THE GENERATION OF FUZZY SETS AND THE CONSTRUCTION OF CHARACTERIZING FUNCTIONS OF FUZZY DATA

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Abstract. Measurement results contain different kinds of uncertainty. Besides systematic errors and random errors individual measurement results are also subject to another type of uncertainty, so-called *fuzziness*. It turns out that special fuzzy subsets of the set of real numbers $\mathbb{R}$ are useful to model fuzziness of measurement results. These fuzzy subsets x^* are called *fuzzy numbers*. The membership functions of fuzzy numbers have to be determined. In the paper first a characterization of membership function is given, and after that methods to obtain special membership functions of fuzzy numbers, so-called *characterizing functions* describing measurement results are treated.

1. Introduction

A critical point in fuzzy set theory is "how to obtain the membership function". There are some methods for special situations, where this is possible in a natural way. Especially, for special fuzzy subsets of the set of real numbers $\mathbb{R}$, and for special fuzzy subsets of $\mathbb{R}^2$.

Construction of membership functions of data is useful in many field of sciences [2, 4, 8]. There are methods based on probabilistic theory [1]. The fuzzy set concept was firstly introduced by L. Zadeh [7], description of the fuzzy set theory can be found in monographs, for example in [3].

2. δ-Cuts and Generating Families

Let A^* be a fuzzy subset of a universal set M, which means, that there exists a function, called *membership function* $\mu\colon M \to [0,1]$ associating each object in M with a real number in the interval [0,1]. Then for $\delta \in (0,1]$ the so-called δ-Cut $\mathcal{C}_\delta[A^*]$ is defined based on the membership function $\mu(\cdot)$ of A^* in the following way:

$$\mathcal{C}_\delta[A^*] := \{x \in M\colon\ \mu(x) \geq \delta\}$$

Denoting by $\mathbb{1}_A(\cdot)$ the indicator function of a classical set $A \subseteq M$, where

$$\mathbb{1}_A(x) := \left\{ \begin{array}{ll} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{array} \right\} \qquad \forall x \in M,$$

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the family of δ-Cuts $\big(\mathcal{C}_\delta[A^*];\ \delta \in (0,1]\big)$ determines the membership function $\mu(\cdot)$ of A^* by the following lemma:

Lemma 2.1.

$$\mu(x) = \max\big\{\delta \cdot \mathbb{1}_{\mathcal{C}_\delta[A^*]}(x):\ \delta \in [0,1]\big\} \quad \forall\, x \in M$$

This lemma is well known. Moreover the following holds true for all $\delta \in (0,1]$:

$$\mathcal{C}_\delta[A^*] = \bigcap_{0<\beta<\delta} \mathcal{C}_\beta[A^*]$$

Now one could think that all nested families $\big(A_\delta;\ \delta \in (0,1]\big)$ of classical subsets of M, i.e. $A_{\delta_1} \supseteq A_{\delta_2}$ for $\delta_1 < \delta_2$, are already the δ-Cuts of a fuzzy subset of M. This is not true as can be seen by simple examples (compare Example 2.3).

Definition 2.2. Given a nested family $\big(A_\delta;\ \delta \in (0,1]\big)$ of classical subsets of M, a fuzzy subset A^* of M is generated, whose membership function $\mu(\cdot)$ is defined in the following way:

$$\mu(x) := \sup\{\delta \cdot \mathbb{1}_{A_\delta}(x):\ \delta \in (0,1]\} \qquad \forall\, x \in M$$

Example 2.3. Let M be the set of real numbers and let the nested family $\big(A_\delta;\ \delta \in (0,1]\big)$ of classical subsets of M be defined in following way:

$$A_\delta := \left\{ \begin{array}{ll} [0,\ 2] & \text{for } \delta \in \big(0, \frac{1}{2}\big) \\ \big[\frac{1}{2},\ \frac{3}{2}\big] & \text{for } \delta \in \big[\frac{1}{2}, 1\big] \end{array} \right\}$$

Then the generated fuzzy subset A^* has the following membership function $\mu(\cdot)$ according to Definition 2.2:

$$\mu(x) = \left\{ \begin{array}{ll} 0 & \text{for } x \notin [0,2] \\ \frac{1}{2} & \text{for } x \in \big[0, \frac{1}{2}\big) \cup \big(\frac{3}{2}, 2\big] \\ 1 & \text{for } x \in \big[\frac{1}{2}, \frac{3}{2}\big] \end{array} \right\} \qquad \forall\, x \in M$$

The δ-Cuts for this fuzzy subset A^* are the following:

$$\mathcal{C}_\delta[A^*] = \left\{ \begin{array}{ll} [0,\ 2] & \text{for } \delta \in \big(0, \frac{1}{2}\big] \\ \big[\frac{1}{2},\ \frac{3}{2}\big] & \text{for } \delta \in \big(\frac{1}{2}, 1\big] \end{array} \right\} \qquad \forall\, \delta \in (0,1]$$

Now the set $A_{\frac{1}{2}}$ is equal to the interval $\big[\frac{1}{2},\ \frac{3}{2}\big]$ and the δ-Cut $\mathcal{C}_{\frac{1}{2}}[A^*]$ is equal to the interval $[0,\ 2]$.

For the membership function generated by a nested family $\big(A_\delta;\ \delta \in (0,1]\big)$ the following holds:

Theorem 2.4. *Let $\mathcal{C}_\delta[A^*]$ be the δ-Cut of a fuzzy set A^* with membership function $\mu(\cdot)$ generated by a nested family of subsets $\big(A_\delta;\ \delta \in (0,1]\big)$ of a set M according to Definition 2.2. Then for any $\delta \in (0,1]$ the following holds true:*

$$\mathcal{C}_\delta[A^*] = A_\delta \qquad \textit{if and only if} \qquad A_\delta = \bigcap_{0<\beta<\delta} A_\beta$$

Proof. Firstly extend the nested family of subsets $\big(A_\delta;\ \delta \in (0,1]\,\big)$ by the element $A_0 := M$. Then the proof is made in three steps.

