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ON FUZZY HYPERIDEALS OF Γ-HYPERRINGS

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Abstract. The aim of this paper is the study of fuzzy Γ-hyperrings. In this regard the notion of ν-fuzzy hyperideals of Γ-hyperrings are introduced and basic properties of them are investigated. In particular, the representation theorem for ν-fuzzy hyperideals are given and it is shown that the image of a ν-fuzzy hyperideal of a Γ-hyperring under a certain conditions is two-valued. Finally, the product of ν-fuzzy hyperideals are studied.

1. Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups, began to analysis their properties and applied them to groups, rational algebraic functions [16]. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties and applied mathematics (for example see [5], [6], [22]).

Also, following the introduction of fuzzy sets by L. A. Zadeh in 1965 [23], the fuzzy set theory were developed by Zadeh himself and many researchers in mathematics and it was applied in many pure and applied areas. For example the concept of a fuzzy group was introduced by A. Rosenfeld and the notion of fuzzy ideal in a ring introduced and studied by W. J. Liu [15]. Recently fuzzy set theory have been had good develop in hyperstructures theory (for example see [7], [8], [9], [10], [11],[24]).

The notion of $\Gamma-$rings introduced by N. Nobosawa in [19] and immediately after him in 1966, Barnes extended this notion and obtained more results [4]. Kyuno investigated the new aspects of Γ-rings such as, prime Γ-rings and left and right unities of Γ-rings. Also in recent years Ozturk, Y. B. Jun and C. Y. Lee in [12] and [20] applied the concept of fuzzy sets to the theory of Γ-rings.

In this paper, first we introduce the notion of (ν-)fuzzy hyperideals of Γ-hyperrings and, then we obtain some related basic results. We characterize (ν-)fuzzy hyperideals based on their level subsets and associate a new (ν-fuzzy) hyperideal from a given fuzzy hyperideal of a Γ-hyperring. In particular, we show that under certain conditions ν-fuzzy hyperideals of Γ-hyperrings are two-valued. Finally we describe ν-fuzzy hyperideals of product of Γ-hyperrings.

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2. **Preliminaries**

In this section we gather all definitions and simple properties of $\Gamma-$hyperrings that we require in the next notions.

Let H be a nonempty set. A map $+ : H \times H \longrightarrow P_*(H)$ is called *hyperoperation* or *join operation*, where $P_*(H)$ denotes the set of all nonempty subsets of H.

Definition 2.1. [6] A nonempty set M together a hyperoperation + is called a *polygroup* if the following conditions are satisfied:
(1) for all $x, y, z \in M$, $(x + y) + z = x + (y + z)$;
(2) for all $x \in M$ there exist an unique element $e \in M$ such that $e+x = x = x+e$ (we denote e by 0) ;
(3) for all $x \in M$ there exists an unique element $x' \in M$ such that $e \in x+x' \cap x'+x$ (we denote x' by $-x$);
(4) for all $x, y, z \in M$, $z \in x + y \Longrightarrow x \in z - y \Longrightarrow y \in z - x$.
By $U <_p M$, we mean U is a subpolygroup of M. We denote the set of all subpolygroup of M, by $SP(M)$. A *canonical hypergroup* is a commutative polygroup.

Definition 2.2. [15, 19] An algebraic structure $(R, +, .)$ is called a *hyperring* if the following statements are satisfied:
(i) $(R, +)$ is a canonical hypergroup ;
(ii) $(R, .)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x.0 = 0 = 0.x$;
(iii) The multiplication is distributive with respect to the hyperoperation +, i.e., $x.(y + z) = x.y + x.z$ and $(y + z).x = y.x + z.x \ \forall x, y, z \in R$.

Remark 2.3. (i) It can be easily proved that zero is unique.
(ii) For simplicity of notation, sometimes we write xy instead of $x.y$ in Definition 2.2.
(iii) If $A, B \subseteq R$ and $x \in R$, then $A + B = \bigcup\{a + b | a \in A, b \in B\}$. Also, $A + x$ is used for $A + \{x\}$.
(iv) By axioms of Definition 2.2, it is easy to see that, $-(-x) = x$ and $-(x+y) = -x - y$, where $-A = \{-a \ | \ a \in A\}$. Also, $(a + b).(c + d) \subseteq a.c + b.c + a.d + b.d.$

Definition 2.4. Let R be a hyperring. Then
(i) R is *commutative* if $x.y = y.x \ \forall x, y \in R$;
(ii) R is called *with identity*, if there exists an element, say $1 \in R$, such that $1.x = x = x.1, \forall x \in R$;
(iii) A nonempty subset A of R is said to be a subhyperring of R if $(A, +, .)$ is itself a hyperring. If $R \setminus \{0\}$ is a multiplicative group, then $(R, +, .)$ is a hyperfield.

