$-contractive mappings has been recently introduced by Wardowski in [17], as a generalization of fuzzy contractions of Gregori and Sapena.

Definition 1.4. [17] Denote by $\mathcal{H}$ the family of all onto and strictly decreasing mappings $\eta : (0, 1] \to [0, \infty)$. Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in (0, 1)$ satisfying

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t)),\ \forall\ x, y \in X\ \forall\ t > 0.$$

For $\eta(t) = \frac{1}{t} - 1$ one obtains the class of fuzzy contractive mappings introduced by Gregori and Sapena in [5].

If $\eta \in \mathcal{H}$ then $\eta(1) = 0$ and η is continuous.

In [17] Wardowski formulated the conditions guaranteeing the existence and the uniqueness of the fixed point of a fuzzy $\mathcal{H}$-contractive mapping in M-complete fuzzy metric spaces in the sense of George and Veeramani.

Theorem 1.5. [17] *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $T : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that:*

(a) $\prod_{i=1}^{k} M(x, Tx, t_i) \neq 0$, *for all $x \in X$, $k \in \mathbb{N}$ and any sequence $(t_n) \subseteq (0, \infty)$, $t_n \downarrow 0$;*
(b) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, *for all $r, s \in \{M(x, Tx, t) : x \in X, t > 0\}$;*
(c) $\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\}$ *is bounded for all $x \in X$ and any sequence $(t_n) \subseteq (0, \infty)$, $t_n \downarrow 0$.*

Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $(T^n x_0)_{n \in \mathbb{N}}$ converges to x^*.

In our paper we present Wardowski's result in connection with the structure of the t-norm of the space (see also [13]). We also consider Ćirić's concept of quasi-contractions ([1]) in fuzzy metric setting and prove a fixed point theorem for this class of contractions in M-complete fuzzy metric spaces. It is worth mentioning that all earlier similar results refers to quasi-contractions in non-Archimedean (probabilistic) metric spaces.

2. Main Results

Our first theorem, although less general than Theorem 1.5, reveals the connection between the conditions of Wardowski's theorem and the structure of a strict t-norm.

Recall that a continuous t-norm $*$ is said to be strict if it is strictly increasing in each place on $(0, 1]^2$. Any strict t-norm $*$ is Archimedean, that is, $x * x < x$, for all $x \in (0,1)$ and positive, that is, $\forall a, b \in (0,1] \Rightarrow a * b > 0$. A t-norm $*$ is strict if and only if the semigroups $([0,1], *)$ and $([0,\infty], +)$ are isomorphic, that is, there exists a continuous, strictly decreasing function g from $[0,1]$ to $[0,\infty]$ with $g(0)=\infty, g(1)=0$, such that $x * y = g^{-1}(g(x)+g(y)),\ \forall x, y \in [0,1]$, where g^{-1} is the inverse of g. Such a function g is called an additive generator for $*$ and a t-norm generated by g will be denoted by $*_g$. For example, the t-norm $*_P$ is a strict t-norm generated by the function $g:[0,1] \to [0,\infty]$, $g(0)=\infty$, $g(s) = -\ln s \quad (s \neq 0)$. For more details about t-norms the reader is referred to [6] and [8].

Theorem 2.1. *Let $*_g$ be a strict t-norm. If $(X, M, *)$ is an M-complete fuzzy metric space under a t-norm $* \geq *_g$ and $T: X \to X$ is a $\mathcal{H}$-contractive mapping with respect to g with the property $M(x, Tx, 0+) = \lim_{t\to 0^+} M(x, Tx, t) > 0$ for all $x \in X$, then T has a unique fixed point.*

