APPLICATIONS OF SOFT SETS IN HILBERT ALGEBRAS

Y. B. JUN AND C. H. PARK

Abstract. The concept of soft sets, introduced by Molodtsov [20] is a mathematical tool for dealing with uncertainties, that is free from the difficulties that have troubled the traditional theoretical approaches. In this paper, we apply the notion of the soft sets of Molodtsov to the theory of Hilbert algebras. The notion of soft Hilbert (abysmal and deductive) algebras, soft subalgebras, soft abysms and soft deductive systems are introduced, and their basic properties are investigated. The relations between soft Hilbert algebras, soft Hilbert abysmal algebras and soft Hilbert deductive algebras are also derived.

1. Introduction

Problems in system identification involve characteristics which are essentially non-probabilistic in nature [26]. In response to this situation Zadeh [27] introduced *fuzzy set theory* as an alternative to probability theory and recently outlined a generalization in [28]. Uncertainty occuring in fields such as economics, engineering and environment, cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, and the theories of (intuitionistic) fuzzy sets, vague sets, interval mathematics, and rough sets. However, as is pointed out in [20], none of these theories can handle all problems. Maji et al. [19] and Molodtsov [20] suggested that one reason may be the inadequacy of the parametrization tool of the relevant theory and as a solution Molodtsov [20]introduced the concept of soft sets and suggested several applications. At present, work on the theory of soft sets is progressing rapidly. Maji et al. [19] described the application of soft set theory to a decision making problem and [18] also studied several operations on soft sets. Kovkov et al. [17] considered optimization problems in the framework of the theory of soft sets. Chen et al. [8] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Aktaş and Çağman [1] investigated basic properties of soft sets, and compared soft sets to the related concepts of fuzzy sets and rough sets. They gave a definition of soft groups, and derived their basic properties. Jun and Park [16] applied the notion of soft sets to BCK/BCI-algebras. The algebraic structure of set theories dealing with uncertainties has also been studied and the study of structures of fuzzy sets in algebraic structures has been carried out by several authors (see e.g. [2], [3],

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[9], [10], [15], [25]). Hilbert algebras are also an important branch in algebraic structures. In this paper, we apply the notion of the soft set theory of Molodtsov to the theory of Hilbert algebras, which are a part of propositional logic containing only a logical connective implication, denoted by "·", and the constant 1. We also introduce the concepts of soft Hilbert algebras, soft Hilbert abysmal algebras and soft Hilbert deductive systems and investigate basic properties as intersection, union, morphism. We then apply our results to study (intuitionistic) fuzzy trends together with rough set applications by combining soft sets and several kinds of algebras.

2. **Basic Results on Hilbert Algebras**

A Hilbert algebra can be considered as part of propositional logic containing only a logical connective implication denoted by "·" and the constant 1 which is interpreted as the value "true".
In other words, an algebra $\mathcal{H} := (H, \cdot, 1)$ of type $(2, 0)$ is called a *Hilbert algebra* if it satisfies:

(H1) $(\forall a, b \in H)\ (a \cdot (b \cdot a) = 1)$.
(H2) $(\forall a, b, c \in H)\ ((a \cdot (b \cdot c)) \cdot ((a \cdot b) \cdot (a \cdot c)) = 1)$.
(H3) $(\forall a, b \in H)\ (a \cdot b = b \cdot a = 1 \Rightarrow a = b)$.

If $\mathcal{H} := (H, \cdot, 1)$ is a Hilbert algebra and $\leq$ in $\mathcal{H}$ is a binary relation defined by $a \leq b$ if and only if $a \cdot b = 1$, then $\leq$ is a partial order in $\mathcal{H} := (H, \cdot, 1)$. A mapping f from a Hilbert algebra $\mathcal{G} = (G, \cdot, 1)$ into a Hilbert algebra $\mathcal{H} = (H, \cdot 1)$ is called a *morphism*if $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in G$. Note that if f is a morphism from a Hilbert algebra $\mathcal{G} = (G, \cdot, 1)$ into a Hilbert algebra $\mathcal{H} = (H, \cdot 1)$, then $f(1) = 1$.

In a Hilbert algebra $\mathcal{H} := (H, \cdot, 1)$, we have:

(a1) $x \leq y \cdot x$.
(a2) $x \cdot 1 = 1,\ 1 \cdot x = x$.
(a3) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$.
(a4) $x \leq (x \cdot y) \cdot y$.
(a5) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.
(a6) $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$.
(a7) $x \leq y \Rightarrow z \cdot x \leq z \cdot y,\ y \cdot z \leq x \cdot z$.

The concept of a deductive system on a Hilbert algebra $\mathcal{H} := (H, \cdot, 1)$ was introduced by A. Diego [11] and defined as a subset of H containing 1 and closed under a "deduction", i.e.:

Definition 2.1. A nonempty subset D of a Hilbert algebra $\mathcal{H} := (H, \cdot, 1)$ is called a *deductive system* of $\mathcal{H}$ if

(Di) $1 \in D$,
(Dii) $(\forall x \in D)\ (\forall y \in H)\ (x \cdot y \in D \Rightarrow y \in D)$.