Example 2.5. [18] (i) Let $(A, +, .)$ be a ring and N a normal semigroup of $(A, .)$. Then the multiplicative classes $\overline{x} = xN, x \in A$ form a partition of A. Let $\overline{A} = A/N$ be the set of these classes. If we define the product $\overline{x} \odot \overline{y}$ in $\overline{A}$ of $\overline{x}, \overline{y} \in \overline{A}$ as equal to their product as subsets of A, and their sum $\overline{x} \oplus \overline{y}$ in $\overline{A}$ as the set of all $\overline{z} \in \overline{A}$

contained in their sum as subsets of A, i.e.,

$$\overline{x} \oplus \overline{y} = \{\overline{z} | z \in \overline{x} + \overline{y}\} \text{ and } \overline{x} \odot \overline{y} = \overline{x.y}.$$

Then $(\overline{A}, \oplus, \odot)$ is a hyperring.

(ii) Let R be a commutative ring with identity. Letting $\overline{R} = \{\overline{x} = \{x, -x\} | x \in R\}$. Then $\overline{R}$ is a hyperring with respect to the hyperoperation $\overline{x} \oplus \overline{y} = \{\overline{x+y}, \overline{x-y}\}$ and multiplication $\overline{x} \odot \overline{y} = \overline{x.y}$.

Definition 2.6. (i) A nonempty subset I of a hyperring R is called a (resp. left) *right hyperideal* of R if (resp. $x.r \in I$) $r.x \in I\ \forall r \in R,\ \forall x \in I$;

(ii) I is called a *hyperideal* if I is both left and right hyperideal;

(iii) A proper hyperideal I of R ($I \neq R$) is called a *prime hyperideal* if $a.b \in I$ implies that $a \in I$ or $b \in I$ (for a study of prime hyperideals and prime subhypermodules see [36]). The set of all prime hyperideal of R is called the *prime spectrum* of R and it is denoted by $Spec(R)$.

Definition 2.7. Let $(M, +)$ and $(\Gamma, +)$ be canonical hypergroups. Then M is said to be a $\Gamma-$*hyperring* if there exists a mapping $. : M \times \Gamma \times M \to P_*(M)$ such that the following conditions are satisfied:

(1) $(x + y)\alpha z \subseteq x\alpha z + y\alpha z$, $x\alpha(y + z) \subseteq x\alpha y + x\alpha z,\ \forall x, y, z \in M,\ \forall \alpha \in \Gamma$;

(2) $x(\alpha + \beta)y \subseteq x\alpha y + x\beta y,\ \forall x, y \in M,\ \forall \alpha, \beta \in \Gamma$;

(3) $(x\alpha y)\beta z = x\alpha(y\beta z),\ \forall x, y, z \in M,\ \forall \alpha, \beta \in \Gamma$.

If in Definition 2.2, we replace all inclusions by equality, then M is called a *strong* Γ-*hyperring*.

Definition 2.8. A right (resp. left) *hyperideal* of Γ-hyperring M is a subpolygroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). Also if Δ is a subpolygroup of Γ, then the subpolygroup I of M is said to be a right (left) Δ-hyperideal if $I\Delta M \subseteq I$ (resp. $M\Delta I \subseteq I$). By $U <_h M$, we mean U is a hyperideal of Γ-hyperring M. Also we denote the set of all hyperideals of M by $HI(M)$.

Clearly every hyperideal of a Γ-hyperring is a Δ-hyperideal for some $\Delta \subseteq \Gamma$.

We use $I = [0, 1]$, the real unit interval as a chain with the usual ordering, in which $\bigwedge$ stands for minimum or infimum (inf)(or intersection) and $\bigvee$ stands for maximum or supremum(sup) (or union), for the degree of membership. A fuzzy subset of a given set X is a mapping $\mu : X \longrightarrow I$. We denote the set of all fuzzy subset of X by $FS(X)$, that is $FS(X) = \{\mu |\ \mu : X \longrightarrow [0, 1] \text{ is a function}\}$. For $\mu \in FS(X)$, the level subset of μ is defined by $\mu_t = \{x \in X |\ \mu(x) \geq t\}$. For a fuzzy set μ of X we denote by $Im(\mu)$ the image of μ.

Definition 2.9. [20] Let $(M, +)$ be a canonical hypergroup and $\mu \in FS(M)$. Then μ is a *fuzzy subpolygroup* of M if for all $a, b \in M$ the following conditions hold:

(1) $\bigwedge\limits_{z \in a+b} \mu(z) \geq \mu(a) \wedge \mu(b)$;

(2) $\mu(-a) \geq \mu(a)$.

By $\mu <_{FP} M$, we mean μ is a fuzzy subpolygroup of M. Also we denote the set of all fuzzy subpolygroups of M, by $FP(M)$.

3. ν-Fuzzy Hyperideals of Γ-Hyperrings

In the sequel by M we mean a $\Gamma-$hyperring.

Definition 3.1. (i) A fuzzy subset μ of M is said to be a left (resp. right) *fuzzy hyperideal* of M if and only if for all $x, y \in M$ and $\gamma \in \Gamma$ we have

(1) $\mu \in FP(M)$;

(2) $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(y) \quad$ (resp. $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(x)$).

By $\mu <_{FHI} M$, we mean μ is a fuzzy hyperideal of M. Also we denote the set of all fuzzy hyperideals of M by $FHI(M)$.