- $A_\delta \subseteq \mathcal{C}_\delta[A^*]$ is fulfilled $\forall\, \delta \in (0,1]$.
  For arbitrary $x \in A_\delta$ we have $\delta \cdot \mathbb{1}_{A_\delta} = \delta$ and thus $\sup\big\{\beta \cdot \mathbb{1}_{A_\beta}(x) \colon \beta \in (0,1]\big\} \geq \delta$. Using the definition of $\mu(\cdot)$ we have $\mu(x) \geq \delta$ and from the definition of the δ-Cut we obtain $x \in \mathcal{C}_\delta[A^*]$.
- $A_\delta = \bigcap\limits_{0<\beta<\delta} A_\beta \quad \Rightarrow \quad \mathcal{C}_\delta[A^*] = A_\delta$
  Choose $x \notin A_\delta$. From $A_\delta = \bigcap_{0<\beta<\delta} A_\beta$ we know that there exists $\alpha < \delta$, with $x \notin A_\alpha$ and from the nested structure of the generating family $\big(A_\delta;\ \delta \in (0,1]\,\big)$ we know that $x \notin A_\beta\ \forall\, \beta \in [\alpha, 1]$. We have
  $$\mu(x) = \sup\big\{\beta \cdot \mathbb{1}_{A_\beta}(x) \colon \beta \in [0,1]\big\} \leq \alpha < \delta$$
  and from that $x \notin \mathcal{C}_\delta[A^*]$.
- $\mathcal{C}_\delta[A^*] = A_\delta \quad \Rightarrow \quad A_\delta = \bigcap\limits_{0<\beta<\delta} A_\beta$
  $A_\delta \subseteq \bigcap_{0<\beta<\delta} A_\beta$ holds from the nested structure of the generating family $\big(A_\delta;\ \delta \in (0,1]\,\big)$. Choose $x \notin \mathcal{C}_\delta[A^*]$ and suppose that $A_\delta = \mathcal{C}_\delta[A^*]$. Then
  $$\mu(x) = \sup\big\{\beta \cdot \mathbb{1}_{A_\beta}(x) \colon \beta \in [0,1]\big\} < \delta.$$
  Choose $\alpha \in (\mu(x), \delta)$, then $x \notin A_\alpha$ and hence $x \notin \bigcap_{0<\beta<\delta} A_\beta$.

□

Remark 2.5. For a nested sequence of classical subsets $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n$ of a set M a nested family of subsets $\big(A_\delta;\ \delta \in (0,1]\,\big)$, where $A_{\delta_1} \supseteq A_{\delta_2}$ for $\delta_1 < \delta_2$, can be constructed in the following way:

$$A_\delta = B_i, \text{ where } \delta \in \big(1 - \tfrac{i}{n}, 1 - \tfrac{i-1}{n}\big] \qquad \forall \delta \in (0,1]$$

3. Fuzzy Numbers and Characterizing Functions

In applications, measurement results of continuous quantities do not result in *precise numbers times the measurement unit*, but are always more or less imprecise and we call them fuzzy. For one-dimensional continuous quantities real measurement data are best described by so-called fuzzy numbers.

Definition 3.1. A fuzzy subset x^* of the set of real numbers $\mathbb{R}$ is called *fuzzy number* if its membership function $\xi(\cdot)$ fulfils the following three conditions:

(1) $\xi : \mathbb{R} \to [0,1]$
(2) $\operatorname{supp}[\xi(\cdot)] := \big\{x \in \mathbb{R} \colon\ \xi(x) > 0\big\}$ is a bounded set
(3) $\forall\, \delta \in (0,1]$ the δ-Cut $\mathcal{C}_\delta[\xi(\cdot)]$ is non-empty and a finite union of compact intervals, i.e.
$$\mathcal{C}_\delta[\xi(\cdot)] = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}, b_{\delta,j}]$$

Functions $\xi(\cdot)$ fulfilling conditions (1) – (3) are called *characterizing functions*.

In Figure 1 some examples of characterizing functions are depicted.