Lemma 2.2. [7] *A deductive system D of a Hilbert algebra $\mathcal{H}$ has the following property:*

$$(\forall x \in D)(\forall y \in H)(x \leq y \Rightarrow y \in D).$$

Definition 2.3. [14] We say that a nonempty subset D of a Hilbert algebra $\mathcal{H} := (H, \cdot, 1)$ is an *abysm* of $\mathcal{H}$ if

$$H \cdot D := \{x \cdot y \mid x \in H, y \in D\} = D,$$

Note that both $\{1\}$ and H are abysms of $\mathcal{H}$.

Lemma 2.4. [14] *Every abysm is a subalgebra.*

The converse of Lemma 2.4 is not true in general [14].

Lemma 2.5. [14] *Every deductive system is an abysm.*

The converse of Lemma 2.5 is not true in general [14]. For more details on deductive systems and Hilbert algebras, we refer the reader to [4, 5, 6, 12, 13].

3. **Basic Results on Soft Sets**

Molodtsov [20] defined a soft set as follows: Let U be an initial universe set, E a set of parameters, $\mathrm{P}(U)$ the power set of U and $A \subset E$.

Definition 3.1. [20] A pair (α, A) is called a *soft set* over U, if α is a mapping given by

$$\alpha : A \to \mathrm{P}(U).$$

In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $\alpha(\varepsilon)$ may be considered as the set of ε-approximate elements of a soft set (α, A). As Molodtsov [20]has illustrated, a soft set is not a set.

Definition 3.2. [18] Let (α, A) and (β, B) be two soft sets over a common universe U. The *intersection* of (α, A) and (β, B) is defined to be a soft set (γ, C) satisfying the following conditions:

(1) $C = A \cap B$,
(2) $(\forall e \in C)$ $(\gamma(e) = \alpha(e)$ or $\beta(e)$, (as both are same set)).

We write $(\gamma, C) = (\alpha, A)\widetilde{\cap}(\beta, B)$.

Definition 3.3. [18] Let (α, A) and (β, B) be two soft sets over a common universe U. The *union* of (α, A) and (β, B) is defined to be a soft set (γ, C) satisfying the following conditions:

(1) $C = A \cup B$,
(2) for all $e \in C$,

$$\gamma(e) = \begin{cases} \alpha(e) & \text{if } e \in A \setminus B, \\ \beta(e) & \text{if } e \in B \setminus A, \\ \alpha(e) \cup \beta(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $(\gamma, C) = (\alpha, A)\widetilde{\cup}(\beta, B)$.

Definition 3.4. [18] If (α, A) and (β, B) are two soft sets over a common universe U, then "(α, A) *AND* (β, B)" denoted by $(\alpha, A)\widetilde{\wedge}(\beta, B)$ is defined by $(\alpha, A)\widetilde{\wedge}(\beta, B) = (\gamma, A \times B)$, where $\gamma(a, b) = \alpha(a) \cap \beta(b)$ for all $(a, b) \in A \times B$.

Definition 3.5. [18] If (α, A) and (β, B) are two soft sets over a common universe U, then "(α, A) *OR* (β, B)" denoted by $(\alpha, A)\widetilde{\vee}(\beta, B)$ is defined by $(\alpha, A)\widetilde{\vee}(\beta, B) = (\gamma, A \times B)$, where $\gamma(a, b) = \alpha(a) \cup \beta(b)$ for all $(a, b) \in A \times B$.

Definition 3.6. [18] For two soft sets (α, A) and (β, B) over a common universe U, we say that (α, A) is a *soft subset* of (β, B), if

(1) $A \subset B$,

(2) For every $\varepsilon \in A$, $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are identical approximations.

We write $(\alpha, A)\widetilde{\subset}(\beta, B)$.

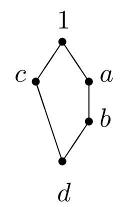
4. Soft Hilbert (Abysmal, Deductive) Algebras

In what follows, unless otherwise specified, $\mathcal{H}$ and A will respectively denote a Hilbert algebra and a nonempty set, and R will refer to an arbitrary binary relation between an element of A and an element of H. In other words, R will denote a subset of $A \times H$. A set-valued function $\alpha : A \to \mathrm{P}(H)$ can be defined as $\alpha(x) = \{y \in H \mid xRy\}$ for all $x \in A$. and it follows that the pair (α, A) is a soft set over $\mathcal{H}$.

Definition 4.1. Let (α, A) be a soft set over $\mathcal{H}$. Then (α, A) is called a *soft Hilbert algebra* (resp. *soft Hilbert abysmal algebra* and *soft Hilbert deductive algebra*) over $\mathcal{H}$ if $\alpha(x)$ is a subalgebra (resp. abysm and deductive system) of $\mathcal{H}$ for all $x \in A$.