Proof. As the proof follows the lines of the proof of Theorem 3.2. in [17], we only sketch it. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$, $x_n = T^n x$ be the sequence of iterates of x. Then, for all $t > 0$, $n \in \mathbb{N}$, $g(M(x_n, x_{n+1}, t)) \leq k^n g(M(x, Tx, t))$. Let $m, n \in \mathbb{N}, m < n$ and $t > 0$ be given and let $\{a_i\}$ be a strictly decreasing sequence of positive numbers with $\sum_{i=1}^{\infty} a_i = 1$. Then

$$\begin{aligned} M(x_m, x_n, t) &\geq M(x_m, x_n, \sum_{i=m}^{n-1} a_i t) \geq \prod_{i=m}^{n-1} M(x_i, x_{i+1}, a_i t) \\ &\geq (*_g)_{i=m}^{n-1} M(x_i, x_{i+1}, a_i t). \end{aligned}$$

This implies

$$\begin{aligned} g(M(x_m, x_n, t)) &\leq \sum_{i=m}^{n-1} g(M(x_i, x_{i+1}, a_i t)) \leq \sum_{i=m}^{n-1} k^i g(M(x, Tx, a_i t)) \\ &\leq g(M(x, Tx, 0+)) \sum_{i=m}^{n-1} k^i, \end{aligned}$$

proving that (x_n) is Cauchy. The fact that the limit of (x_n) is the unique fixed point of T can be easily reproduced from the proof of Theorem 3.2. in [17]. □

Our main theorem is related to the concept of quasi-contraction, initiated by Lj. B. Ćirić in [1]. We define a fuzzy $\mathcal{H}$-quasi-contractive mapping as follows:

Definition 2.2. Let $(X, M, *)$ be a fuzzy metric space. A mapping $T: X \to X$ is said to be fuzzy $\mathcal{H}$-quasi-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in (0,1)$ satisfying the following condition:

$$\begin{aligned} \eta(M(Tx, Ty, t)) \leq k \max\{&\eta(M(x, y, t)), \eta(M(x, Tx, t)), \eta(M(y, Ty, t)), \\ &\eta(M(x, Ty, t)), \eta(M(y, Tx, t))\} \end{aligned} \tag{1}$$

for all $x, y \in X$ and any $t > 0$.

A similar definition, in the setting of non-Archimedean probabilistic Menger spaces, goes back to S.S. Chang (see [2]).

Theorem 2.3. *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $T : X \to X$ be a fuzzy $\mathcal{H}$-quasi-contractive mapping with respect to $\eta \in \mathcal{H}$ such that*

(a) $\tau \geq r * s \Rightarrow \eta(\tau) \leq \eta(r) + \eta(s)$, *for all* $r, s, \tau \in \{M(T^i x, T^j x, t) : x \in X, t > 0, i, j \in \mathbb{N}\}$;

(b) $\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.

Then T has a unique fixed point $x^* \in X$ and for each $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. For $A \subseteq X$ let $\delta_t(A) = \sup\{\eta(M(x, y, t)) : x, y \in A\}$ and for each $x \in X$ let

$$O(x, n) = \{x, Tx, ..., T^n x\} \text{ and } O(x, \infty) = \{x, Tx, ...\}, n \in \mathbb{N}.$$

Let $x \in X$ be arbitrary. Let $n \in \mathbb{N}$ and let $i, j \in \{1, 2, ..., n\}$. Then from (1), we obtain

$$\begin{aligned} \eta(M(T^i x, T^j x, t)) &= \eta(M(TT^{i-1}x, TT^{j-1}x, t)) \\ &\leq k \max\{\eta(M(T^{i-1}x, T^{j-1}x, t)), \eta(M(T^{i-1}x, T^i x, t)), \\ &\eta(M(T^{j-1}x, T^j x, t)), \eta(M(T^{i-1}x, T^j x, t)), \eta(M(T^{j-1}x, T^i x, t))\} \\ &\leq k\delta_t(O(x, n)), \end{aligned}$$

and so

$$\eta(M(T^i x, T^j x, t)) \leq k\delta_t(O(x, n)), \ i, j \in \{1, 2, ..., n\}, \ x \in X. \tag{2}$$