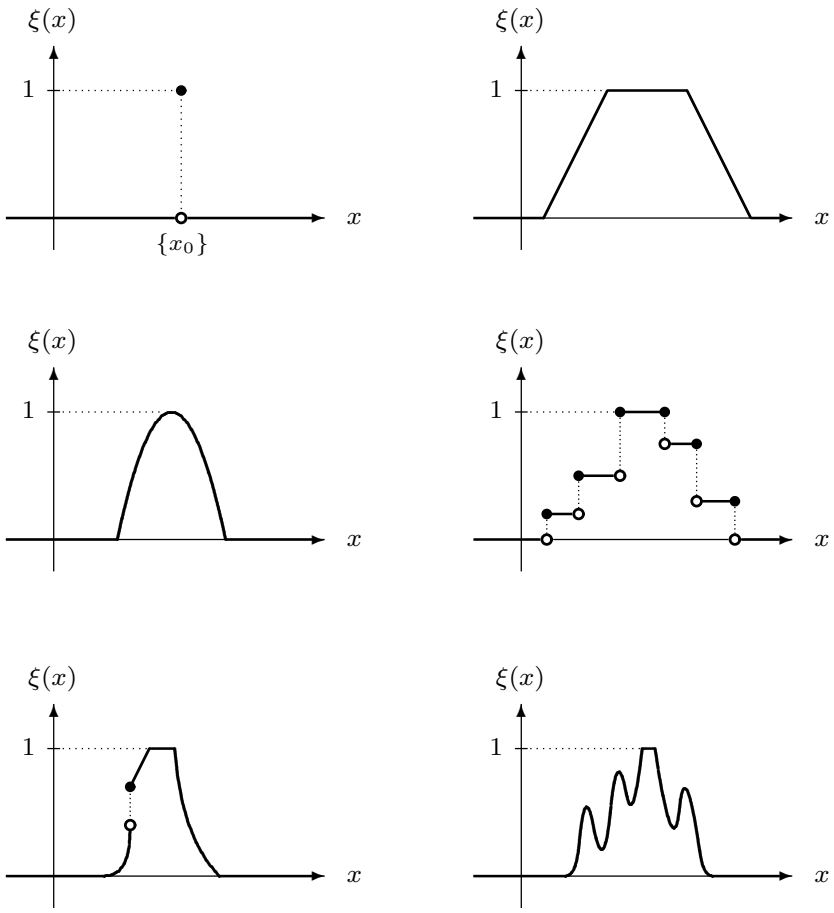


FIGURE 1. Characterizing Functions

An important topic is how to obtain the characterizing function of a fuzzy observed (measured) quantity. There is no general solution to this problem, but some measurement situations allow to generate the characterizing function. This problem is a special case of the problem obtaining the membership function of a fuzzy set. Four references for that are the following: [4] and [5] are good introductions to the problem, [2] and [8] are more recent application-oriented contributions to this topic.

Example 3.2. For a digital measurement equipment the measurement result is a decimal number with finitely many digits. About the remaining infinitely many

digits nothing is known. Therefore the measurement result is an interval $[\underline{x}, \overline{x}]$, where the number $\underline{x}$ is the reading of the instrument and the remaining (infinitely many) decimals are all set to be 0. The number $\overline{x}$ is obtained if the remaining decimals are all set to be 9.

The characterizing function is the indicator function $\mathbb{1}_{[\underline{x},\overline{x}]}(\cdot)$ of the interval $[\underline{x}, \overline{x}]$. This is a special characterizing function.

Example 3.3. Let the result of the measurement of a one-dimensional continuous quantity be a light point on a screen. Then the light intensity $h(x)$ is a real valued, non-negative function of a real variable x. The characterizing function $\xi(\cdot)$ of the measurement result x^* is obtained by normalization in the following way:

$$\xi(x) := \frac{h(x)}{\max\left\{h\left(z\right) :\ z \in \mathbb{R}\right\}} \quad \forall\, x \in \mathbb{R}$$

This function $\xi(\cdot)$ is a characterizing function.

Example 3.4. In case the measurement result is characterized by the (fuzzy) boundary of a color intensity picture, the color intensity $g(x)$ for $x \in \mathbb{R}$ can be used to generate the characterizing function $\xi(\cdot)$ of the observed quantity x^* in the following way:

The so-called *scaled rate of change* of the color intensity transition is used, i.e. the normalized derivative of the function $g(\cdot)$, where the derivative is denoted by $g'(\cdot)$:

$$\xi(x) := \frac{|g'(x)|}{\max\left\{\ |g'\left(z\right)| :\ z \in \mathbb{R}\right\}} \quad \forall\, x \in \mathbb{R}$$

Again by this definition a characterizing function $\xi(\cdot)$ is obtained, which is characterizing the fuzzy measurement result x^*.

Example 3.5. For analogue measurement equipments with pointers the best one can do is to make a photograph of the pointer position on the measurement scale. This is a color intensity picture. Taking the scale on a straight line the color intensity is a fuzzy point on the scale. This situation can be dealt with as in Example 3.3.

For the measurement of a one-dimensional quantity in discrete case (finitely many points) we have to extend it to the whole real line. We can create a partly constant characterizing function (Example 3.6) or a partly linear characterizing function (Example 3.7).

Example 3.6. Let the function $h(\cdot)$ be a non-negative function defined in finitely many points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$ and let the distance between $x_1 < x_2 < \ldots < x_n$ be constant and equal to Δ.

We can extend the function $h(\cdot)$ to a non-negative partially constant function $g(\cdot)$ in the following way:

$$g(x) := \left\{ \begin{array}{cl} 0 & \text{for } x < x_1 - \frac{\Delta}{2} \\ h(x_1) & \text{for } x \in \left[x_1 - \frac{\Delta}{2}, x_1 + \frac{\Delta}{2}\right) \\ \max\{h(x_1), h(x_2)\} & \text{for } x = x_1 + \frac{\Delta}{2} \\ \cdots & \\ h(x_i) & \text{for } x \in \left(x_i - \frac{\Delta}{2}, x_i + \frac{\Delta}{2}\right) \\ \max\{h(x_i), h(x_{i+1})\} & \text{for } x = x_i + \frac{\Delta}{2} \\ \cdots & \\ h(x_n) & \text{for } x \in \left(x_n - \frac{\Delta}{2}, x_n + \frac{\Delta}{2}\right] \\ 0 & \text{for } x > x_n + \frac{\Delta}{2} \end{array} \right\} \qquad \forall\, x \in \mathbb{R}$$

The function $g(\cdot)$ is defined on the whole real line, we can construct the characterizing function by normalization in the same way as in Example 3.3. In Figure 2(b) is shown the results of such construction.

Example 3.7. Let the function $h(\cdot)$ be a non-negative function defined in finitely many points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$, where $x_1 < x_2 < \ldots < x_n$ and $h(x_1) = h(x_n) = 0$.

We can extend the function $h(\cdot)$ to a non-negative partially linear function $g(\cdot)$ in the following way:

$$g(x) := \left\{ \begin{array}{cl} 0 & \text{for } x < x_0 \\ \frac{x - x_1}{x_2 - x_1}\, h(x_2) & \text{for } x \in [x_1, x_2) \\ \cdots & \\ h(x_i) + \frac{x - x_i}{x_{i+1} - x_i}\, (h(x_{i+1}) - h(x_i)) & \text{for } x \in [x_i, x_{i+1}) \\ \cdots & \\ h(x_{n-1}) - \frac{x - x_{n-1}}{x_n - x_{n-1}}\, h(x_{n-1}) & \text{for } x \in [x_{n-1}, x_n] \\ 0 & \text{for } x > x_n \end{array} \right\} \qquad \forall\, x \in \mathbb{R}$$

This situation can be dealt with as in Example 3.3 or Example 3.4. In Figure 2(c) is shown the results of such construction (using the normalization approach).