(ii) A fuzzy subset μ of M is said to be a left (resp. right) ν-*fuzzy hyperideal* of M if and only if for all $x, y \in M$ and $\gamma \in \Gamma$ we have

(1) $\mu \in FP(M)$ and $\nu \in FP(\Gamma)$;

(2) $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(y) \wedge \nu(\gamma) \quad$ (resp. $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(x) \wedge \nu(\gamma)$).

By $\mu <_{FHI_\nu} M$, we mean μ is a ν-fuzzy hyperideal of M. Also we denote the set of all ν-fuzzy hyperideals of M by $FHI_\nu(M)$.

Clearly, every fuzzy hyperideal is a ν-fuzzy hyperideal, for some $\nu \in FP(\Gamma)$, by letting $\nu = \chi_\Gamma$, where χ_Γ denotes the characteristic function of Γ.

Example 3.2. Let $(M, +, \cdot)$ be an hyperring and Γ be an hyperideal of M. Define $\circ : M \times \Gamma \times M \longrightarrow \mathcal{P}^*(M)$ by $(a, \gamma, b) \mapsto a \circ \gamma \circ b = \{z \in M \mid z \in a.\gamma.b\}$. Then it is easy to verify that M is a strong $\Gamma-$hyperring. Also if I and Δ are hyperideals of hyperring $(M, +, .)$ and $\Delta \subseteq \Gamma$, then I is a $\Delta-$hyperideal of $\Gamma-$hyperring M, since $I\Delta M \subseteq I$ and $M\Delta I \subseteq I$. Now define μ and ν on I and Δ respectively as follow:

$$\mu(x) = \begin{cases} 0.8 & \text{if } x \in I, \\ 0 & \text{Otherwise} \end{cases} \qquad \nu(\delta) = \begin{cases} 0.5 & \text{if } \delta \in \Delta, \\ 0 & \text{Otherwise} \end{cases}$$

It is easy to verify that μ and ν are fuzzy subpolygroups of M and Γ respectively. Suppose that $x, y \in M$ and $\delta \in \Delta$ and $z \in x \circ \delta \circ y$. We can consider two cases:

(1) $x \in I$ or $y \in I$ then we can say that $x \circ \delta \circ y \subseteq I$ and so for all $z \in x \circ \delta \circ y$, we have $\mu(z) = 0.8 \geq 0.5 = (\mu(x) \vee \mu(y)) \wedge \nu(\delta)$.

(2) $x, y \notin I$ then $\mu(z) \geq 0 = (\mu(x) \vee \mu(y)) \wedge \nu(\delta)$.

Therefore μ is a $\nu-$fuzzy hyperideal of M as a $\Gamma-$hyperring.

Example 3.3. Let R be a hyperring and let $M_{m,n}\ (R)$ be the set of all matrices by the size $m \times n$ with entries of R . Define $\circ : M_{m,n}\ (R) \times M_{n,m}\ (R) \times M_{m,n}\ (R) \longrightarrow \mathcal{P}^*(M_{m,n}\ (R))$ by:

$$A \circ B \circ C = \{Z \in M_{m,n}(R) \,|Z \in ABC, \ \ A, C \in M_{m,n}(R), \ B \in M_{n,m}\ (R)\} .$$

Then it easy to verify that $M_{m,n}(R)$ is a $M_{n,m}(R)-$hyperring. Also if I and J are hyperideal of hyperring $(R, +, .)$, then it is easy to verify that $M_{m,n}(I)$ is a $M_{n,m}(J)-$hyperideal of $M_{m,n}(R)$ since $M_{m,n}(I) \circ M_{n,m}(J) \circ M_{m,n}(R) \subseteq M_{m,n}(I)$ (by Definition 2.3) and $M_{m,n}(R) \circ M_{n,m}(J) \circ M_{m,n}(I) \subseteq M_{m,n}(I)$ (by Definition

2.3). Now define μ and ν on $M_{m,n}(I)$ and $M_{n,m}(J)$ respectively as follow:

$$\mu(X)=\begin{cases} 4/5 & \text{if } X\in M_{m,n}(I),\\ 7/10 & \text{if } X\notin M_{m,n}(I)\end{cases} \qquad \nu(Y)=\begin{cases} 1/2 & \text{if } Y\in M_{n,m}(J),\\ 1/4 & \text{if } Y\notin M_{n,m}(J)\end{cases}$$

It is routine to check that μ is a $\nu-$fuzzy hyperideal of $M_{m,n}(R)$ as an $M_{n,m}(R)-$hyperring.

Lemma 3.4. Let μ be a ν-fuzzy hyperideal of M. Then $\mu(x)\leq\mu(0_M)$, for all $x\in M$.

Proof. For any $x\in M$ we have $0_M\in x-x$. Thus $\mu(0_M)\geq\ \mu(x)\wedge\mu(-x)\ =\mu(x)$. □

Theorem 3.5.(Representation Theorem) Let μ be a fuzzy set in a Γ-hyperring M. Then μ is a left (resp. right) ν-fuzzy hyperideal of M if and only if each level subset μ_t of μ is a left (resp. right) ν_t-hyperideal of M, for each $t\in[0,\mu(0_M)\wedge\nu(0_\Gamma)]$.