Example 4.2. (1) Let $H = \{a, b, c, d, 1\}$ be a set with the following Cayley table and Hasse diagram:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	1	1



Then $\mathcal{H} := (H, \cdot, 1)$ is a Hilbert algebra [14]. Let (α, A) be a soft set over $\mathcal{H}$, where $A = H$ and $\alpha : A \to \mathrm{P}(H)$ is a set-valued function defined by

$$\alpha(x) = \{y \in H \mid xRy \Leftrightarrow (y \cdot x) \cdot x \in \{1, a\}\}$$

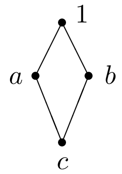
for all $x \in A$. Then $\alpha(1) = \alpha(a) = H$, $\alpha(b) = \{1, a, c\}$, $\alpha(c) = \{1, a, b\}$, and $\alpha(d) = \{1, a, b, c\}$ are abysms of $\mathcal{H}$. Hence (α, A) is a soft Hilbert abysmal algebra over $\mathcal{H}$, and by Lemma 2.4, (α, A) is a soft Hilbert algebra over $\mathcal{H}$. Now let (β, A) be a soft set over $\mathcal{H}$, where $A = H$ and $\beta : A \to \mathrm{P}(H)$ is a set-valued function defined by

$$\beta(x) = \{y \in H \mid xRy \Leftrightarrow (x \cdot y) \cdot y \in \{1, a\}\}$$

for all $x \in A$. Then $\beta(1) = \beta(a) = H$, $\beta(b) = \{1, a, c, d\}$, $\beta(c) = \{1, a, b, d\}$, and $\beta(d) = \{1, a\}$ are abysms of $\mathcal{H}$. Therefore (β, A) is a soft Hilbert abysmal algebra over $\mathcal{H}$. However, because $d \cdot b = 1 \in \beta(b)$ and $b \notin \beta(b)$, $\beta(b) = \{1, a, c, d\}$ is not a deductive system of $\mathcal{H}$ and hence (β, A) is not a soft Hilbert deductive algebra over $\mathcal{H}$.

(2) Let $G = \{1, a, b, c\}$ be a set with the following Cayley table and Hasse diagram.

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1



Then $\mathcal{G} := (G, \cdot, 1)$ is a Hilbert algebra [14]. Let (γ, A) be a soft set over $\mathcal{G}$, where $A = G$ and $\gamma : A \to \mathrm{P}(G)$ is a set-valued function defined by

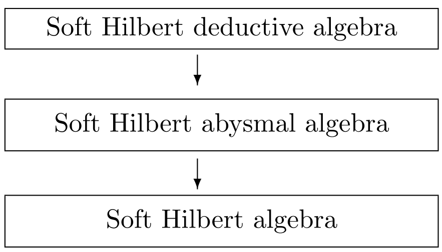
$$\gamma(x) = \{y \in G \mid xRy \Leftrightarrow (y \cdot x) \cdot x \in \{1, b\}\}$$

for all $x \in A$. Then $\gamma(1) = \gamma(b) = G$ and $\gamma(a) = \gamma(c) = \{1, b\}$ are deductive systems of $\mathcal{G}$. Therefore (γ, A) is a soft Hilbert deductive algebra over $\mathcal{G}$. Let (δ, A) be a soft set over $\mathcal{G}$, where $A = G$ and $\delta : A \to \mathrm{P}(G)$ is a set-valued function defined by

$$\delta(x) = \{y \in G \mid xRy \Leftrightarrow x \cdot y \in \{1, c\}\}$$

for all $x \in A$. Then $\delta(1) = \{1, c\}$, $\delta(a) = \{1, a\}$, $\delta(b) = \{1, b\}$, and $\delta(c) = H$ are subalgebras of $\mathcal{G}$. Hence (δ, A) is a soft Hilbert algebra over $\mathcal{G}$. However, $\delta(1) = \{1, c\}$ is not an abysm of $\mathcal{G}$, and so (δ, A) is not a soft Hilbert abysmal algebra over $\mathcal{G}$.

Using Lemmas 2.4 and 2.5, we have the following relations.



As Example 4.2 shows, none of these implications is reversible in general. Let A be a fuzzy subalgebra of $\mathcal{H}$ with membership function μ_A and consider the following family of t-level sets for the function μ_A:

$$\alpha(t) = \{x \in X \mid \mu_A(x) \geq t\},\ t \in [0, 1].$$

Then $\alpha(t)$ is a subalgebra of $\mathcal{H}$. If we know the family α, we can find the functions $\mu_A(x)$ as follows:

$$\mu_A(x) = \sup\{t \in [0, 1] \mid x \in \alpha(t)\}.$$

Thus, every fuzzy subalgebra A may be considered as the soft Hilbert algebra $(\alpha, [0, 1])$.