Remark 3.8. Many measurement results are in one of the forms given by the above examples.

Another problem is measurement results of vector-valued quantities. This is explained in the next section.

4. Fuzzy Vectors and Vector-characterizing Functions

For vector-valued continuous quantities $\underline{x} = (x_1, \ldots, x_k)$, measurement results are again subject to variability, errors, and fuzziness. The mathematical models for errors are stochastic quantities, and that for fuzziness so-called *fuzzy vectors*.

Definition 4.1. A fuzzy subset $\underline{x}^*$ of the k-dimensional Euclidean space $\mathbb{R}^k$ is called *fuzzy vector* if its membership function $\zeta(\cdot, \ldots, \cdot)$ fulfils the following conditions:

(1) $\zeta : \mathbb{R}^k \to [0, 1]$
(2) $\mathrm{supp}\,[\zeta\,(\cdot, \ldots, \cdot)] := \left\{\underline{x} \in \mathbb{R}^k \colon\ \zeta\,(\underline{x}) > 0\right\}$ is a bounded subset of $\mathbb{R}^k$
(3) $\forall\ \delta \in (0, 1]$ the δ-Cut $\mathcal{C}_\delta\,[\zeta\,(\cdot, \ldots, \cdot)]$ is non-empty and a finite union of compact connected subsets of $\mathbb{R}^k$

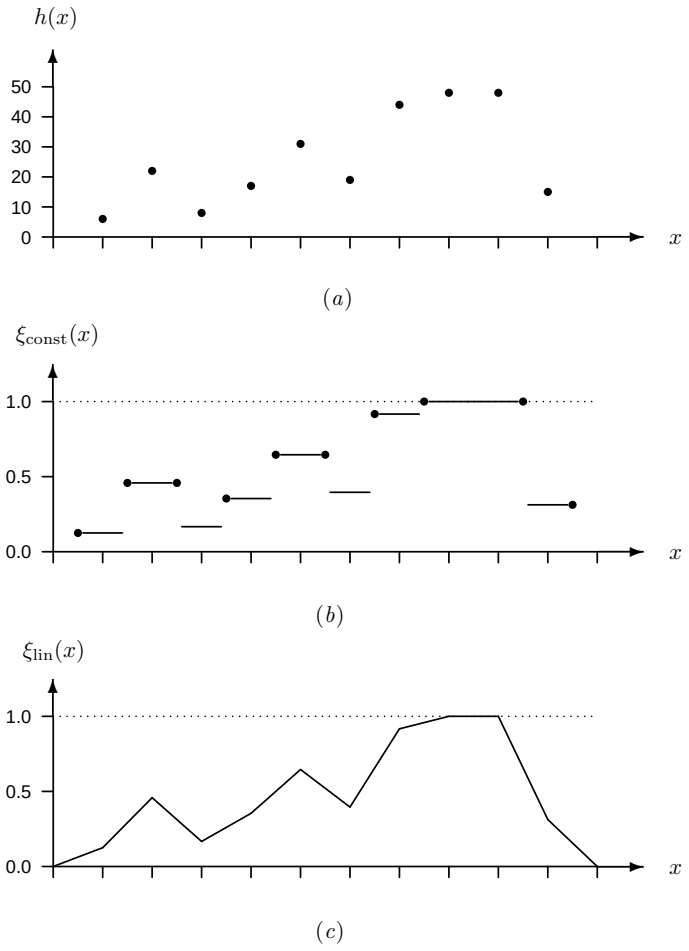


FIGURE 2. Construction of Characterizing Functions

- (a) Selected points from the function $h(\cdot)$, from which the characterizing function is constructed in Examples 3.6 and 3.7
- (b) Characterizing function constructed from the selected points in (a) according to Example 3.6
- (c) Characterizing function constructed from the selected points in (a) according to Example 3.7

Functions $\zeta(\cdot, \ldots, \cdot)$ fulfilling the conditions in Definition 4.1 are called *vector-characterizing functions.*

In Figure 3 examples of vector-characterizing functions are depicted.

Again, it is important how to obtain the vector-characterizing function $\zeta(\cdot, \ldots, \cdot)$ of a fuzzy vector.

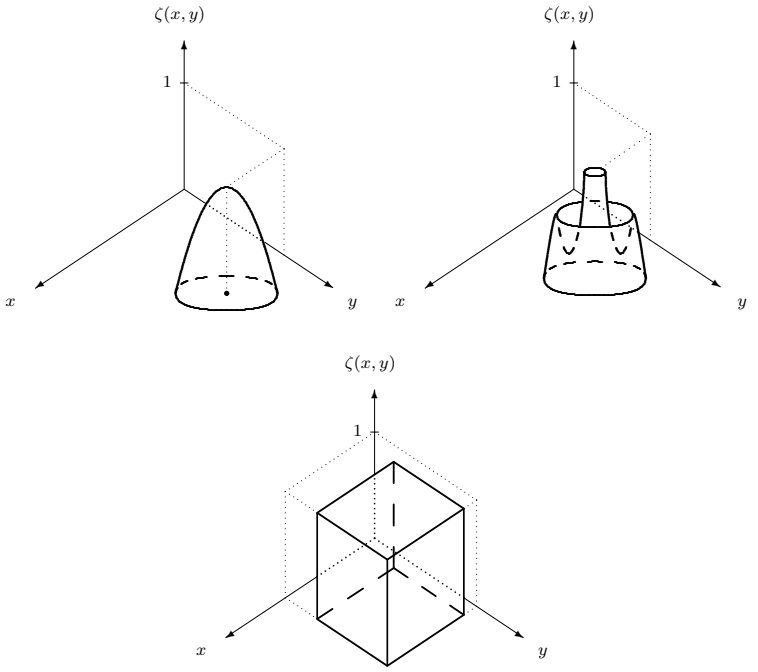


FIGURE 3. Vector-characterizing Functions

Example 4.2. In the situation of 2-dimensional continuous quantities where the so-called measurement results are presented on a screen, we can obtain the vector-characterizing function in the following way:

Let $h(x, y)$ be the light intensity of the light point $\underline{x}^* = (x, y)^*$. Then the function $h(\cdot, \cdot)$ is non-negative and bounded on the whole plane. Therefore, the values $\zeta(x, y)$ of the vector-characterizing function $\zeta(\cdot, \cdot)$ are given by

$$\zeta(x, y) := \frac{h(x, y)}{\max\{h(u, v) \colon (u, v) \in \mathbb{R}^2\}} \qquad \forall\, (x, y) \in \mathbb{R}^2.$$

The function $\zeta(\cdot, \cdot)$ defined above fulfils the conditions for vector-characterizing functions. In Figures $4(a)$ and $5(a)$ examples are shown.