Now, if $\delta_t(O(x, n)) = \eta(M(T^{i_0}x, T^{j_0}x, t))$ for some $i_0, j_0 > 1$, then from (2) it follows $\delta_t(O(x, n)) \leq k\delta_t(O(x, n))$, that is, $\delta_t(O(x, n)) = 0$ and thus $\eta(M(T^i x, T^j x, t)) = 0$, $\forall i, j \leq n$. Particularly, $\eta(M(x, Tx, t)) = 0$, which implies $M(x, Tx, t) = 1$. From (GV2) it follows that $x = Tx$, that is, x is a fixed point for T. In the contrary case,

$$\delta_t(O(x, n)) = \eta(M(x, T^l x, t)), \tag{3}$$

for some $l \leq n$. Then, by choosing a strictly decreasing sequence of positive numbers $\{a_i\}$ with $\sum_{i=1}^{\infty} a_i = 1$, from (3), we deduce

$$\begin{aligned} \delta_t(O(x, n)) &= \eta(M(x, T^l x, t)) = \eta(M(x, T^l x, \sum_{i=1}^{\infty} a_i t)) \\ &\leq \eta(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)) + \eta(M(Tx, T^l x, \sum_{i=1}^{j} a_i t)), \ \forall \ j \end{aligned}$$

and so

$$\begin{aligned} \delta_t(O(x, n)) &\leq \limsup_{j\to\infty} \eta(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)) + \eta(M(Tx, T^l x, t)) \\ &\leq \limsup_{j\to\infty} \eta(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)) + k\delta_t(O(x, n)). \end{aligned}$$

Then

$$\delta_t(O(x, n)) \leq \frac{1}{1-k} \limsup_{j\to\infty} \eta(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)). \tag{4}$$

Let $n, m, n < m$ be any natural numbers. From (2), we get

$$\eta(M(T^n x, T^m x, t)) = \eta(M(TT^{n-1}x, T^{m-n+1}T^{n-1}x, t)) \leq k\delta_t(O(T^{n-1}x, m-n+1)). \tag{5}$$

From (3), there exists $k_1 \leq m-n+1$ such that

$$\delta_t(O(T^{n-1}x, m-n+1)) = \eta(M(T^{n-1}x, T^{k_1}T^{n-1}x, t)). \tag{6}$$

From (2),(5) and (6), we get

$$\begin{aligned}\eta(M(T^n x, T^m x, t)) &\leq k\eta(M(T^{n-1}x, T^{k_1}T^{n-1}x, t)) \\ = k\eta(M(TT^{n-2}x, T^{k_1+1}T^{n-2}x, t)) &\leq k^2\delta_t(O(T^{n-2}x, k_1+1)) \\ &\leq k^2\delta_t(O(T^{n-2}x, m-n+2)).\end{aligned}$$

Proceeding in this manner, we obtain

$$\eta(M(T^n x, T^m x, t)) \leq k^n\delta_t(O(x, m)). \tag{7}$$

From (4) and (7) it follows

$$\eta(M(T^n x, T^m x, t)) \leq \frac{k^n}{1-k}\limsup_{j\to\infty}\eta(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)). \tag{8}$$

From (8) and (b), we have

$$\lim_{m,n\to\infty}\eta(M(T^n x, T^m x, t)) = 0,$$

and so $\lim_{m,n\to\infty} M(T^n x, T^m x, t) = 1$. Thus, $(x_n)_{n\in\mathbb{N}}, x_n = T^n x$ is a Cauchy sequence. By the completeness of X there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Let $t > 0$ be given. Then, for each $\epsilon > 0$ and $n \in \mathbb{N}$, we have

$$M(x^*, Tx^*, t+\epsilon) \geq M(x^*, T^{n+1}x^*, \epsilon) * M(Tx^*, T^{n+1}x^*, t)$$

and hence

$$\begin{aligned}\eta(M(x^*, Tx^*, t+\epsilon)) &\leq \eta(M(x^*, T^{n+1}x^*, \epsilon)) + \eta(M(Tx^*, T^{n+1}x^*, t)) \\ &\leq \eta(M(x^*, T^{n+1}x^*, \epsilon)) + k\max\{\eta(M(x^*, T^n x^*, t)), \eta(M(x^*, Tx^*, t)), \\ &\quad \eta(M(T^n x^*, T^{n+1}x^*, t)), \eta(M(x^*, T^{n+1}x^*, t)), \eta(M(T^n x^*, Tx^*, t))\}.\end{aligned}$$