Proof. Suppose that μ is a left(resp. right) ν-fuzzy hyperideal of M and let $\mu_t\neq\emptyset$. We have $\mu_t\subseteq M$, then for any $x,y,z\in\mu_t$, $(x+y)+z=x+(y+z)$. We show that

$$\forall a\in\mu_t,\ \exists 0_M\in\mu_t:\ a+0_M=a.$$

Since $a\in\mu_t$ and $\mu_t\subseteq M$, so $a\in M$ then there exists an unique $0_M\in M$ such that $a+0_M=a$. Also we have $0_M\in a-a$, thus $\mu(0_M)\geq\mu(a)\wedge\mu(-a)\geq t$, therefore $0_M\in\mu_t$. Similarly for all $x\in\mu_t$, there exists $-x\in\mu_t$, such that $0_M\in x-x$. We now show that

$$M\nu_t\mu_t\subseteq\mu_t \text{ (resp. } \mu_t\nu_tM\subseteq\mu_t).$$

Let $m\in M,\gamma\in\nu_t$, $u\in\mu_t$, and $z\in m\gamma u$, then we have

$$\mu(z)\geq\bigwedge_{z\in m\gamma u}\mu(z)\geq\mu(u)\wedge\nu(\gamma)\geq t;$$

thus $z\in\mu_t$. Therefore $M\nu_t\mu_t\subseteq\mu_t$. Similarly we can prove that $\mu_t\nu_tM\subseteq\mu_t$.

Conversely, suppose that μ_t is a left (resp. right) ν_t-hyperideal of M. We show that for all $a,b\in M$, $\bigwedge\limits_{z\in a+b}\mu(z)\geq\mu(a)\wedge\mu(b)$.

If $a,b\in M$, then there exist $t_1,t_2\in[0,1]$, $\mu(a)=t_1$, $\mu(b)=t_2$. Put $t=t_1\wedge t_2$, thus $a,b\in\mu_t$, and $a+b\subseteq\mu_t$. Also if $z\in a+b$, we have $\mu(z)\geq t=\mu(a)\wedge\mu(b)$, therefore $\bigwedge\limits_{z\in a+b}\mu(z)\geq\mu(a)\wedge\mu(b)$. Obviously for all $x\in M$, we have $\mu(x)\geq\mu(-x)$.

Let $x,y\in M$ and $\gamma\in\Gamma$ and $\mu(y)=t_1$ and $\nu(\gamma)=t_3$. Put $t=t_1\wedge t_3$, thus $y\in\mu_t$ and $\gamma\in\nu_t$. So $x\gamma y\subseteq\mu_t$, since μ_t is a ν_t-hyperideal. Then for all $z\in x\gamma y$ we have

$$\mu(z)\geq t=\mu(y)\wedge\nu(\gamma).$$

Similarly, we obtain that $\bigwedge\limits_{z\in x\gamma y}\mu(z)\geq\mu(x)\wedge\nu(\gamma)$. This completes the proof. □

Example 3.6.

(1) Let $I_1 \subset I_2 \subset ... \subset I_n \subset ...$ be a strictly increasing sequence of left (resp. right)hyperideals of an arbitrary Γ-hyperring M and $\{t_j\}_{j=1}^{\infty}$ be a strictly increasing sequence in $[0, 1]$. Define μ on M as follows:

$$\mu(x) = t_j \text{ if } x \in I_j \backslash I_{j-1}, \ \ where \ \ t_{j-1} < t_j, j = 1, 2, ... \ \ and \ \ \mu(x) = 0, if x \in M \backslash \cup_{j=1}^{\infty} I_j,$$

It is easy to verify that $\mu_{t_{j+1}} \subseteq \mu_{t_j}$ and the only level subsets of M are M, and $\mu_{t_j} = I_j, j = 1, 2, ...$ Then by Theorem 3.5 μ is a left (resp. right) fuzzy hyperideal of M.

(2) Let $I_1 \supset I_2 \supset ... \supset I_n \supset ...$ be a strictly decreasing sequence of left (resp. right)hyperideals of an arbitrary Γ-hyperring M and $\{t_j\}_{j=1}^{n}$ be a strictly decreasing sequence in $[0, 1]$.

Define fuzzy subset μ on V by $\mu(x) = t_{j-1}$, if $x \in I_{j-1} \backslash I_j$ where

$$t_{j-1} > t_j, j = 1, 2, 3, ... \text{ and } \mu(x) = 1 \text{ if } x \in \cap_j^{\infty} I_j.$$

Again by Theorem 3.5 it is easy to verify that μ is a left (resp. right) fuzzy hyperideal of M, since the only level subsets of M are M and $\mu_{t_j} = I_j, j = 1, 2, ...$

(3) In Example 3.3, μ is ν-fuzzy hyperideal of $M_{m,n}(R)$, since $\mu_{4/5} = M_{m,n}(R)$ and $\mu_{7/10} = M_{m,n}(I)$ and $\nu_{4/5} = \nu_{7/10} = \nu_{1/4} = M_{n,m}(J)$, which are hyperideals.