Example 4.3. In case of 2-dimensional quantities, where the light intensities are defined on a finite number of points, the vector-characterizing function can be obtained in the following way:

Let $I = \{(x_1, y_1), \ldots, (x_1, y_m),\ \ldots,\ (x_n, y_1), \ldots,\ (x_n, y_m)\} \subset \mathbb{R}^2$ be a set with $n, m \in \mathbb{N}$, where x_i and y_j are linearly ordered, and the points (x_i, y_j) are equally

spaced in the plane:

$$\begin{array}{ll} \exists \Delta_x \in \mathbb{R}^+ & \forall\, i \in \{1,\dots,n-1\} \quad x_{i+1} = x_i + \Delta_x \\ \exists \Delta_y \in \mathbb{R}^+ & \forall\, j \in \{1,\dots,m-1\} \quad y_{j+1} = y_j + \Delta_y \end{array}$$

Let $h(\cdot,\cdot)$ be a non-negative function defined on the set I and let $h(x_1,\cdot) = h(\cdot,y_1) = h(x_n,\cdot) = h(\cdot,y_m) = 0$. Examples of such functions are depicted in Figures 4(b) and 5(b).

Then we can create a partly constant function $g(\cdot,\cdot)$ in the following way:

$$g(x,y) := \left\{ \begin{array}{l} h(x_i,y_j) \\ \qquad \text{for } x \in (x_i - \frac{1}{2}\Delta_x, x_i + \frac{1}{2}\Delta_x) \text{ and } y \in (y_j - \frac{1}{2}\Delta_y, y_j + \frac{1}{2}\Delta_y) \\ \\ \max\{h(x_i,y_j), h(x_{i+1},y_j)\} \\ \qquad \text{for } x = x_i + \frac{1}{2}\Delta_x \text{ and } y \in (y_j - \frac{1}{2}\Delta_y, y_j + \frac{1}{2}\Delta_y) \\ \\ \max\{h(x_i,y_j), h(x_i,y_{j+1})\} \\ \qquad \text{for } x \in (x_i - \frac{1}{2}\Delta_x, x_i + \frac{1}{2}\Delta_x) \text{ and } y = y_j + \frac{1}{2}\Delta_y \\ \\ \max\{h(x_i,y_j), h(x_{i+1},y_j), h(x_i,y_{j+1}), h(x_{i+1},y_{j+1})\} \\ \qquad \text{for } x = x_i + \frac{1}{2}\Delta_x \text{ and } y = y_j + \frac{1}{2}\Delta_y \\ \\ \qquad\qquad\qquad \text{where } i \in \{1,\dots,n-1\},\ j \in \{1,\dots,m-1\} \\ \\ 0 \qquad \text{for } x \notin (x_1,x_n) \text{ or } y \notin (y_1,y_m) \end{array} \right\}$$

$$\forall\, (x,y) \in \mathbb{R}^2$$

The vector-characterizing function $\zeta(\cdot,\cdot)$ can be obtained from the function $g(\cdot,\cdot)$ by normalization:

$$\zeta(x,y) := \frac{g(x,y)}{\max\left\{g(u,v) \colon (u,v) \in \mathbb{R}^2\right\}} \qquad \forall\, (x,y) \in \mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above again fulfils the conditions for vector-characterizing functions. In Figures 4(c) and 5(c) are shown the results of this construction.

Example 4.4. The so-called partly linear approach is a little more complicated. In the one-dimensional case we have simply connected two neighbouring points. In the two dimensional case there are four neighbouring points arranged in a rectangle. We want to interpolate a plane through four edge points of this rectangle, but this is not possible in all cases. We will create a new point in the centre of each rectangle and set it's value as mean of the values at the edges of the rectangle.

Let the set I and the function $h(\cdot,\cdot)$ be defined in the same way as in the previous example.

We define $x_{i+\frac{1}{2}} := x_i + \frac{1}{2}\Delta_x$, $y_{j+\frac{1}{2}} := y_j + \frac{1}{2}\Delta_y$, and define the set J in the following way:

$$J := \left\{ \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}\right) \colon i \in \{1,\dots,n-1\},\ j \in \{1,\dots,m-1\} \right\} \subset \mathbb{R}^2$$

We extend the function $h(\cdot,\cdot)$ to the set J in the following way:

$$h(x,y) = \frac{1}{4}\left(\begin{array}{l} h(x-\frac{1}{2}\Delta_x, y-\frac{1}{2}\Delta_y) + h(x-\frac{1}{2}\Delta_x, y+\frac{1}{2}\Delta_y) + \\ \qquad\qquad + h(x+\frac{1}{2}\Delta_x, y-\frac{1}{2}\Delta_y) + h(x+\frac{1}{2}\Delta_x, y+\frac{1}{2}\Delta_y) \end{array} \right)$$

$$\forall\, (x,y) \in J$$

Then we can create a partly linear function $g(\cdot,\cdot)$ in the following way:

$$g(x,y) := \begin{cases} h(x_i,y_j) + \frac{(x-x_i)\Delta_y-(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_j)-h(x_i,y_j))+ \\ \qquad +\frac{2(y-y_j)}{\Delta_y}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_j)) \\ \text{for } x > x_i \text{ and } \frac{x-x_i}{x_{i+1}-x_i} \le \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} > \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_{i+1},y_j) + \frac{(x-x_{i+1})\Delta_y+(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_{j+1})-h(x_{i+1},y_j))- \\ \qquad -\frac{2(x-x_{i+1})}{\Delta_x}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_{i+1},y_j)) \\ \text{for } y \le y_{j+1} \text{ and } \frac{x-x_i}{x_{i+1}-x_i} \le \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} \le \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_i,y_{j+1}) + \frac{(x-x_i)\Delta_y+(y-y_{j+1})\Delta_x}{\Delta_x\Delta_y}(h(x_{i+1},y_{j+1})-h(x_i,y_{j+1}))- \\ \qquad -\frac{2(y-y_{j+1})}{\Delta_y}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_{j+1})) \\ \text{for } x \le x_{i+1} \text{ and } \frac{x-x_i}{x_{i+1}-x_i} > \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} \le \frac{y-y_j}{y_{j+1}-y_j} \\ \\ h(x_i,y_j) + \frac{-(x-x_i)\Delta_y+(y-y_j)\Delta_x}{\Delta_x\Delta_y}(h(x_i,y_{j+1})-h(x_i,y_j))+ \\ \qquad +\frac{2(x-x_i)}{\Delta_x}(h(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}})-h(x_i,y_j)) \\ \text{for } y > y_i \text{ and } \frac{x-x_i}{x_{i+1}-x_i} > \frac{y-y_j}{y_{j+1}-y_j} \text{ and } \frac{x-x_{i+1}}{x_i-x_{i+1}} > \frac{y-y_j}{y_{j+1}-y_j} \\ \\ \text{where } i \in \{1,\dots,n-1\},\ j \in \{1,\dots,m-1\} \\ \\ 0 \quad \text{for } x \notin (x_1,x_n) \text{ or } y \notin (y_1,y_m) \end{cases}$$

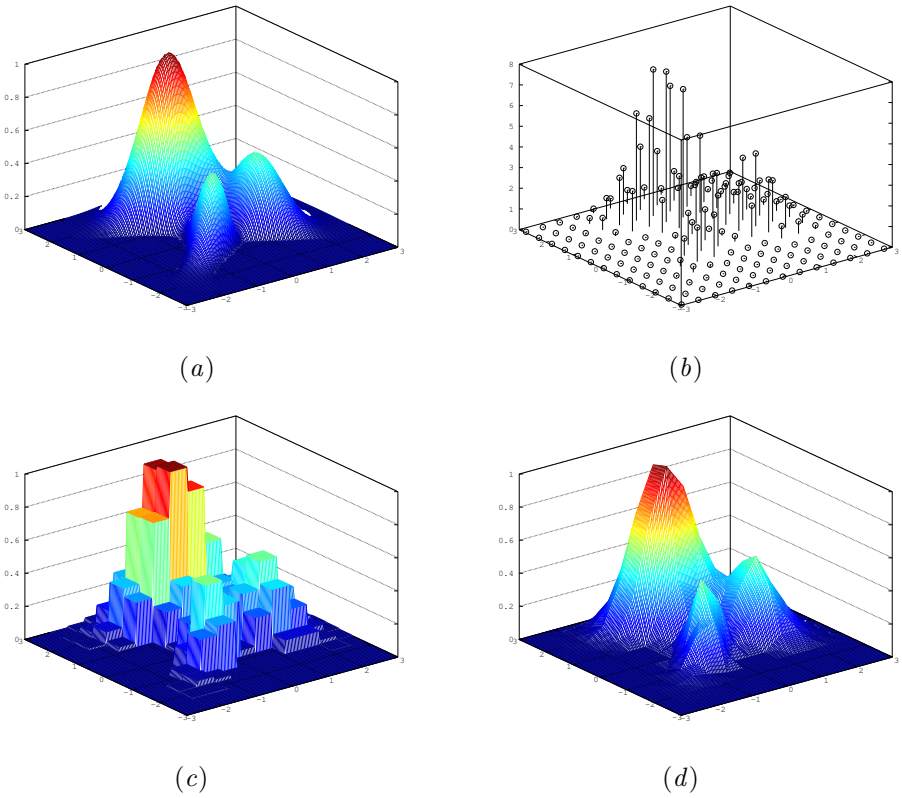
$$\forall\,(x,y) \in \mathbb{R}^2$$

The vector-characterizing function $\zeta(\cdot,\cdot)$ can be obtained again from the function $g(\cdot,\cdot)$ by normalization:

$$\zeta(x,y) := \frac{g(x,y)}{\max\{g(u,v)\colon (u,v) \in \mathbb{R}^2\}} \qquad \forall\,(x,y) \in \mathbb{R}^2$$

The function $\zeta(\cdot,\cdot)$ defined above fulfils the conditions for vector-characterizing functions. In Figures 4(d) and 5(d) the results of this construction are given.

$$h(x,y) = \max \left\{ \begin{array}{l} 3\left(1-x\right)^2 e^{\left(-x^2-(y+1)^2\right)} - 10\left(\frac{1}{5}x - x^3 - y^5\right) e^{\left(-x^2-y^2\right)} \\ \qquad\qquad -\frac{1}{3}e^{\left(-(x+1)^2-y^2\right)} - \frac{1}{100} \\ 0 \end{array} \right\}$$