Theorem 4.3. *Let* (α, A) *be a soft Hilbert algebra* (*resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra*) *over* $\mathcal{H}$. *If* B *is a subset of* A, *then* $(\alpha|_B, B)$ *is a soft Hilbert algebra* (*resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra*) *over* $\mathcal{H}$.

Proof. Straightforward. □

The following example shows that there exists a soft set (α, A) over $\mathcal{H}$ such that

(1) (α, A) is not a soft Hilbert deductive algebra (resp. soft Hilbert abysmal algebra) over $\mathcal{H}$.
(2) there exists a subset B of A such that $(\alpha|_B, B)$ is a soft Hilbert deductive algebra (resp. soft Hilbert abysmal algebra) over $\mathcal{H}$.

Example 4.4. Let (β, A) be the soft set over $\mathcal{H}$ of Example 4.2(1). Note that (β, A) is not a soft Hilbert deductive algebra over $\mathcal{H}$. However, if we take $B = \{a, d\} \subset A$, then $(\beta|_B, B)$ is a soft Hilbert deductive algebra over $\mathcal{H}$. Now let (δ, A) be a soft set over $\mathcal{G}$ given in Example 4.2(2). Then (δ, A) is not a soft Hilbert abysmal algebra over $\mathcal{G}$, but if $B = \{a, b, c\}$ is a subset of A, then $(\delta|_B, B)$ is a soft Hilbert abysmal algebra over $\mathcal{G}$.

Theorem 4.5. *Let (α, A) and (β, B) be two soft Hilbert algebras (resp. soft Hilbert abysmal algebras and soft Hilbert deductive algebras) over $\mathcal{H}$. If $A \cap B \neq \emptyset$, then the intersection $(\alpha, A)\widetilde{\cap}(\beta, B)$ is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$.*

Proof. By Definition 3.2, we have $(\alpha, A)\widetilde{\cap}(\beta, B) = (\gamma, C)$, where $C = A \cap B$ and $\gamma(x) = \alpha(x)$ or $\beta(x)$ for all $x \in C$. Now $\gamma : C \to \mathrm{P}(H)$ is a mapping, and therefore (γ, C) is a soft set over $\mathcal{H}$. Since (α, A) and (β, B) are soft Hilbert algebras (resp. soft Hilbert abysmal algebras and soft Hilbert deductive algebras) over $\mathcal{H}$, hence either $\gamma(x) = \alpha(x)$ or $\gamma(x) = \beta(x)$ is a subalgebra (resp. abysm and deductive system) of $\mathcal{H}$, for all $x \in C$. Hence $(\gamma, C) = (\alpha, A)\widetilde{\cap}(\beta, B)$ is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$. □

Corollary 4.6. *Let (α, A) and (β, A) be two soft Hilbert algebras (resp. soft Hilbert abysmal algebras and soft Hilbert deductive algebras) over $\mathcal{H}$. Then their intersection $(\alpha, A)\widetilde{\cap}(\beta, A)$ is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$.*

Proof. Straightforward. □

Theorem 4.7. *Let (α, A) and (β, B) be two soft Hilbert algebras (resp. soft Hilbert abysmal algebras and soft Hilbert deductive algebras) over $\mathcal{H}$. If A and B are disjoint, then the union $(\alpha, A)\widetilde{\cup}(\beta, B)$ is also a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$.*

Proof. By Definition 3.3, we have $(\alpha, A)\widetilde{\cup}(\beta, B) = (\gamma, C)$, where $C = A \cup B$ and for every $e \in C$,

$$\gamma(e) = \begin{cases} \alpha(e) & \text{if } e \in A \setminus B, \\ \beta(e) & \text{if } e \in B \setminus A, \\ \alpha(e) \cup \beta(e) & \text{if } e \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, hence either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. Let $x \in A \setminus B$. Since (α, A) is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$, hence $\gamma(x) = \alpha(x)$ is a subalgebra (resp. abysm and deductive system) of $\mathcal{H}$. Now suppose $x \in B \setminus A$. Then, since (β, B) is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$, it follows that $\gamma(x) = \beta(x)$ is a subalgebra (resp. abysm and deductive system) of $\mathcal{H}$. Hence $(\gamma, C) = (\alpha, A)\widetilde{\cup}(\beta, B)$ is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$. □

Theorem 4.8. *If (α, A) and (β, B) are soft Hilbert algebras (resp. soft Hilbert abysmal algebras and soft Hilbert deductive algebras) over $\mathcal{H}$, then $(\alpha, A)\widetilde{\wedge}(\beta, B)$ is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$.*

Proof. By Definition 3.4, we have

$$(\alpha, A)\widetilde{\wedge}(\beta, B) = (\gamma, A \times B),$$

where $\gamma(x, y) = \alpha(x) \cap \beta(y)$ for all $(x, y) \in A \times B$. Since $\alpha(x)$ and $\beta(y)$ are subalgebras (resp. abysms and deductive systems) of $\mathcal{H}$, the intersection $\alpha(x) \cap \beta(y)$ is also a subalgebra (resp. abysm and deductive system) of $\mathcal{H}$. Hence $\gamma(x, y)$ is a subalgebra (resp. abysm and deductive system) of $\mathcal{H}$ for all $(x, y) \in A \times B$, and so $(\alpha, A)\widetilde{\wedge}(\beta, B) = (\gamma, A \times B)$ is a soft Hilbert algebra (resp. soft Hilbert abysmal algebra and soft Hilbert deductive algebra) over $\mathcal{H}$. □