Lemma 3.7. If $\mu \in FHI_\nu(M)$ and $\bigwedge_{t \in x-y} \mu(t) = \mu(0_M)$, then $\mu(x) = \mu(y)$.

Proof. We have

$$\mu(x) \geq \bigwedge_{t \in x-y+y} \mu(t) \geq (\bigwedge_{t' \in x-y} \mu(t')) \wedge \mu(y) = \mu(0_M) \wedge \mu(y) = \mu(y).$$

Then, $\mu(x) \geq \mu(y)$. Similarly, we have $\mu(y) \geq \mu(x)$. Therefore $\mu(x) = \mu(y)$. $\square$

In next propositions we construct new (ν-fuzzy) hyperideals by given fuzzy hyperideals of Γ-hyperrings.

Proposition 3.8. Let μ be a left (resp. right) ν-fuzzy hyperideal of M and $\mu(0_M) = \nu(0_\Gamma)$. Then the set

$$M_\mu = \{x \in M | \ \mu(x) = \mu(0_M)\}$$

is a left (resp. right) $\nu_{\mu(0_M)}$-hyperideal of M.

Proof. A direct verification shows that M_μ is a canonical hypergroup and $M_\mu \subseteq M$. We show that $M\nu_{\mu(0_M)}M_\mu \subseteq M_\mu$. Let $z \in x\gamma y$ such that $x \in M, \gamma \in \nu_{\mu(0_M)}$ and $y \in M_\mu$. We have $\mu(z) \geq \mu(y) \wedge \nu(\gamma) \geq \mu(0_M)$. Then by Lemma 3.4, $\mu(z) = \mu(0_M)$, thus $z \in M_\mu$. Similarly, we obtain $M_\mu \nu_{\mu(0_M)} M \subseteq M_\mu$. $\square$

Proposition 3.9. Let μ be a left (resp. right) ν-fuzzy hyperideal of M, then

$$supp(\mu) = \{x \in M | \ \mu(x) > 0\}$$

is a left (resp. right) $supp(\nu)$-hyperideal of M.

Proof. The proof is similar to the proof of Proposition 3.5 by some modification. □

Proposition 3.10. If μ is a ν-fuzzy hyperideal of Γ-hyperring M, then

$$R(\mu)(x) = \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(z) \mid z \in nx, \exists n \in \mathbb{N}\}\}$$

is a ν-fuzzy hyperideal of M.

Proof. Let $z \in x + y$. We prove that

$$R(\mu)(z) \geq R(\mu)(x) \wedge R(\mu)(y).$$

For this we have

$$\begin{aligned}
R(\mu)(z) &= \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(a)|\ a \in nz, \exists n \in \mathbb{N}\}\} \\
&\geq \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(a)|a \in nx + ny, \exists n \in \mathbb{N}\}\} \qquad (\text{since } z \in x + y) \\
&\geq \bigvee_{n\in\mathbb{N}} [\{\bigwedge\{\mu(t_1)|t_1 \in nx, \exists n \in \mathbb{N}\}\} \wedge \{\bigwedge\{\mu(t_2)|t_2 \in ny, \exists n \in \mathbb{N}\}\}] \\
&= [\bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(t_1)|t_1 \in nx, \exists n \in \mathbb{N}\}\}] \wedge [\bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(t_2)|t_2 \in ny, \exists n \in \mathbb{N}\}\}] \\
&= R(\mu)(x) \wedge R(\mu)(y).
\end{aligned}$$

Also we have

$$\begin{aligned}
R(\mu)(x) &= \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(z)|z \in nx, \exists n \in \mathbb{N}\}\} \\
&\geq \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(-z)| -z \in n(-x), \exists n \in \mathbb{N}\}\} \\
&= R(\mu)(-x).
\end{aligned}$$

Now suppose that $z \in x\gamma y$. We prove that

$$R(\mu)(z) \geq (R(\mu)(x) \vee R(\mu)(y)) \wedge \nu(\gamma).$$

For this we have

$$\begin{aligned}
R(\mu)(z) &= \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(a)|a \in nz, \exists n \in \mathbb{N}\}\} \\
&\geq \bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(a)|a \in (nx)\gamma y, \exists n \in \mathbb{N}\}\} \qquad (\text{since } z \in x\gamma y) \\
&\geq [\bigvee_{n\in\mathbb{N}} \{\bigwedge\{\mu(b)|b \in nx, \exists n \in \mathbb{N}\}\}] \wedge \nu(\gamma) \qquad (\text{since } \mu \in FHI(M)) \\
&= R(\mu)(x) \wedge \nu(\gamma).
\end{aligned}$$

Similarly, we can prove that $R(\mu)(z) \geq R(\mu)(y) \wedge \nu(\gamma)$. Therefore $R(\mu) \in FHI_\nu(M)$. □

Proposition 3.11. Let $\mu \in FHI_\nu(M)$ and $\mu^+(x) = \mu(x) + 1 - \mu(0_M)$.

(i) Then μ^+ is a ν-fuzzy hyperideal of M.

(ii) If $\mu(0_M) = \nu(0_\Gamma)$, then μ^+ is a ν^+-fuzzy hyperideal of M, where $\nu^+(x) = \nu(x) + 1 - \nu(0_\Gamma)$.