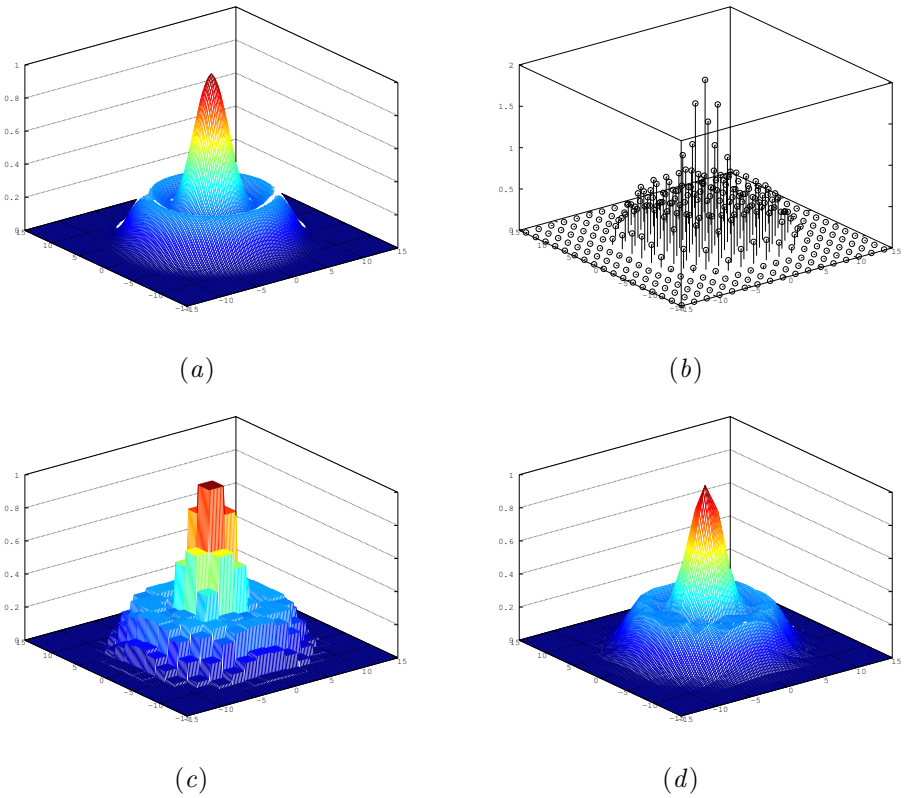


(a) (b) (c) (d)

FIGURE 4. Two-dimensional Vector-characterizing Functions

(a) Vector-characterizing function constructed from the function $h(\cdot,\cdot)$ according to Example 4.2
(b) Selected points from the function $h(\cdot,\cdot)$, from which the vector-characterizing function is constructed in Examples 4.3 and 4.4
(c) Vector-characterizing function constructed from the selected points in (b) according to Example 4.3
(d) Vector-characterizing function constructed from the selected points in (b) according to Example 4.4

$$h(x,y) = \left\{ \begin{array}{ll} 1 + \frac{1}{3}\pi & \text{for } x = y = 0 \\ \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} + \frac{1}{12}\left(4\pi - \sqrt{x^2+y^2}\right) & \text{for } 0 < \sqrt{x^2+y^2} \leq 4\pi \\ 0 & \text{for } 4\pi < \sqrt{x^2+y^2} \end{array} \right\}$$



(a) (b) (c) (d)

FIGURE 5. Two-dimensional Vector-characterizing Functions

(a) Vector-characterizing function constructed from the function $h(\cdot,\cdot)$ according to Example 4.2
(b) Selected points from the function $h(\cdot,\cdot)$, from which the vector-characterizing function is constructed in Examples 4.3 and 4.4
(c) Vector-characterizing function constructed from the selected points in (b) according to Example 4.3
(d) Vector-characterizing function constructed from the selected points in (b) according to Example 4.4

5. Vectors of Fuzzy Numbers

Definition 5.1. A n-dimensional *vector of fuzzy numbers* $(x_1^*, \ldots, x_n^*)$ is a vector containing n fuzzy numbers $x_1^*, \ldots, x_n^*$. It is determined by n characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$ belonging to the fuzzy numbers $x_1^*, \ldots, x_n^*$.

Definition 5.2. A function $T : [0,1] \times [0,1] \to [0,1]$ is called *triangular norm* or *t-norm*, if and only if $\forall\, x, y, z \in [0,1]$ the following conditions are fulfilled:

(1) $T(x,y) = T(y,x)$, i.e. T is commutative
(2) $T(T(x,y),z) = T(x,T(y,z))$, i.e. T is associative
(3) $T(x,1) = x$, i.e. 1 is neutral to T
(4) $x \leq y \Rightarrow T(x,z) \leq T(y,z)$, i.e. T is monotone

Combination of fuzzy numbers into a fuzzy vector is possible based on t-norms. For two fuzzy numbers a^* and b^* with corresponding characterizing functions $\xi_{a^*}(\cdot)$, $\xi_{b^*}(\cdot)$ a fuzzy vector $(a,b)^*$ is given by its vector-characterizing function $\zeta_{(a,b)^*}(\cdot,\cdot)$, whose values $\zeta_{(a,b)^*}(x,y)$ are defined based on a t-norm T by:

$$\zeta_{(a,b)^*}(x,y) := T(\xi_{a^*}(x), \xi_{b^*}(y)) \qquad \forall\, (x,y) \in \mathbb{R}^2$$

In the general case of n fuzzy numbers $x_1^*, \ldots, x_n^*$ with corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$ a fuzzy vector $(x_1, \ldots, x_n)^*$ is given by its vector-characterizing function $\zeta_{(x_1,\ldots,x_n)^*}$ whose values $\zeta_{(x_1,\ldots,x_n)^*}(x_1, \ldots, x_n)$ are defined based on a t-norm T by using its associativity:

$$\zeta_{(x_1,\ldots,x_n)^*}(x_1, \ldots, x_n) := T(\xi_1(x_1), T(\ldots, T(\xi_{n-1}(x_{n-1}), \xi_n(x_n)) \ldots))$$

$$\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$$

For details see [6].

Remark 5.3. A fuzzy vector can be obtained from a vector of fuzzy numbers using a t-norm. For statistical applications the most important t-norm is the minimum t-norm.