Definition 4.9. A soft Hilbert algebra (soft Hilbert abysmal algebra and soft Hilbert deductive algebra) (α, A) over $\mathcal{H}$ is said to be *trivial* (resp. *whole*) if $\alpha(x) = \{1\}$ (resp. $\alpha(x) = H$) for all $x \in A$.

Let $f : \mathcal{H} \to \mathcal{G}$ be a mapping of Hilbert algebras. For a soft set (α, A) over $\mathcal{H}$, $(f(\alpha), A)$ is a soft set over $\mathcal{G}$ where $f(\alpha) : A \to \mathrm{P}(G)$ is defined by $f(\alpha)(x) = f(\alpha(x))$ for all $x \in A$.

Lemma 4.10. *Let $f : \mathcal{H} \to \mathcal{G}$ be a morphism of Hilbert algebras. If (α, A) is a soft Hilbert algebra over $\mathcal{H}$, then $(f(\alpha), A)$ is a soft Hilbert algebra over $\mathcal{G}$. Moreover, if f is onto and (α, A) is a soft Hilbert abysmal algebra (resp. soft Hilbert deductive algebra) over $\mathcal{H}$, then $(f(\alpha), A)$ is a soft Hilbert abysmal algebra (resp. soft Hilbert deductive algebra) over $\mathcal{G}$.*

Proof. For every $x \in A$, since $\alpha(x)$ is a subalgebra of $\mathcal{H}$, hence $f(\alpha)(x) = f(\alpha(x))$ is a subalgebra of $\mathcal{G}$ and its morphic image is also a subalgebra of $\mathcal{G}$. Hence $(f(\alpha), A)$ is a soft Hilbert algebra over $\mathcal{G}$. Assume that f is onto and (α, A) is a soft Hilbert abysmal algebra (resp. soft Hilbert deductive algebra) over $\mathcal{H}$. Then $\alpha(x)$ is an abysm (resp. deductive system) of $\mathcal{H}$. Since f is onto, it follows that $f(\alpha)(x) = f(\alpha(x))$ is an abysm (resp. deductive system) of $\mathcal{G}$ for all $x \in A$ so that $(f(\alpha), A)$ is a soft Hilbert abysmal algebra (resp. soft Hilbert deductive algebra) over $\mathcal{G}$. □

Theorem 4.11. *Let $f : \mathcal{H} \to \mathcal{G}$ be a morphism of Hilbert algebras and let (α, A) be a soft set over $\mathcal{H}$.*

(1) *If (α, A) is a soft Hilbert algebra over $\mathcal{H}$ such that $\alpha(x) = \ker(f)$ for all $x \in A$, then $(f(\alpha), A)$ is the trivial soft Hilbert algebra over $\mathcal{G}$.*
(2) *Assume that f is onto and (α, A) is a whole soft Hilbert abysmal algebra (resp. whole soft Hilbert deductive algebra) over $\mathcal{H}$. Then $(f(\alpha), A)$ is a*

whole soft Hilbert abysmal algebra (*resp. whole soft Hilbert deductive algebra*) *over* $\mathcal{G}$.

Proof. (1) Assume that $\alpha(x) = \ker(f)$ for all $x \in A$. Then $f(\alpha)(x) = f(\alpha(x)) = \{1\}$ for all $x \in A$. Hence by Lemma 4.10 and Definition 4.9, $(f(\alpha), A)$ is the trivial soft Hilbert algebra over $\mathcal{G}$.

(2) Suppose that f is onto and (α, A) is whole. Then $\alpha(x) = H$ for all $x \in A$, and so $f(\alpha)(x) = f(\alpha(x)) = f(H) = G$ for all $x \in A$. It follows from Lemma 4.10 and Definition 4.9 that $(f(\alpha), A)$ is a whole soft Hilbert abysmal algebra (resp. whole soft Hilbert deductive algebra) over $\mathcal{G}$. □

Definition 4.12. Let S be a subalgebra of $\mathcal{H}$. A subset D of H is called

(1) a *subalgebra* of $\mathcal{H}$ related to S (briefly, S-*subalgebra*) if

$$(\forall x, y \in S)(x \cdot y \in D), \tag{4.1}$$

(2) an *abysm* of $\mathcal{H}$ related to S (briefly, S-*abysm*) if

$$S \cdot D = D, \tag{4.2}$$

(3) a *deductive system* of $\mathcal{H}$ related to S (briefly, S-*deductive system*) if it satisfies (Di) and

$$(\forall x \in D)(\forall y \in S)(x \cdot y \in D \ \Rightarrow \ y \in D). \tag{4.3}$$