Proof. (i) Let $z \in x + y$, then we have

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(0_M) \\ &\geq (\mu(x) \wedge \mu(y)) + 1 - \mu(0_M) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= (\mu(x) + 1 - \mu(0_M)) \wedge (\mu(y) + 1 - \mu(0_M)) \\ &= \mu^+(x) \wedge \mu^+(y). \end{aligned}$$

Also we have

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(0_M) \\ &\geq \mu(-z) + 1 - \mu(0_M) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= \mu^+(-z). \end{aligned}$$

Now suppose $z \in x\gamma y$, then we have

$$\mu^+(z) = \mu(z) + 1 - \mu(0_M) \geq (\mu(x) \wedge \nu(\gamma)) + 1 - \mu(0_M). \tag{1}$$

We consider the following cases.

Case 1. If $\mu(x) \geq \nu(\gamma)$, then

$$(\mu(x) \wedge \nu(\gamma)) + 1 - \mu(0_M) = \nu(\gamma) + 1 - \mu(0_M) \tag{2}$$

we have $\mu(x) + 1 - \mu(0_M) \geq \mu(x) \geq \nu(\gamma)$, then

$$(\mu(x) + 1 - \mu(0_M)) \wedge \nu(\gamma) = \nu(\gamma). \tag{3}$$

Then from (1), (2) and (3) it is concluded that $\mu^+(z) \geq \nu(\gamma) + 1 - \mu(0_M) \geq \nu(\gamma)$. Thus $\mu^+(z) \geq \mu^+(x) \wedge \nu(\gamma)$.

Case 2. If $\mu(x) \leq \nu(\gamma)$, then

$$\begin{aligned} \mu^+(z) &\geq (\mu(x) \wedge \nu(\gamma)) + 1 - \mu(0_M) \\ &= \mu(x) + 1 - \mu(0_M) \\ &= \mu^+(x) \\ &\geq \mu^+(x) \wedge \nu(\gamma). \end{aligned}$$

Similarly to the both cases 1 and 2 we can obtain $\mu^+(z) \geq \mu^+(y) \wedge \nu(\gamma)$. Thus

$$\mu^+(z) \geq (\mu^+(x) \vee \mu^+(y)) \wedge \nu(\gamma).$$

Therefore $\mu^+ \in FHI_\nu(M)$.

(ii) Let μ is a ν-fuzzy hyperideal of $\Gamma-$hyperring M and $\mu(0_M) = \nu(0_\Gamma)$ and $z \in x - y$, for all $x, y \in M$. Obviously $\mu^+(z) \geq \mu^+(x) \wedge \mu^+(y)$. Suppose that $z \in x\gamma y$, for $x, y \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(0_M) \\ &\geq \{[\mu(x) \vee \mu(y)] \wedge \nu(y)\} + 1 - \mu(0_M) \\ &= [(\mu(x) + 1 - \mu(0_M)) \vee (\mu(y) + 1 - \mu(0_M))] \wedge (\nu(\gamma) + 1 - \mu(0_M)) \\ &= [\mu^+(x) \vee \mu^+(y)] \wedge \nu^+(\gamma) \quad \text{(since } \mu(0_M) = \nu(0_\Gamma)\text{)}. \end{aligned}$$

Therefore μ^+ is a ν^+-fuzzy hyperideal of M. □

Proposition 3.13. Let M be a Γ-hyperring and $\mu \in FHI_\nu(M)$.

(i) If $f : [0, \mu(0_M) \vee \nu(0_\Gamma)] \longrightarrow [0, 1]$ is an increasing map, then $\mu_f : M \longrightarrow [0, 1]$ defined by $\mu_f(x) = f(\mu(x))$ for all $x \in M$ is a ν_f-fuzzy hyperideal of M, where $\nu_f : \Gamma \longrightarrow [0, 1]$ is defined by $\nu_f(\gamma) = f(\nu(\gamma))$ for all $\gamma \in \Gamma$.

(ii) If $\mu(0_M) = \nu(0_\Gamma)$ and $\widetilde{\mu} : M \longrightarrow [0, 1]$ defined by $\widetilde{\mu}(x) = \mu(x)\mu(0_M)$ for all $x \in M$ is a $\widetilde{\nu}$-fuzzy hyperideal of M, where $\widetilde{\nu} : \Gamma \longrightarrow [0, 1]$ is defined by $\widetilde{\nu}(\gamma) = \nu(\gamma)\nu(0_\Gamma)$ for all $\gamma \in \Gamma$.

Proof. (i) Let $z \in x + y$ then $\mu(z) \geq \mu(x) \wedge \mu(y)$. Since f is increasing then, $f(\mu(z)) \geq f(\mu(x)) \wedge f(\mu(y))$, therefore $\mu_f(z) \geq \mu_f(x) \wedge \mu_f(y)$. Also we have

$$\mu_f(z) = f(\mu(z)) \geq f(\mu(-z)) = \mu_f(-z).$$

Suppose that $z \in x\gamma y$, then we have

$$\begin{aligned} \mu(z) \quad &\geq \quad (\mu(x) \vee \mu(y)) \wedge \nu(\gamma) \qquad \text{(since } \mu \in FHI_\nu(M)\text{)} \\ &\Longrightarrow \quad f(\mu(z)) \geq [f(\mu(x)) \vee f(\mu(y))] \wedge f(\nu(\gamma)) \quad \text{(since } f \text{ is increasing)} \\ &\Longrightarrow \quad \mu_f(z) \geq (\mu_f(x) \vee \mu_f(y)) \wedge \nu_f(\gamma). \end{aligned}$$

Therefore $\mu_f \in FHI_{\nu_f}(M)$.