In Figure 6 examples for the above construction using the following t-norms are given:

Minimum t-norm:

$$T_{min}(x,y) = \min\{x,y\} \qquad \forall\, (x,y) \in [0,1]^2$$

Product t-norm:

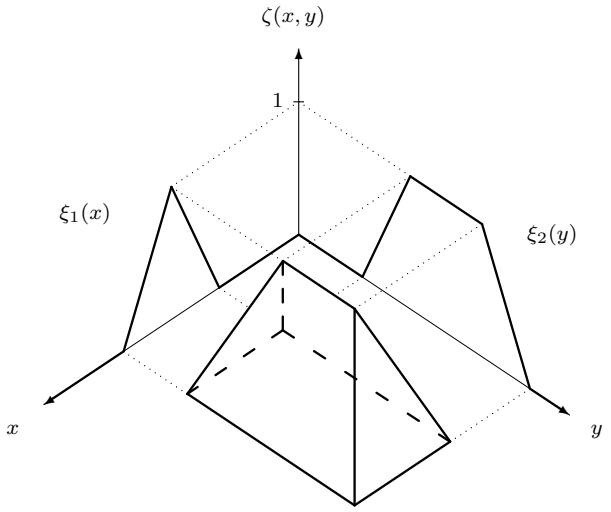
$$T_{prod}(x,y) = x \cdot y \qquad \forall\, (x,y) \in [0,1]^2$$

Limited sum t-norm:

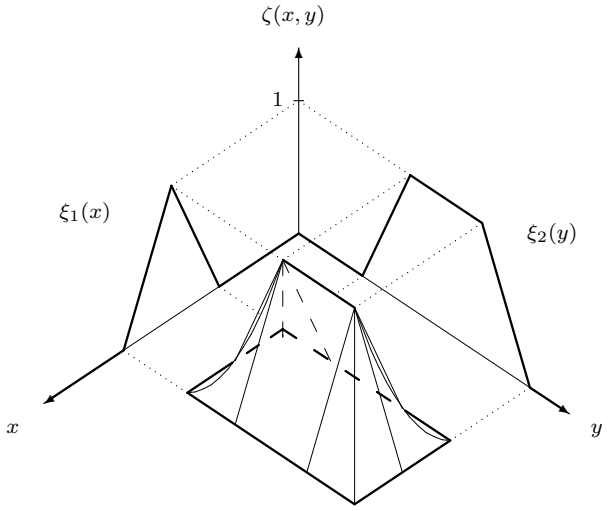
$$T_{lsum}(x,y) = \max\{x + y - 1, 0\} \qquad \forall\, (x,y) \in [0,1]^2$$

Drastic product t-norm:

$$T_{dp}(x,y) = \min\{x,y\} \cdot \mathbb{1}_{\{\max\{x,y\}\}}(1) \qquad \forall\, (x,y) \in [0,1]^2$$

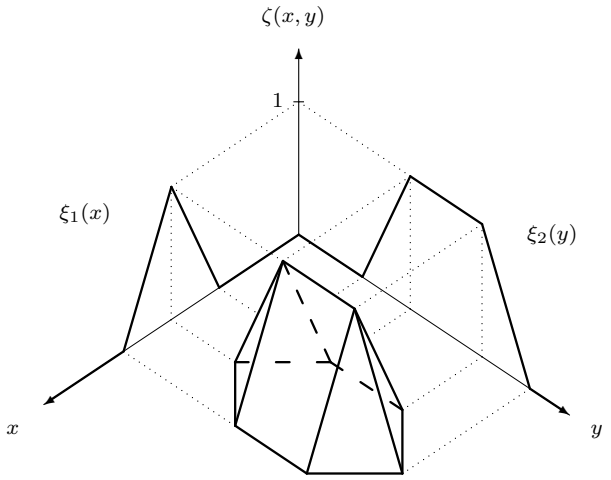


(a) Combination of Two Fuzzy Numbers Using the Minimum t-norm

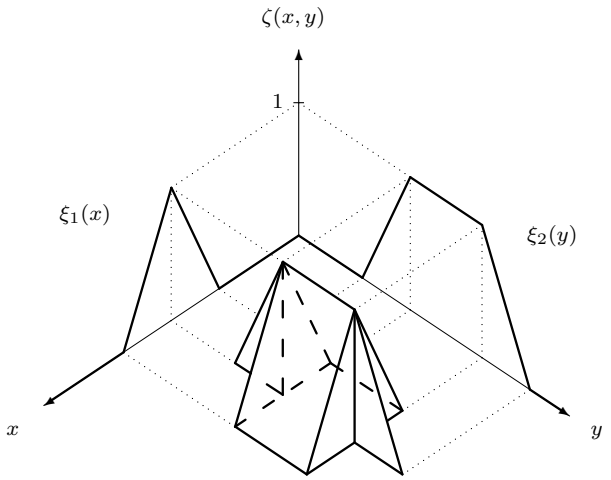


(b) Combination of Two Fuzzy Numbers Using the Product t-norm

FIGURE 6. Combinations of Fuzzy Numbers



(c) Combination of Two Fuzzy Numbers Using the Limited Sum t-norm



(d) Combination of Two Fuzzy Numbers Using the Drastic Product t-norm

FIGURE 6. Combinations of Fuzzy Numbers

6. Conclusion

There is no general rule of how to obtain the membership function of a fuzzy set. But in this paper for different important situations methods for the construction of the characterizing function of a fuzzy number as well as for the construction of vector-characterizing functions of fuzzy vectors are given.

In Section 2 of this article the δ-Cuts are introduced and an universal approach for the construction of the membership function based on a nested family of sets $\big(A_\delta;\ \delta \in (0,1]\,\big)$ is explained. In Theorem 2.4 a necessary and sufficient condition for the nested family of sets to be equal to the δ-Cuts of the generated fuzzy set is given.

In Section 3 fuzzy numbers are introduced and different methods are proposed how to obtain the characterizing function for one-dimensional measured quantities. Similarly in Section 4 fuzzy vectors are introduced, and different methods to obtain the vector-characterizing function for multi-dimensional data are proposed and graphically demonstrated. In Section 5 t-norms are used to describe how to combine fuzzy numbers into a fuzzy vector. Using the above described principles the characterizing function of a fuzzy number or the vector-characterizing function of a fuzzy vector can be obtained for different types of measurement data.

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