Definition 4.13. Let (α, A) and (β, B) be two soft Hilbert algebras over $\mathcal{H}$. Then we say that (α, A) is a *soft subalgebra* $((\alpha, A)\widetilde{<}(\beta, B))$ (resp. *soft abysm* $((\alpha, A)\widetilde{\ll}(\beta, B))$ and *soft deductive system* $((\alpha, A)\widetilde{\prec}(\beta, B)))$ of (β, B) if it satisfies the following conditions:

(1) $A \subset B$,
(2) $\alpha(x)$ is a $\beta(x)$-subalgebra (resp. $\beta(x)$-abysm and $\beta(x)$-deductive system) for all $x \in A$.

The following implications are easily verified.

$$(\alpha, A)\widetilde{\prec}(\beta, B) \ \Rightarrow \ (\alpha, A)\widetilde{\ll}(\beta, B) \ \Rightarrow \ (\alpha, A)\widetilde{<}(\beta, B). \tag{4.4}$$

The following example shows that the reverse implications (4.4) are not true in general.

Example 4.14. (1) For any $a \in \mathcal{H}$ and subset D of $\mathcal{H}$, let

$$\tfrac{a}{D} = \{x \in X \mid a \cdot x \in D\}. \tag{4.5}$$

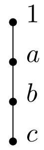
Note that if D is a deductive system of $\mathcal{H}$, then $\frac{a}{D}$ is a deductive system of $\mathcal{H}$. Let (α, A) be the soft Hilbert algebra over $\mathcal{H}$ of Example 4.2(1). Let $B = \{a, c, d\}$ be a subset of A and let $\beta : B \to \mathrm{P}(H)$ be a set-valued function defined by

$$\beta(x) = \{y \in H \mid xRy \Leftrightarrow y \in \tfrac{x}{\{1,a,b\}}\}$$

for all $x \in B$. Then $\beta(a) = \{1, a\}$, $\beta(c) = \{1, a, b, c\}$ and $\beta(d) = H$ are deductive systems of $\alpha(a)$, $\alpha(c)$ and $\alpha(d)$, respectively. Hence (β, B) is a soft deductive system of (α, A).

(2) Let $H = \{1, a, b, c\}$ be a set with the following Cayley table and Hasse diagram.

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1



Then $\mathcal{H} := (H, \cdot, 1)$ is a Hilbert algebra. Let (α, A) be a soft set over $\mathcal{H}$ (where $A = H$) and $\alpha : A \to \mathrm{P}(H)$ be a set-valued function defined by

$$\alpha(x) = \{y \in H \mid xRy \Leftrightarrow x \in \tfrac{y \cdot x}{\{1,a\}}\}$$

for all $x \in A$. Then $\alpha(1) = \alpha(a) = H$, $\alpha(b) = \{1, a\}$, and $\alpha(c) = \{1, a, b\}$ are deductive systems of $\mathcal{H}$. Hence (α, A) is a soft Hilbert deductive algebra over $\mathcal{H}$. Let (β, B) be a soft set over $\mathcal{H}$, where $B = H$ and $\beta : B \to \mathrm{P}(H)$ is a set-valued function defined by

$$\beta(x) = \{y \in H \mid xRy \Leftrightarrow y \in \tfrac{x \cdot y}{\{1,a\}}\}$$

for all $x \in B$. Then $\beta(1) = \beta(a) = H$, $\beta(b) = \{1, a, c\}$, and $\beta(c) = \{1, a\}$. Hence (β, B) is a soft Hilbert abysmal algebra over $\mathcal{H}$. Now, since $c \in \beta(b)$ and $c \cdot b = 1 \in \beta(b)$, but $b \notin \beta(b)$ it follos that $\beta(b) = \{1, a, c\}$ is not a deductive system of $\mathcal{H}$. Hence (β, B) is not a soft Hilbert deductive algebra over $\mathcal{H}$. Let $C = \{1, a, b\}$ and let (γ, C) be a soft set over $\mathcal{H}$, where $\gamma : C \to \mathrm{P}(H)$ is a set-valued function defined by

$$\gamma(x) = \{y \in H \mid xRy \Leftrightarrow y \in \tfrac{x}{\{1,b\}}\}$$

for all $x \in C$. Then $\gamma(1) = \{1, b\}$ and $\gamma(a) = \gamma(b) = \{1, a, b\}$ are, respectively, a $\beta(1)$-abysm and a $\beta(a) = \beta(b)$-abysm. Hence (γ, C) is a soft abysm of (β, B). However, $b \cdot a \in \gamma(1)$ and $a \notin \gamma(1)$ and hence $\gamma(1) = \{1, b\}$ is not a $\beta(1)$-deductive system. It follows that (γ, C) is not a soft deductive system of (β, B) .