(ii) Let $z \in x + y$ then we have

$$\begin{aligned} \widetilde{\mu}(z) &= \mu(z)/\mu(0_M) \\ &= (1/\mu(0_M))\mu(z) \\ &\geq (1/\mu(0_M))(\mu(x) \wedge \mu(y)) \qquad \text{(since } \mu \in FHI_\nu(M)\text{)} \\ &= (\mu(x)/\mu(0_M)) \wedge (\mu(y)/\mu(0_M)) \\ &= \widetilde{\mu}(x) \wedge \widetilde{\mu}(y). \end{aligned}$$

Also we have

$$\begin{aligned} \widetilde{\mu}(z) &= (1/\mu(0_M))\mu(z) \\ &\geq (1/\mu(0_M))\mu(-z) \qquad \text{(since } \mu \in FHI_\nu(M)\text{)} \\ &= \widetilde{\mu}(-z). \end{aligned}$$

Suppose that $z \in x\gamma y$, then we have

$$\begin{aligned}
\widetilde{\mu}(z) &= (1/\mu(0_M))\mu(z) \\
&\geq (1/\mu(0_M))[(\mu(x) \vee \mu(y)) \wedge \nu(\gamma)] \qquad (\text{since } \mu \in FHI_\nu(M)) \\
&= [\widetilde{\mu}(x) \vee \widetilde{\mu}(y)] \wedge \widetilde{\nu}(\gamma).
\end{aligned}$$

Therefore, $\widetilde{\mu} \in FHI_{\widetilde{\nu}}(M)$. □

In the next theorem, we prove that under certain conditions, fuzzy hyperideal of Γ-hyperring is two-valued.

Theorem 3.14. Let $\mu \in FHI_{\eta'}(M)$, $\eta = 1/2\eta'$ and μ be maximal in the set $X = \{\nu \in FHI_\eta(M) \mid \nu(x) = 1,\ \exists x \in M\}$ under conclusion. Then μ is two-valued fuzzy hyperideal of M and it takes just 0 and 1.

Proof. Clearly $\mu \in FHI_\eta(M)$. We know that there exists $x \in M$ such that $\mu(x) = 1$, thus $\mu(0_M) \geq \mu(x) = 1$, hence $\mu(0_M) = 1$.

Let $x \in M$ be such that $\mu(x) \neq 1$. We show that $\mu(x) = 0$. Suppose that there exists $a \in M$ such that $0 < \mu(a) < 1$. Define $\nu : M \to [0,1]$ by $\nu(x) = 1/2(\mu(x) + \mu(a))$, for all $x \in M$. We show that $\nu \in FHI_\eta(M)$. Suppose that $z \in x + y$, then we have

$$\begin{aligned}
\nu(z) &= 1/2(\mu(z) + \mu(a)) \\
&\geq 1/2[(\mu(x) \wedge \mu(y)) + \mu(a)] \qquad (\text{since } \mu \in FHI_\eta(M)) \\
&= 1/2[(\mu(x) + \mu(a)) \wedge (\mu(y) + \mu(a))] \\
&= 1/2(\mu(x) + \mu(a)) \wedge 1/2(\mu(y) + \mu(a)) \\
&= \nu(x) \wedge \nu(y).
\end{aligned}$$

Also it is easy to verify that if $z \in M$, then $\nu(z) \geq \nu(-z)$. Now suppose $z \in x\gamma y$, we prove $\nu(z) \geq \nu(x) \wedge \eta(\gamma)$. We have

$$\begin{aligned}
\nu(z) &= 1/2(\mu(z) + \mu(a)) \\
&\geq 1/2[(\mu(x) \wedge \eta'(\gamma)) + \mu(a)] \qquad (\text{since } \mu \in FHI_{\eta'}(M)) \\
&= 1/2[\mu(x) + \mu(a)] \ \wedge \ 1/2[\eta'(\gamma)) + \mu(a)] \\
&= \nu(x) \wedge (\eta(\gamma) + 1/2\mu(a)) \\
&\geq \nu(x) \wedge \eta(\gamma).
\end{aligned}$$

Similarly we can prove that $\nu(z) \geq \nu(y) \wedge \eta(\gamma)$. Therefore $\nu \in FHI_\eta(M)$.