Letting $n \to \infty$ (having in mind Lemma 1.3) we obtain

$$\eta(M(x^*, Tx^*, t+\epsilon)) \leq k\eta(M(x^*, Tx^*, t)),$$

and so

$$\eta(M(x^*, Tx^*, t)) = \lim_{\epsilon\to 0^+}\eta(M(x^*, Tx^*, t+\epsilon)) \leq k\eta(M(x^*, Tx^*, t)).$$

Thus $\eta(M(x^*, Tx^*, t)) = 0$, implying $M(x^*, Tx^*, t) = 1$.

To show the uniqueness assume that y^* is a fixed point of T. Then, for all $t > 0$,

$$\begin{aligned} \eta(M(x^*, y^*, t)) &= \eta(M(Tx^*, Ty^*, t)) \\ &\leq k \max\{\eta(M(x^*, y^*, t)), \eta(M(x^*, Tx^*, t)), \\ &\eta(M(y^*, Ty^*, t)), \eta(M(x^*, Ty^*, t)), \eta(M(y^*, Tx^*, t))\} \\ &= k\eta(M(x^*, y^*, t)). \end{aligned}$$

This gives $M(x^*, y^*, t) = 1$, that is, $x^* = y^*$. □

We illustrate our result by the following example.

Example 2.4. Let $X = [0, 1]$, and let $M(x, y, t) = (\frac{t+1}{t+2})^{|x-y|}$ for all $x, y \in X$ and each $t > 0$. Then (X, M, T_P) is a M-complete fuzzy metric space. Define the map $T : X \to X$ by

$$T(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } 0 < x \leq 1. \end{cases}$$

Obviously, $\frac{1}{2}$ is the unique fixed point of T.

We show that T is not a fuzzy $\mathcal{H}$-contractive map. On the contrary, assume that there exists $\eta \in \mathcal{H}$ such that

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t)) \ \forall\ x, y \in X \ \forall\ t > 0, \tag{9}$$

where $k \in [0, 1)$ is a constant. Let $x = 0$, $t = 1$ and let $0 < y \leq 1$. Then from (9), we get $\eta((\frac{2}{3})^{\frac{1}{4}}) \leq k\eta((\frac{2}{3})^y)$ and so

$$\eta((\frac{2}{3})^{\frac{1}{4}}) \leq k \lim_{y \to 0^+} \eta((\frac{2}{3})^y) = k\eta(1) = 0,$$

a contradiction. Thus, we cannot invoke Theorem 1.5 to show that the mapping T has a fixed point.

On the other hand, from the equality

$$\ln M(\frac{1}{4}, \frac{1}{2}, t) = \frac{1}{2} \ln M(0, \frac{1}{2}, t), \forall t > 0$$

it immediately follows that if $\eta(s) = -lns \quad (s \in (0, 1])$, then for each $x, y \in X$ and any $t > 0$,

$$\begin{aligned} \eta(M(Tx, Ty, t)) \leq \frac{1}{2} \max\{&\eta(M(x, y, t)), \eta(M(x, Tx, t)), \\ &\eta(M(y, Ty, t)), \eta(M(x, Ty, t)), \eta(M(y, Tx, t))\}, \end{aligned}$$

that is, T is a fuzzy $\mathcal{H}$-quasi-contractive mapping with respect to η.

As $g(s) = -\ln s$ is the generator of the strict t-norm $*_P$, all the conditions of Theorem 2.3 are fulfilled.

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A. Amini-Harandi, Department of Mathematics, University of Isfahan, Isfahan, 81745-163, Iran

E-mail address: a.amini@sci.ui.ac.ir

D. Mihet*, West University of Timisoara, Faculty of Mathematics and Computer Science, Bv. V. Parvan 4, 300223, Timisoara, Romania

E-mail address: mihet@math.uvt.ro

*Corresponding author