(3) Let (γ, A) be a soft Hilbert algebra over $\mathcal{G}$ given in Example 4.2(2). Take a subset $D = \{1, a, b\}$ of A. Let $\delta : D \to \mathrm{P}(G)$ be a set-valued function defined by

$$\delta(x) = \{y \in G \mid xRy \Leftrightarrow y \in \tfrac{x}{\{1,c\}}\}$$

for all $x \in D$. Then $(\delta, D)\widetilde{\precsim}(\gamma, A)$. Since $\delta(1) = \{1, c\}$ is not a $\gamma(1)$-abysm, (δ, D) is not a soft abysm of (γ, A).

Theorem 4.15. *Let (α, A) and (β, A) be two soft Hilbert algebras over $\mathcal{H}$. Then*

$$(\forall x \in A)(\alpha(x) \subset \beta(x) \Rightarrow (\alpha, A)\widetilde{<}(\beta, A)). \tag{4.6}$$

Proof. Straightforward. □

Theorem 4.16. *Let (α, A) be a soft Hilbert algebra over $\mathcal{H}$ and let (β_1, B_1) and (β_2, B_2) be soft subalgebras* (resp. *soft abysms and soft deductive systems*) *of (α, A). Then*

(1) $(\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2)\widetilde{<}(\alpha, A)$ (resp. $(\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2)\widetilde{\ll}(\alpha, A)$ *and* $(\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2)\widetilde{\precsim}(\alpha, A)$).

(2) *If* $B_1 \cap B_2 = \emptyset$, *then* $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2)\widetilde{<}(\alpha, A)$ (resp. $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2)\widetilde{\lll}(\alpha, A)$ *and* $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2)\widetilde{\prec}(\alpha, A)$).

(3) $(\beta_1, B_1)\widetilde{\wedge}(\beta_2, B_2)\widetilde{<}(\gamma, A \times A)$ (resp. $(\beta_1, B_1)\widetilde{\wedge}(\beta_2, B_2)\widetilde{\lll}(\gamma, A \times A)$ *and* $(\beta_1, B_1)\widetilde{\wedge}(\beta_2, B_2)\widetilde{\prec}(\gamma, A \times A)$), *where* $(\gamma, A \times A) = (\alpha, A)\widetilde{\wedge}(\alpha, A)$.

Proof. (1) By Definition 3.2, we have

$$(\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2) = (\beta, B),$$

where $B = B_1 \cap B_2$ and $\beta(x) = \beta_1(x)$ or $\beta_2(x)$ for all $x \in B$. Obviously, $B \subset A$. Let $x \in B$. Then $x \in B_1$ and $x \in B_2$. If $x \in B_1$, then since $(\beta_1, B_1)\widetilde{<}(\alpha, A)$ (resp. $(\beta_1, B_1)\widetilde{\lll}(\alpha, A)$ and $(\beta_1, B_1)\widetilde{\prec}(\alpha, A)$), hence $\beta(x) = \beta_1(x)$ is an $\alpha(x)$-subalgebra (resp. $\alpha(x)$-abysm and $\alpha(x)$-deductive system). If $x \in B_2$, then since $(\beta_2, B_2)\widetilde{<}(\alpha, A)$ (resp. $(\beta_2, B_2)\widetilde{\lll}(\alpha, A)$ and $(\beta_2, B_2)\widetilde{\prec}(\alpha, A)$), hence $\beta(x) = \beta_2(x)$ is an $\alpha(x)$-subalgebra (resp. $\alpha(x)$-abysm and $\alpha(x)$-deductive system). It follows that $(\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2) = (\beta, B)\widetilde{<}(\alpha, A)$ (resp.

$$(\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2) = (\beta, B)\widetilde{\lll}(\alpha, A) \text{ and } (\beta_1, B_1)\widetilde{\cap}(\beta_2, B_2) = (\beta, B)\widetilde{\prec}(\alpha, A)).$$

(2) Assume that $B_1 \cap B_2 = \emptyset$. We can write $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2) = (\beta, B)$ where $B = B_1 \cup B_2$ and

$$\beta(x) = \begin{cases} \beta_1(x) & \text{if } x \in B_1 \setminus B_2, \\ \beta_2(x) & \text{if } x \in B_2 \setminus B_1, \\ \beta_1(x) \cup \beta_2(x) & \text{if } x \in B_1 \cap B_2 \end{cases}$$

for all $x \in B$. Since $(\beta_i, B_i))\widetilde{<}(\alpha, A)$ (resp. $(\beta_i, B_i))\widetilde{\lll}(\alpha, A)$ and $(\beta_i, B_i))\widetilde{\prec}(\alpha, A)$) for $i = 1, 2$, $B = B_1 \cup B_2 \subset A$ and $\beta_i(x)$ is an $\alpha(x)$-subalgebra (resp. $\alpha(x)$-abysm and $\alpha(x)$-deductive system) for all $x \in B_i$, $i = 1, 2$. Since $B_1 \cap B_2 = \emptyset$, $\beta(x)$ is an $\alpha(x)$-subalgebra (resp. $\alpha(x)$-abysm and $\alpha(x)$-deductive system) for all $x \in B$. Therefore $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2) = (\beta, B)\widetilde{<}(\alpha, A)$ (resp. $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2) = (\beta, B)\widetilde{\lll}(\alpha, A)$ and $(\beta_1, B_1)\widetilde{\cup}(\beta_2, B_2) = (\beta, B)\widetilde{\prec}(\alpha, A)$).