Hence, by Proposition 3.10, $\nu^+ \in FHI_\eta(M)$. Also we have

$$\begin{aligned}
\nu^+(x) &= \nu(x) + 1 - \nu(0_M) \\
&= 1/2(\mu(x) + \mu(a)) + 1 - 1/2(\mu(0_M) + \mu(a)) \\
&= 1/2(\mu(x) + 1). \qquad (\text{since } \mu(0_M) = 1)
\end{aligned}$$

So we have

$$\nu^+(0_M) = 1/2(\mu(0_M) + 1) = 1/2(1 + 1) = 1.$$

Thus $\nu^+ \in X$. Also we have

$$\nu^+(0_M) = 1 > \nu^+(a) = 1/2(\mu(a) + 1) > \mu(a) \neq 1.$$

Hence ν^+ is non-constant and $\nu^+(a) > \mu(a)$. So μ is not maximal, this is a contradiction. Therefore there is not any $a \in M$ such that $0 < \mu(a) < 1$. □

4. Fuzzy Product of ν-Fuzzy Hyperideals

Suppose that $(M_i, +_i)_{i\in I}$ is a family of canonical hypergroups. Then $\prod_{i\in I} M_i = \{(x_i)_{i\in I} \mid x_i \in M_i\}$, the cartesian product of $(M_i, +_i)_{i\in I}$, with following hyperoperation is a canonical hypergroup:

$$(x_i)_{i\in I} + (y_i)_{i\in I} = \{(z_i)_{i\in I} \mid z_i \in x_i +_i y_i\}.$$

It is easy to verify that if M_i is a Γ_i-hyperring, then $\prod_{i\in I} M_i$ is $\prod_{i\in I} \Gamma_i$-hyperring by the following rule:

$$\circ : (\prod_{i\in I} M_i) \times (\prod_{i\in I} \Gamma_i) \times (\prod_{i\in I} M_i) \longrightarrow P^*(\prod_{i\in I} M_i),$$

which is defined by

$$(x_i)_{i\in I} \circ (\gamma_i)_{i\in I} \circ (y_i)_{i\in I} = \{(z_i)_{i\in I} \mid z_i \in x_i\gamma_i y_i, \forall i \in I\}.$$

Notation. In the next proposition, by $\prod_{i\in I} \mu_i$, we mean the fuzzy product of μ_is, which is defined as follows:

$$(\prod_{i\in I} \mu_i)((x_i)_{i\in I}) = \bigwedge_{i\in I} \mu_i(x_i).$$

In the next proposition we describe fuzzy hyperideals of product of Γ-hyperrings.

Proposition 4.1. Let μ_i be ν_i-fuzzy hyperideal of M_i as Γ_i-hyperring ($\forall i \in I$). Then $\prod_{i\in I} \mu_i$ is a $\prod_{i\in I} \nu_i$- fuzzy hyperideal of $\prod_{i\in I} M_i$ as $\prod_{i\in I} \Gamma_i$-hyperring.

Proof. Suppose $(z_i)_{i\in I} \in (x_i)_{i\in I} + (y_i)_{i\in I}$. Then $z_i \in x_i +_i y_i$, so $\mu_i(z_i) \geq \mu_i(x_i) \wedge \mu_i(y_i)$, for all $i \in I$. Also we have

$$\begin{aligned}
(\prod_{i\in I} \mu_i)((z_i)_{i\in I}) &= \bigwedge_{i\in I} \mu_i(z_i) \\
&\geq \bigwedge_{i\in I} (\mu_i(x_i) \wedge \mu_i(y_i)) \qquad (\text{since } \mu_i \in FHI_{\nu_i}(M_i)) \\
&= (\bigwedge_{i\in I} \mu_i(x_i)) \wedge (\bigwedge_{i\in I} \mu_i(y_i)) \\
&= (\prod_{i\in I} \mu_i)((x_i)_{i\in I}) \wedge (\prod_{i\in I} \mu_i)((y_i)_{i\in I}).
\end{aligned}$$

Also it is easy to verify that $(\prod_{i\in I}\mu_i)((x_i)_{i\in I}) \geq (\prod_{i\in I}\mu_i)(-(x_i)_{i\in I})$.

Suppose that $(z_i)_{i\in I} \in (x_i)_{i\in I}(\gamma_i)_{i\in I}(y_i)_{i\in I}$, then we have $z_i \in x_i\gamma_i y_i,\ \forall i \in I$

$$\Longrightarrow \quad \mu_i(z_i) \geq (\mu_i(x_i) \vee \mu_i(y_i)) \wedge \nu_i(\gamma_i),\ \forall i \in I \qquad (\text{since } \mu_i \in FHI_{\nu_i}(M_i))$$

$$\Longrightarrow \quad \bigwedge_{i\in I}\mu_i(z_i) \geq \bigwedge_{i\in I}[(\mu_i(x_i) \vee \mu_i(y_i)) \wedge \nu_i(\gamma_i)]$$

$$\Longrightarrow \quad (\prod_{i\in I}\mu_i)((z_i)_{i\in I}) \geq [(\prod_{i\in I}\mu_i)((x_i)_{i\in I}) \vee (\prod_{i\in I}\mu_i)((y_i)_{i\in I})] \wedge (\prod_{i\in I}\nu_i)((\gamma_i)_{i\in I}).$$

Therefore $\prod_{i\in I}\mu_i$ is a $\prod_{i\in I}\nu_i$-fuzzy hyperideal of $\prod_{i\in I}M_i$. □

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