(3) Note that $(\gamma, A \times A)$ is a soft Hilbert algebra over $\mathcal{H}$ (see Theorem 4.8). We can write

$$(\beta_1, B_1)\widetilde{\wedge}(\beta_2, B_2) = (\delta, B),$$

where $B = B_1 \times B_2$ and $\delta(x_1, x_2) = \beta_1(x_1) \cap \beta_2(x_2)$ for all $(x_1, x_2) \in B$. Obviously, $B \subset A \times A$ and $\delta(x_1, x_2) = \beta_1(x_1) \cap \beta_2(x_2)$ is a $\gamma(x_1, x_2) = \alpha(x_1) \cap \alpha(x_2)$-subalgebra (resp. $\gamma(x_1, x_2) = \alpha(x_1) \cap \alpha(x_2)$-abysm and $\gamma(x_1, x_2) = \alpha(x_1) \cap \alpha(x_2)$-deductive system). Hence, we have (3). □

Theorem 4.17. *Let* $f : \mathcal{H} \to \mathcal{G}$ *be a morphism of Hilbert algebras and let* (α, A) *and* (β, B) *be soft Hilbert algebras over* $\mathcal{H}$. *Then*

$$(\alpha, A)\widetilde{<}(\beta, B) \Rightarrow (f(\alpha), A)\widetilde{<}(f(\beta), B). \tag{4.7}$$

Proof. Assume that $(\alpha, A)\widetilde{<}(\beta, B)$. Let $x \in A$. Then $A \subset B$ and $\alpha(x)$ is a subalgebra of $\beta(x)$. Since f is a homomorphism, $f(\alpha)(x) = f(\alpha(x))$ is a subalgebra of $f(\beta(x)) = f(\beta)(x)$, and therefore $(f(\alpha), A)\widetilde{<}(f(\beta), B)$. □

Theorem 4.18. *Let $f : \mathcal{H} \to \mathcal{G}$ be an onto morphism of Hilbert algebras and let (α, A) and (β, B) be soft Hilbert algebras over $\mathcal{H}$. Then*

$$(\alpha, A)\widetilde{\prec}(\beta, B) \Rightarrow (f(\alpha), A)\widetilde{\prec}(f(\beta), B), \tag{4.8}$$

$$(\alpha, A)\widetilde{\lll}(\beta, B) \Rightarrow (f(\alpha), A)\widetilde{\lll}(f(\beta), B). \tag{4.9}$$

Proof. Assume that f is onto and $(\alpha, A)\widetilde{\prec}(\beta, B)$ (resp. $(\alpha, A)\widetilde{\lll}(\beta, B)$). Let $x \in A$. Then $A \subset B$ and $\alpha(x)$ is a deductive system (resp. abysm) of $\beta(x)$. Since f is an onto morphism, $f(\alpha)(x) = f(\alpha(x))$ is a deductive system (resp. abysm) of $f(\beta(x)) = f(\beta)(x)$, and therefore $(f(\alpha), A)\widetilde{\prec}(f(\beta), B)$ (resp. $(f(\alpha), A)\widetilde{\lll}(f(\beta), B)$). □

Corollary 4.19. *Let $f : \mathcal{H} \to \mathcal{G}$ be a morphism of Hilbert algebras and let (α, A) and (β, B) be soft Hilbert algebras over $\mathcal{H}$. Then*

$$(\alpha, A)\widetilde{\prec}(\beta, B) \text{ (or, } (\alpha, A)\widetilde{\lll}(\beta, B)) \Rightarrow (f(\alpha), A)\widetilde{<}(f(\beta), B). \tag{4.10}$$

Proof. Straightforward. □

5. Conclusions and Future Work

The soft set theory of Molodtsov [20] offers a general mathematical tool for dealing with uncertain, fuzzy, or vague objects. Molodtsov in [20] has given several possible applications of soft set theory. In this paper, we have studied an application of soft set theory to a Hilbert algebra which is a well known algebraic structure. Based on these results, we plan to study applications of soft set theory in other algebraic structures using the fuzzy and rough techniques of Zadeh [27] and Pawlak [21]. We also plan to use of the results of this paper to solve decision making problems and study (intuitionistic) fuzzy trends. The role of this paper is to supply algebraic trends by combining the notion of soft sets with algebras and we hope that researchers can apply these to applications in the real world.

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Young Bae Jun, Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Korea
E-mail address: skywine@gmail.com

Chul Hwan Park*, Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea
E-mail address: skyrosemary@gmail.com

*Corresponding author